AN INTRODUCTION TO THE PRACTICAL USE OF MUSIC-MATHEMATICS

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1. The Basics

We begin by assigning an integer mod 12 to each named note of the chromatic scale. We arbitrarily set $C = 0$, and then then increase or decrease by an integer for each half step.

C	$C\sharp/D\flat$	D	$D\sharp/E\flat$	E	F	$F\sharp/G\flat$
0	1	2	3	4	5	6

G	$G\sharp/A\flat$	A	$A\sharp/B\flat$	B	C	$C\sharp/D\flat$
7	8	9	10	11	0	1

Table 1. $\mathbb{Z}_{12}$ and the Chromatic Scale

An interval of 12 is obviously an octave, 7 a perfect fifth, 5 a perfect fourth, 4 a major third, 3 a minor third, and 6 a tritone. From the chart we can also see that a C above an F is a perfect fifth, a very consonant or aesthetically pleasant interval, but an F above a C is a perfect fourth, which is still consonant, if slightly less so. Now I will explain how we may move these notes about in $\mathbb{Z}_{12}$.

2. Translations and Inversions

Next we look at two functions, T_n and I_n. T_n translates (or musically transposes) a note or a group of notes by n (or n half-steps).

Definition 2.1. Let n be an integer mod 12. Then $T_n(x) = x + n$ maps $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$ and is a *translation of x by n*. Also, given a sequence of notes $A = \langle x_1, x_2, x_3 \rangle$ played simultaneously or consecutively, $T_n(A) = \langle x_1 + n, x_2 + n, x_3 + n \rangle$.

Example 2.2. $T_7(\langle 0, 4, 7 \rangle) = \langle 7, 11, 2 \rangle$. In musical terms, this is the transposition of a C triad by a fifth to a G triad.

Here it is worth noting that when referring to a major triad, a capital letter is used, whereas a minor triad bears a lower case letter. So $C = \langle 0, 4, 7 \rangle$ whereas $c = \langle 0, 3, 7 \rangle$.

Definition 2.3. Let n be an integer mod 12. Then $I_n(x) = -x + n$ maps $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$ and is an *inversion of x about n*.

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Example 2.4. $I_0(4) = -4 = 8$, $I_7(2) = -2 + 7 = 5$, and $I_0(6) = 6$. This last example illustrates the significance of the tritone. Furthermore, $I_0(\langle 0, 4, 7\rangle) = \langle 0, 8, 5\rangle$ and $I_7(\langle 0, 4, 7\rangle) = \langle 7, 3, 0\rangle$.

3. The T/I Group

Now, examining the intervalic structure of these chords, we see that major triads are a major third and a minor third, or 4,3, while minor triads are a minor third and a major third, 3,4. Translations maintain the intervalic structure, sending a major triad to a major triad and a minor to a minor. Inversions, on the other hand, flip the order of the intervals and thus the chord type as well, major to minor or vice versa. So between these two functions, we can form all 24 major and minor triads from any one of those 24. Below[1] we have these triads in an array.

Prime Forms	Inverted Forms
$C = \langle 0, 4, 7\rangle$	$\langle 0, 8, 5\rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8\rangle$	$\langle 1, 9, 6\rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9\rangle$	$\langle 2, 10, 7\rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10\rangle$	$\langle 3, 11, 8\rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11\rangle$	$\langle 4, 0, 9\rangle = a$
$F = \langle 5, 9, 0\rangle$	$\langle 5, 1, 10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1\rangle$	$\langle 6, 2, 11\rangle = b$
$G = \langle 7, 11, 2\rangle$	$\langle 7, 3, 0\rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3\rangle$	$\langle 8, 4, 1\rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4\rangle$	$\langle 9, 5, 2\rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5\rangle$	$\langle 10, 6, 3\rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6\rangle$	$\langle 11, 7, 4\rangle = e$

FIGURE 1. The T/I Group

This set was formed from translated and inverted forms of the C major triad, but as stated above, it can be formed from any triad via translations and inversions. If we compose these translations and inversions, we see that this array actually maintains the axioms of a group, treating composition as the operation. Now we will advance to specific compositions of T_n and I_n and their relation to musical harmonies.

