Colimits and Homological Algebra

Andy Kiersz

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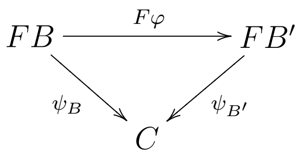
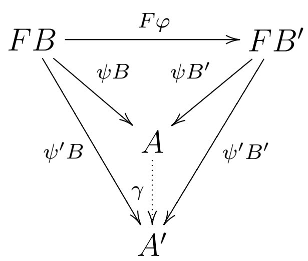
1 Colimits

We begin our discussion by defining the notion of a diagram. Let $\mathcal{A}$ be a category, and let $\mathcal{B}$ be a small category. A *diagram* in $\mathcal{A}$, based on $\mathcal{B}$ is a functor $F : \mathcal{B} \to \mathcal{A}$.

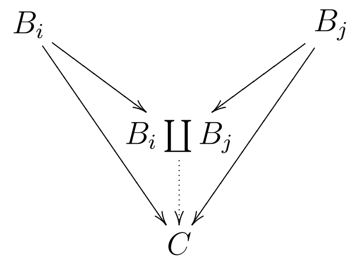
If C is an object of $\mathcal{A}$ and $F : \mathcal{B} \to \mathcal{A}$ is a diagram, we define a *morphism* $\psi : F \to C$ to be a collection $\{\psi_B\}$ of morphisms $\psi_B : B \to C$ for each object B of $\mathcal{B}$, such that if $\varphi : B \to B'$ is any morphism in $\mathcal{B}$, the following diagram commutes:



The *colimit* $\lim_{\rightarrow} F$ of F is an object A of $\mathcal{A}$ and a morphism $\psi : F \to A$ which satisfies the universal property that, given any morphism ψ' from F to an object A', there exists a unique morphism $\gamma : A \to A'$ making the following diagram commute:

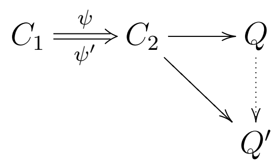


Two common and useful examples of colimits are the *coproduct* and *coequalizer*. We define the coproduct of a collection of objects $\{B_i\}$ of $\mathcal{A}$, written as $\coprod_i B_i$, to be the colimit of the diagram consisting of the objects B_i with no morphisms other than the identity morphisms. The coproduct thus satisfies, in the case of two objects, the familiar universal diagram:



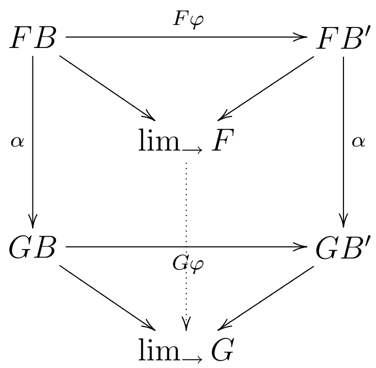
Examples of the coproduct are the disjoint union of sets or spaces, and the direct sum of abelian groups or, more generally, modules over a commutative ring.

The *coequalizer* of two morphisms $\psi, \psi' : C_1 \to C_2$ is the colimit of the diagram consisting of C_1, C_2, their identity morphisms, and the morphisms ψ and ψ', resulting in the again familiar universal diagram:



An example of a coequalizer is the cokernel of the difference of two homomorphisms in the category of abelian groups or modules over a commutative ring: If $f, g : A \to B$ are homomorphisms from the abelian group A to the abelian group B, the coequalizer of f and g will be the group $B/im(f - g)$.

If $F, G : \mathcal{B} \to \mathcal{A}$ are two functors, and $\alpha : F \to G$ is a natural transformation between these functors, then the following diagram commutes:



and we define $\lim_{\to}\alpha : \lim_{\to} F \to \lim_{\to} G$ to be the map induced by the vertical maps $FB \to \lim_{\to} G$, indicated by the dotted line in the above diagram. In this way, we see that $\lim_{\to}$ acts as a functor from the functor category $Fun(\mathcal{B}, \mathcal{A}) \to \mathcal{A}$.

2 Colimits and Exactness

One of the basic concepts of homological algebra is that of the short exact sequence. Let A be a commutative ring. We define a *complex* to be a set of A-modules M_i with homomorphisms $d_i : M_i \to M_{i-1}$ such that $\text{im}(d_i) \subseteq \ker(d_{i-1})$. Complexes are often represented by a diagram:

$$\cdots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

A complex is said to be *exact* if $\forall i, \text{im}(d_i) = \ker(d_{i-1})$. A *short exact sequence* is thus an exact complex of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

We can get much knowledge out of the fact that this sequence is exact. The kernel of α is the image of the zero homomorphism, hence $\ker(\alpha) = 0$, and so α is injective. Similarly, $\text{im}(\beta) = \ker(0) = C$, so β is surjective. Finally, we see that $\ker(\beta) = \text{im}(\alpha) \cong A$, so that $C \cong B/A$.