4. The PLR Group

P, L, and R are three functions, $T/I \to T/I$, which are both mathematically and musically significant. In what follows, x is a member of the T/I group above, and its type is either major (left column) or minor (right).

[1]From Thomas Fiore's paper *Music and Mathematics*, which can be found at http://www.math.uchicago.edu/ fiore/1/musictotal.pdf.

$P(x)$ maps x to the triad of opposite type with the first and third notes switched. Musically, P maintains the perfect fifth interval, but changes the inner major third to a minor one. P maps x to its parallel major or minor, C to c or a to A.

$L(x)$ maps x to the triad of opposite type with second and third notes switched. Musically, L switches the tonic note for the leading tone, C to e or A to $c\sharp/d\flat$.

$R(x)$ maps x to the triad of opposite type with the first and second notes switched. Musically, R maps x to its relative major or minor, C to a or A to $f\sharp/g\flat$.

Let us examine these a little further, namely via a few more examples. Each function maintains two notes of the given triad, which gives them credibility as changes that would sound "musical." Each function switches these two notes, and each switches a different pair. Furthermore each function is bijective. Thus one method for finding $P(x)$ for a given triad x is to look in the opposite column for the one triad with the outer pair of notes switched. There will be only one, by bijectivity, so this method will always work. However, it requires sifting through the table of the T/I group.

However, each of these functions is a composition of translations and an inversion, and as explained above, these maintain and reverse, respectively, the intervalic structure of a triad. Thus P, L, or R each reverse the order of intervals, from 4, 3 to 3, 4. For example,

Example 4.1. $C = \langle 0,4,7\rangle$, $P(C) = c = \langle 7,3,0\rangle = \langle 0,3,7\rangle$.

Thus the different intervals are clear. But taking c in its form found in the T/I array, $\langle 7,3,0\rangle$, we see that the intervals actually go from 4, 3 to -4, -3. This is true of all three functions, as they are each formed from translations and an inversion, and because each retains two notes, we have a faster method of calculating the output of a function.

For P, the parallel major or minor, we switch the first and third. $C = \langle 0,4,7\rangle$, so $P(C) = \langle 7, X, 0\rangle$. Now, since the intervals for C were 4, 3, we know that $P(C)$ should have intervals -4, -3. Thus $X = 7-4 = 0+3 = 3$. $P(C) = c = \langle 7,3,0\rangle$ as confirmed by the previous method. L switches the second and third notes, and R the first and second. A few more examples:

Example 4.2. $L(g) = L(\langle 2,10,7\rangle) = \langle X,7,10\rangle$. The intervals were -4, -3, so they must now be 4, 3. $X = 3$. $\langle 3,7,10\rangle = B\flat$. Correct!

Example 4.3. $R(A) = R(\langle 9,1,4\rangle) = \langle 1,9,X\rangle$. The intervals were 4, 3, so they must now be -4, -3. $X = 6.\langle 1,9,6\rangle = f\sharp$. Correct again.

This relatively simplifies the calculation of the functions, but what of the practical use. Given a chord progression, how can we find the compositions of P, L and R that take one chord to the next? If only one function is necessary then it should be easy to recognize, as there will be notes in common. This is the case with the superb example from Beethoven's 9th Symphony[2]. The chord progression in measures 143-176 of the second movement is as follows:

$$C, a, F, d, B\flat, g, E\flat, c, A\flat, f, D\flat, b\flat, G\flat, e\flat, B, g\sharp, E, c\sharp, A$$

[2]This was an example presented by Thomas Fiore in class but cited back to Richard Cohn's articles from 1991, 1992 and 1997

The sequence of functions needed to create this chord progression starting at C is beautifully simple:

$$R, L, R, L, R, L, R, L, R, L, R, L, R, L, R, L, R, L$$

To make it a little more clear:

$$\begin{array}{cc}
C \to R(C) = a & a \to L(a) = F \\
F \to R(F) = d & d \to L(d) = B\flat \\
B\flat \to R(B\flat) = g & g \to L(g) = E\flat \\
E\flat \to R(E\flat) = c & c \to L(c) = A\flat \\
A\flat \to R(A\flat) = f & f \to L(f) = D\flat \\
D\flat \to R(D\flat) = b\flat & b\flat \to L(b\flat) = G\flat \\
G\flat \to R(G\flat) = e\flat & e\flat \to L(e\flat) = B \\
B \to R(B) = g\sharp & g\sharp \to L(g\sharp) = E \\
E \to R(E) = c\sharp & c\sharp \to L(c\sharp) = A
\end{array}$$

But this example features simple, though elegant, sequences of PLR functions. What of chord changes that require compositions of P, L, and R? There is a simpler, even direct, way to find the path. Here the Oettingen/Riemann graphical *Tonnetz* (”tone-network”) makes its entrance, a visual lattice that allows us to find the PLR composition for any given chord change.

5. **The Construction of the *Tonnetz***

We begin with a lattice. Place $C = 0$ in the middle. Then vertically, at every point, go up or down by a major third, an interval of 4. Then, horizontally go up by a minor third to the right, and down by a minor third to the left, an interval of 3. It is clear that going diagonally changes by an interval of 7, or a perfect fifth. Figure 2, on the next page, shows what I am describing[3].

Thus an upper triangle, with legs above the diagonal, is a major triad, whereas a lower triangle is a minor triad. P, L and R each maintain two notes of the chord, so each one flips the third note across the opposite side of the triangle, as shown on the graph. Therefore, to find the functions needed to change one chord to another, we find each chord, and then the shortest path between the two.

This method becomes even simpler if we redraw the graph. A vertex in the center of each triangle represents the triad that is formed by that triangle. These vertices can be connected by 3 lines that each represent P, L or R. Figure 3, on the next page, is an image of this graph[4].

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[3]Cohn, Rick. ”Neo-Riemannian Operations, Parsimonious Trichords, and Their ’Tonnetz’ Representations.” Journal of Music Theory Vol. 41, No. 1. (1997): p. 15.

[4]Douthett, Jack and Peter Steinbach. ”Parsimonious Graphs: A Study in Parsimony, Contextual Transformations and Modes of Limited Transposition.” Journal of Music Theory Vol. 42, No. 2 (1998): p. 249.