We want to see when a colimit in the category of A-modules preserves exactness. We say that a functor F from the category of A-modules to itself is *right-exact* if it takes short exact sequences of modules to sequences that are exact only at the right end, so that for the short exact sequence above we would have

$$FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC \longrightarrow 0$$

and we define a *left-exact* functor similarly. An *exact* functor is one that is both left- and right-exact.

Proposition 1 $\lim_{\rightarrow}$ *is a right-exact functor. In other words, let $\mathcal{A}$ be the category of A-modules, and let $\mathcal{B}$ be some small category. Let $F, G, H : \mathcal{B} \rightarrow \mathcal{A}$ be functors and let $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$ be natural transformations such that, $\forall B \in \mathcal{B}$,*

$$0 \longrightarrow FB \xrightarrow{\alpha} GB \xrightarrow{\beta} HB \longrightarrow 0$$

is exact. Then,

$$\lim_{\rightarrow} F \xrightarrow{\lim_{\rightarrow}\alpha} \lim_{\rightarrow} G \xrightarrow{\lim_{\rightarrow}\beta} \lim_{\rightarrow} H \longrightarrow 0$$

is a right exact sequence.

Proof: We will show that $\lim_{\rightarrow}\beta$ is an *epimorphism*. An epimorphism is a morphism $\gamma : A \rightarrow B$ such that, if $\eta_1, \eta_2 : B \rightarrow C$ are morphisms with $\eta_1\gamma = \eta_2\gamma$ then $\eta_1 = \eta_2$. In the category of modules over a commutative ring, epimorphisms are surjective functions and vice versa. Let $f, g : \lim H \rightarrow C$ so that $f \lim_{\rightarrow}\beta = g \lim_{\rightarrow}\beta$. f and g thus induce maps $HB \rightarrow C$ as we see in the following diagram:

$$\begin{array}{ccc}
GB & \xrightarrow{\phi} & \lim_{\rightarrow} G \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\lim_{\rightarrow}\beta} \\
HB & \xrightarrow{\psi} & \lim_{\rightarrow} H \\
& {\scriptstyle f\psi}\searrow\!\!\searrow{\scriptstyle g\psi} & {\scriptstyle g}\Downarrow{\scriptstyle f} \\
& & C
\end{array}$$

Because everything in this diagram commutes, we have $f\psi\beta = f(\lim_{\rightarrow}\beta)\phi = g(lim_{\rightarrow}\beta)\phi = g\psi\beta$. Since β is surjective, we have $f\psi = g\psi$. But then this forces $f = g$, since the definition of the colimit states that the morphism between $\lim_{\rightarrow} H$ and C that makes the diagram commute is unique, and so $\lim_{\rightarrow} \beta$ is an epimorphism, and thus surjective, making this part of the sequence exact.

We now check exactness at $\lim_{\rightarrow} G$. Consider $\lim_{\rightarrow} \beta \circ \lim_{\rightarrow} \alpha$. Because the following diagram commutes for all $B \in \mathcal{B}$,

$$
\begin{array}{ccc}
FB & \xrightarrow{\psi} & \lim_{\rightarrow} F \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \lim_{\rightarrow} \alpha} \\
GB & \xrightarrow{\psi'} & \lim_{\rightarrow} G \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \lim_{\rightarrow} \beta} \\
HB & \xrightarrow{\psi''} & \lim_{\rightarrow} H
\end{array}
$$

we have $\psi'' \circ \beta \circ \alpha = \lim_{\rightarrow} \beta \circ \lim_{\rightarrow} \alpha \circ \psi$. But we know that this must equal 0, since $\beta \circ \alpha = 0$ because that sequence is exact. Since 0 satisfies $0 \circ \psi = 0$ for all ψ, and because maps from the colimit are unique, we have $\lim_{\rightarrow} \beta \circ \lim_{\rightarrow} \alpha$, so that $\operatorname{im} \lim_{\rightarrow} \alpha \subset \ker \lim_{\rightarrow} \beta$. Let $x \in \lim_{\rightarrow} G$.