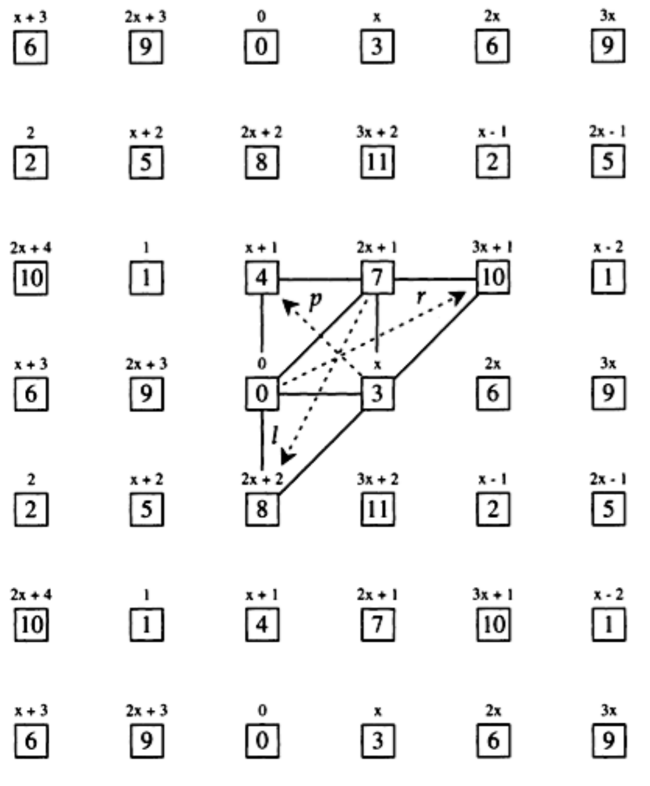

FIGURE 2. Oettingen and Riemann's *Tonnetz*

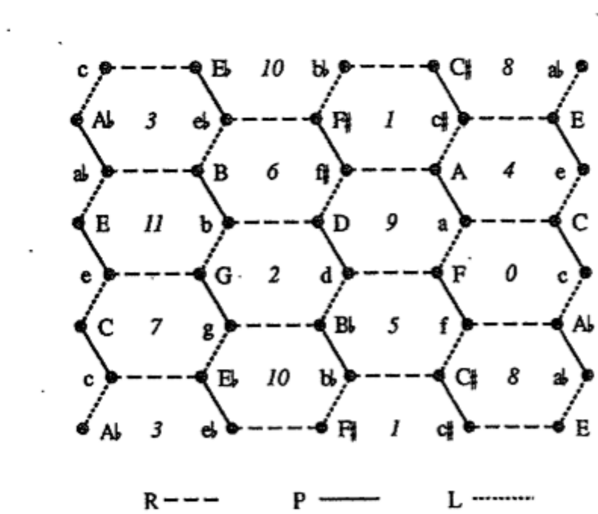

FIGURE 3. The Redrawn "Chickenwire" *Tonnetz*

6. The *Tonnetz* in Practice

Now the chords are explicitly named, and the paths are even more obvious. Let us try a few randomly chosen examples:

Example 6.1. Find the path from C to g. From the Chickenwire *Tonnetz* we see the shortest path to be L, R, P or equivalently P, R, L. Let's examine more thoroughly. From the lattice figure, we see $L(C) = \langle 11, 7, 4\rangle = e$. $R(\langle 11, 7, 4\rangle) = \langle 7, 11, 2\rangle = G$. Finally, $P(\langle 7, 11, 2\rangle) = \langle 2, 10, 7\rangle = g$. The second and third step are very intuitive musically, as e is the relative (R) minor of G, and g is the parallel (P) minor of G.

Example 6.2. Find the path from A to D (a universally common chord change): From chickenwire figure we see the path is R, L. Very simple. Let us explore this diagram a little more thoroughly.

Theorem 6.3. Given any triad, there is one and only one other triad to which the shortest path is length five functions.

Pf: Fix X, a triad. For the first case, let it be major. The fact that P, L and R all switch the type of chord is essential here. Thus:

Chords distance 1 from X: 3 minor chords, one along each path

Chords distance 2 from X: Each minor chord branches to a pair: 6 major chords

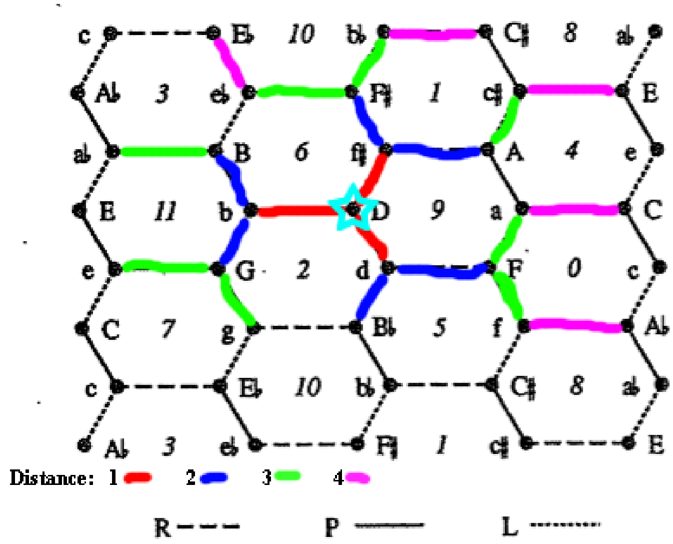

Figure 4. The Various Paths From a Major Chord

Chords distance 3 from X: There should be 12 chords. But every other pair of those six secondary paths form the opposite sides of a hexagon, and can thus branch to one common triad. So we have 3 fewer points. Then, following the path of L and P repeated, the same note is three steps away in either direction. See below. That is one less point. This leaves us with 8 minor chords. 11 of the total 12 minor chords have been accounted for.

$$\begin{array}{rr} X = \langle x, x+4, x+7 \rangle & \text{Other path} \\ P(X) = \langle x+7, x+3, x \rangle & L(X) = \langle x+11, x+7, x+4 \rangle \\ L \circ P(X) = \langle x+8, x, x+3 \rangle & P \circ L(X) = \langle x+4, x+8, x+11 \rangle \\ P \circ L \circ P(X) = \langle x+3, x-1, x+8 \rangle & L \circ P \circ L(X) = \langle x+15, x+11, x+8 \rangle \\ = \langle x+3, x+11, x+8 \rangle & = \langle x+3, x+11, x+8 \rangle \end{array}$$

Chords distance 4 from X: 5 major chords:

Those notes distance 3 which can branch to an R-path include the following, in descending order: LPL, LRL, LRP (or PRL), and PRP. These each have a distinct R-path. That accounts for 4 major chords.

Going the other direction from X, the path $RLPL$ is equivalent to $RPLP$ as shown above. This is our final major chord. All others of distance 4 starting with R are equivalent to the four chords listed first.

If X is minor, the argument is nearly identical. We can avoid that, however, if we simply flip the diagram. See below:

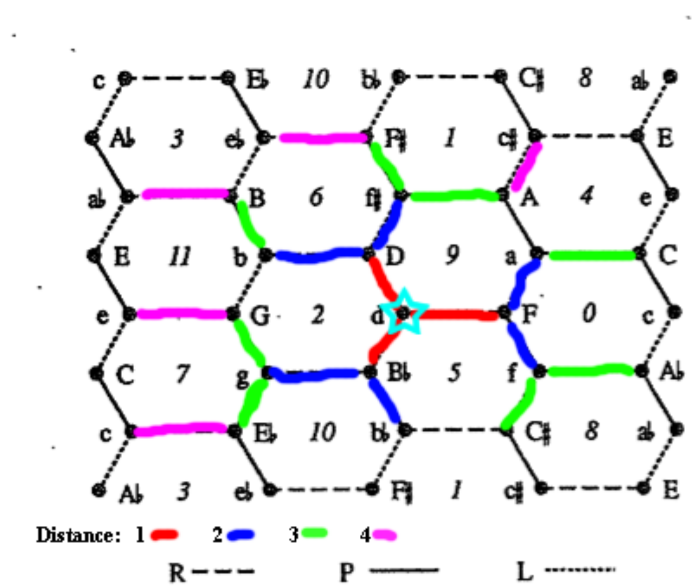

FIGURE 5. The Various Paths From a Minor Chord

7. **The Beatles: Practical Analysis and the PLR Group**

Here, to explore the musical potential of the PLR Group, I will provide a chronological sampling of the songs of The Beatles. The analysis is simple. Just as with Beethoven's progression, I will take the chords from these songs, and find the corresponding path on the *Tonnetz*.