Suppose we have $x \in \ker \lim_{\rightarrow} \beta$. We can take some $b \in GB$ such that $\beta(b) = 0$. Since the lower square on the above diagram commutes, this gives us $\psi'(b) = x$. Because α is surjective, we have some $a \in FB$ with $\alpha(a) = b$. So, because the top square commutes, we have $x = \lim_{\rightarrow} \alpha(\psi(a))$, and so $x \in \operatorname{im} \lim_{\rightarrow} \alpha$, giving exactness. □

We can see, by example, that not all colimits are left-exact. Recall that the coequalizer of two morphisms $f, g : A \rightarrow B$ in the category of abelian groups is $\operatorname{coker}(f - g)$. Consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 4\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 0 \\
 & & {\scriptstyle i}\Downarrow\Downarrow{\scriptstyle 0} & & {\scriptstyle \cong}\Downarrow\Downarrow{\scriptstyle 0} & & {\scriptstyle \phi}\Downarrow\Downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & 2\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

The horizontal rows are clearly exact. For the vertical columns, representing the two morphisms we will find the coequalizer of, 0 is the zero homomorphism, i is the inclusion map, $\cong$ is the natural isomorphism, and π represents a map for which 0 and 2 are mapped to 0, and 1 and 3 are mapped to 1. We see that each of the squares in the diagram is natural very straightforwardly. Taking the coequalizer of the sequence, we get the following:

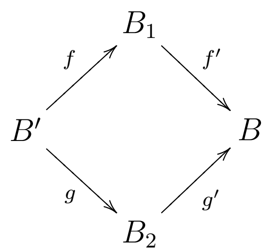
$$0 \longrightarrow 2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

The map from $\mathbb{Z}/2\mathbb{Z}$ to 0 cannot be injective, and so this sequence is not left-exact.

To examine when colimits are left-exact, we introduce the concept of a filtered limit. A category $\mathcal{B}$ is *filtered* if it satisfies the following two conditions:

1. For any two objects B_1 and B_2 of $\mathcal{B}$ there exists an object B of $\mathcal{B}$ with morphisms $B_1 \to B$ and $B_2 \to B$, and

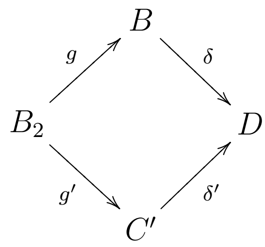
2. For every two morphisms with the same source $f : B' \to B_1$ and $g : B' \to B_2$, there exists an object B and morphisms $f' : B_1 \to B$ and $g' : B_2 \to B$ such that the following diagram commutes:



Proposition 2 *If $\mathcal{B}$ is a filtered small subcategory of the category of A-modules, then $\lim_{\rightarrow} \mathcal{B}$ is the disjoint union $\bigcup_{B \in \mathcal{B}} B/\sim$ where $b_1 \sim b_2$ for $b_1 \in B_1$ and $b_2 \in B_2$ iff there exist morphisms $f_i : B_i \to B$ such that $f_1(b_1) = f_2(b_2)$*

Proof: Let $X = \bigcup_{B \in \mathcal{B}} B/\sim$. We first see that X is in fact an A-module. First note that, for $x \in B_1, f : B_1 \to B$, $x \sim f(x)$, since $f(x) = 1_B(f(x))$, and so f and 1_B are the morphisms satisfying the equivalence relation. It follows immediately that scalar multiplication, defined by $r[x] = [rx]$, since for a commutative ring, scalar multiplication is a morphism in the category of that ring's modules. Finally, for $x \in B_1$ and $y \in B_2$, we define $[x] + [y] = [f(x) + g(y)]$, where $f : B_1 \to B$ and $g : B_2 \to B$ are morphisms to a common module, which exist since $\mathcal{B}$ is filtered. We check to see that this addition is well defined. Say $x' \in B_1'$ with $x \sim x'$. Since they are equivalent, we have some $C \in \mathcal{B}$ and $\phi : B_1 \to C, \psi : B_1' \to C$ with

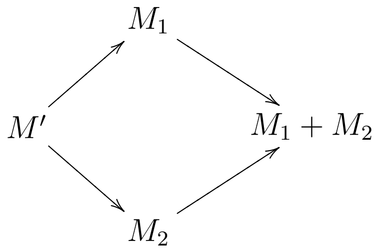
$\phi(x) = \psi(x')$. From the first part of the definition of the filtered category, we have $f' : C \to C'$ and $g' : B_2 \to C'$ for some $C' \in \mathcal{B}$ so that $f'(\psi(x')) \in C'$ and $g'(y) \in C'$, allowing us to add these two quantities. But we see that, by the second part of the definition of a filtered category, that there exists $D \in \mathcal{B}$ and morphisms $\delta : B \to D$ and $\delta' : C' \to D$ with the following diagram commuting:



Since this diagram commutes, $\delta(f(x) + g(y)) = \delta'(f'(\psi(x')) + g'(y))$, so these two are equivalent and this addition is well defined. Hence X is an object of the category A-mod.
We use the universal property to show X is the colimit. Given an A-module C and maps $\psi_i : B_i \to C$, we clearly have an induced map $\psi : \bigcup_{B \in \mathcal{B}} B \to C$. If the ψ_i commute with all the morphisms in $\mathcal{B}$, making the ψ_i a morphism from the diagram $\mathcal{B} \to C$ as described above, then we see that ψ respects the equivalence relation $\sim$, so that ψ descends to a unique map $X \to C$, satisfying the colimit universal property. $\square$
Using this, we get a useful way to look at A-modules in terms of their submodules:

Proposition 3 *An A-module M can be expressed as the filtered colimit of the subcategory $\mathcal{B}$ of A-Mod, consisting of finitely generated submodules of M as objects and inclusion maps as morphisms.*

Proof: We first see that $\mathcal{B}$ is filtered. Let M_1, M_2 be finitely generated submodules of M. We see that $M_1 \subseteq M_1 + M_2$ and $M_2 \subseteq M_1 + M_2$, satisfying the first criterion. Say $M' \subseteq M_1 \cap M_2$, so that we have morphisms from M' to M_1 and M' to M_2. Then the following diagram, with arrows being inclusion maps, commutes:



thus satisfying the second criterion for being a filtered subcategory.
Let $X = \bigcup_{N \in \mathcal{B}} N/\sim$, with $\sim$ being the equivalence relation described in the last theorem. Let $\phi : M \to X$ be given by $\phi(m) = [m]$. ϕ is surjective, since any $[m] \in X$ has preimage $m \in M$. Let $a \in \ker(\phi)$. Since $[a] = [0]$, there are some finitely generated submodules M_1,M_2, and M' of M and injections $i : M_1 \to M'$ and $j : M_2 \to M'$ with $a \in M_1$ and $i(a) = j(0) = 0$. Since i is injective, this means $a = 0$, so ϕ is injective. Thus, $X \cong M$ and by the uniqueness of the colimit, $X = M$. □
We now finish our discussion of filtered colimits by exploring our original motivation of finding exact colimits:

Proposition 4 *Filtered colimits are exact. That is, let $\mathcal{A}$ be the category of A-modules; let $\mathcal{B}$ be a filtered category; let $F, G, H : \mathcal{B} \to \mathcal{A}$ be functors; and let $\alpha : F \to G$ and $\beta : G \to H$ be natural transformations such that, for every object $B \in \mathcal{B}$*

$$0 \longrightarrow FB \xrightarrow{\alpha} GB \xrightarrow{\beta} HB \longrightarrow 0$$

is exact. Then,

$$0 \longrightarrow \lim_{\to} FB \xrightarrow{\lim_{\to}\alpha} \lim_{\to} GB \xrightarrow{\lim_{\to}\beta} \lim_{\to} HB \longrightarrow 0$$

is also an exact sequence.

Proof: Since we showed above that any colimit is right-exact, all we need to show is that $\lim_{\to} \alpha$ is injective. Let $x \in \bigcup_{B \in \mathcal{B}} FB/\sim$ such that x goes to zero in $\bigcup_{B \in \mathcal{B}} GB/\sim$ under $\lim_{\to} \alpha$. We can lift x to some representative element $b \in FB$ for some $B \in \mathcal{B}$. So, since $\lim_{\to} \alpha(x) = 0$, we have $\alpha(b) \sim 0$. This gives us a morphisms $f : B \to B'$ and $g : B'' to B'$ such that

$Gf(\alpha(b)) = Gg(0)$. But since $Gg(0) = 0$, we see that $Gf(\alpha(b)) = 0$. Since the diagram

$$\begin{array}{ccc} FB & \xrightarrow{\alpha} & GB \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB' & \xrightarrow[\alpha]{} & GB' \end{array}$$

commutes, and since α is injective, $Ff(b) = 0$, so we have $b \sim 0$, so that $[b] = [0]$ in $\bigcup_{B \in \mathcal{B}} FB/\sim$, so $\lim_{\rightarrow} \alpha$ is injective. $\square$