Example 7.1. "Please Please Me" (1963)

```
E                                 A    E    G  A  B
  Last night I said these words to my girl
E                       A   E  (riff 1)
  I know you never even try girl


    A                 F#m
Come on (come on) Come on (come on)
 C#m               A
Come on (come on) Come on (come on)
         E         A        B         E (riff 2)
Please please me oh yeah like I please you.
```

Written sequentially, the chord progression is as follows:

$$E, \mathbf{A}, E, \mathbf{G}, A, \mathbf{B}, E, \mathbf{A}, E$$

$$\mathbf{A}, f\sharp, \mathbf{c\sharp}, A, \mathbf{E}, A, \mathbf{B}, E$$

From Figure 3, the "Chickenwire" *Tonnetz*, the corresponding sequence of PLR functions is (bold face functions go from bold face chords):

$$R \circ L, \mathbf{L} \circ \mathbf{R}, P \circ R, \mathbf{L} \circ \mathbf{R} \circ \mathbf{L} \circ \mathbf{R}, L \circ R \circ L \circ R, \mathbf{R} \circ \mathbf{L}, R \circ L, \mathbf{L} \circ \mathbf{R}$$

$$R \circ L^{5}, \mathbf{R}, R \circ L, \mathbf{L}, L \circ R, \mathbf{R} \circ \mathbf{L}, L \circ R \circ L \circ R, \mathbf{R} \circ \mathbf{L}$$

From this we can observe a few things. Transitions from one major chord to another, or generally between two chords of the same type, always require a composition of functions. Thus, the fact that the entire first chorus is major chords results in a rather cumbersome list of functions. Also, adjacent chords, separated by either an interval of 1 or 2 always require lengthy compositions, of 3 functions or more. For instance, **G**, A, **B** corresponds to $\mathbf{L} \circ \mathbf{R} \circ \mathbf{L} \circ \mathbf{R}$, $L \circ R \circ L \circ R$. We can see this in Figures 4 and 5, where of the 5 triads that are 4 compositions away, 4 of them are within a whole step, an interval of 2. Let us examine another song, further down the road and see how the functions play out.

[5]This function $R \circ L$ is from the E to the $\mathbf{A}$, connecting the two lines.

Example 7.2. "Yesterday" (1965)

```
G         F#m
Yesterday,
          B7              Em      Em/D  C
All my troubles seemed so far away
        D                       G          G/F#
Now it looks as though they're here to stay
    Em    A7      C  G
Oh, I believe in yesterday

F#m B7  Em  Em/D  C  G/B     Am        D        G
Why she had to    go I don't know she wouldn't say
F#m  B7   Em  Em/D  C    G/B    Am       D     G
I   said something wrong now I long for yesterday
```

Here is the chord progression:

$$G, \mathbf{f}\sharp, B7^{6}, \mathbf{e}, C, \mathbf{D}, G, \mathbf{e}, A7, \mathbf{C}, G$$

$$\mathbf{f}\sharp, B7, \mathbf{e}, C, \mathbf{G}, a, \mathbf{D}, G$$

$$\mathbf{f}\sharp, \ldots$$

And the sequence of PLR functions:

$$L \circ R \circ L, \mathbf{P} \circ \mathbf{R} \circ \mathbf{L}, R \circ L \circ P, \mathbf{L}, L \circ R \circ L \circ R, \mathbf{R} \circ \mathbf{L}, R, \mathbf{P} \circ \mathbf{R} \circ \mathbf{L}, P \circ R, \mathbf{L} \circ \mathbf{R}$$

$$L \circ R \circ L, \mathbf{P} \circ \mathbf{R} \circ \mathbf{L}, R \circ L \circ P, \mathbf{L}, L \circ R, \mathbf{R} \circ \mathbf{L} \circ \mathbf{R}, P \circ R \circ L, \mathbf{R} \circ \mathbf{L}$$

$$L \circ R \circ L \ldots$$

This sequence of functions is even more convoluted than the previous example. Let us examine another example before exploring this issue more deeply.

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[6] With extended chords ($B7, Fmaj7, A9$) we ignore the additional notes and treat it as only its base triad.

Example 7.3. "Here, There and Everywhere" (1966)

```
G             Bm         Bb
  To lead a better life,
                         Am7  D7
  I need my love to be here.

G     Am7   Bm          C          G     Am7
Here,      making each day of the year
 Bm           C           F#m7          B7
Changing my life with a wave of her hand,
 F#m7  B7   Em                Am         Am7   D7
Nobody can deny that there's something there.

            Bb          Gm
I want her everywhere
       Cm               D7             Gm  (riff 1)
And if she's beside me I know I need never care.
Cm                D7
But to love her is to need her

G         Am7   Bm            C         G     Am7
Everywhere,    knowing that love is to share,
```

The chord progression:

$$G, \mathbf{b}, B\flat, \mathbf{a7}, D7$$
$$\mathbf{G}, a7, \mathbf{b}, C, \mathbf{G}, a7, \mathbf{b}, C, \mathbf{f\sharp 7}, B7$$
$$\mathbf{f\sharp 7}, B7, \mathbf{e}, a, \mathbf{D7}$$
$$B\flat, \mathbf{g}, c, \mathbf{D7}, g, \mathbf{c}, D7$$
$$\mathbf{G}, \dots$$

And the function sequence:

$$L, \mathbf{R \circ P \circ L}, L \circ R \circ L, \mathbf{P \circ R \circ L}$$
$$R \circ L, \mathbf{R \circ L \circ R}, R \circ L \circ R \circ L, \mathbf{L \circ R \circ L}, L \circ R, \mathbf{R \circ L \circ R}, R \circ L \circ R \circ L, \mathbf{L \circ R \circ L}, R \circ P \circ R, \mathbf{P \circ R \circ L}$$
$$L \circ R \circ P, \mathbf{P \circ R \circ L}, R \circ L \circ P, \mathbf{L \circ R}, P \circ R \circ L$$
$$\mathbf{P \circ L}, R, \mathbf{L \circ R}, R \circ L \circ R \circ L \circ P, \mathbf{P \circ L \circ R}, L \circ R, \mathbf{R \circ L \circ R \circ L \circ P},$$
$$R \circ L, \dots$$

Again we obtain a chaotic sequence. There may a creative way, however, to look at this a little more clearly. If we look at every other triad, the functions become much simpler. For example:

$$G, B\flat, D7 \Longrightarrow P \circ R, L \circ P$$
$$a7, C, a7, C \Longrightarrow R, R, R$$
$$\mathbf{G}, \mathbf{b}, \mathbf{G}, \mathbf{b} \Longrightarrow \mathbf{L}, \mathbf{L}, \mathbf{L}$$
$$B\flat, c, g \Longrightarrow R \circ L \circ R, R \circ L$$

This does not work for every sequence in the song, but musically this method makes some sense. Because of the major key, any ascending sequence $(G, a7, b, C)$ will modulate between major and minor. This might make functions simpler, but because of the musical adjacency of sequential notes, we end up with lengthy compositions of PLR functions.

Furthermore, in terms of tension and its release, a cornerstone of all music, it is logical that adjacent chords would be less consonant and that those a chord apart would be more consonant. Thus this follows from the consonance of individual PLR functions.

8. **Conclusion**

We have now seen the underlying framework of a mathematical music system. The way in which functions may be set up, and how the three in particular carry especial significance within tonal music. In addition, they exhibit very interesting mathematical characteristics: the formation of a group, the lattice structure that so naturally maps them, and their relatively compact pathways within the *Tonnetz*.

I then studied a few musical examples, in hopes of finding some some practical way of analyzing these songs and a pattern or general trend. I did to a limited extent, but only by thinking in musical terms and using this knowledge to explain the results of the PLR functions. In short, this method carries more mathematical interest as a musical equivalent than actual musical practicality.