FORBIDDEN MINORS AND MINOR-CLOSED GRAPH PROPERTIES

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Abstract. Kuratowski's Theorem gives necessary and sufficient conditions for graph planarity—that is, embeddability in $\mathbb{R}^2$. This motivates the question: what are the conditions for embeddability on arbitrary surfaces? Is there a "Kuratowski-type" theorem for every surface? This problem and a class of similar problems are answered positively by the Graph Minor Theorem. We introduce the concept of graph minors, then discuss the Robertson-Seymour Theorem and derive the Graph Minor Theorem from it. We then discuss some consequences of the Graph Minor Theorem.

Contents

1. Graphs and notions of subgraphs

First, recall:

Definition 1.1. A graph is a pair $G = (V, E)$. V is called the vertex set of G. E, called the edge set of G, consists of 2-subsets of V.

For our purposes, all graphs are finite ($|V| < \infty$).

Graphs are drawn with a collection of points representing the vertices and a line segment connecting vertices $u, v \in V$ whenever $\{u, v\} \in E$.

Some important graphs are:

- K_n**:** The complete graph has $|V| = n$, $E = \big\{\{u, v\} : u \neq v \in V\big\}$.
- $K_{k,\ell}$**:** The complete bipartite graph has $V = K \amalg L$, $|K| = k$, $|L| = \ell$, $E = \big\{\{u, v\} : u \in K, v \in L\big\}$.
  Complete tripartite graphs $K_{k,\ell,m}$ are defined similarly.
- C_n**:** A cycle has $V = \{v_1, \dots, v_n\}$, $E = \big\{\{v_i, v_{i+1}\} : 1 \leq i \leq n-1\big\} \cup \big\{\{v_1, v_n\}\big\}$.

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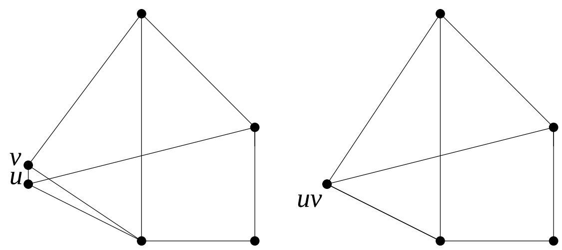

FIGURE 1. A contraction: $G = (V, E) \rightsquigarrow C_{uv}(G) = (V', E')$.

P**:** The Petersen graph is pictured in Figure 2.

Now, consider a graph $G = (V, E)$.

Intuitively, a contraction of an edge in a graph is simply "sliding" the vertices of an edge together until they coincide, as in Figure 1. Of course, this definition can be made rigorous:

Definition 1.2. The contraction of an edge $\{u, v\}$ is the graph $C_{uv}(G) = (V', E')$, where $V' = V \setminus \{u, v\} \cup \{uv\}$ and $E' = E \setminus \big\{\{x, u\}, \{x, v\} : x \in V\big\} \cup \big\{\{x, uv\} : x \in V,\ \{x, u\} \in E \text{ or } \{x, v\} \in E\big\}$.

We define several other simple graph operations, whose intuitive definitions are clear enough from their names:

Definition 1.3. The deletion of a *vertex* $v \in V$ is the graph $D_v(G) = (V', E')$, where $V' = V \setminus \{v\}$ and $E' = E \setminus \big\{\{x, v\} : x \in V\big\}$.

The deletion of an *edge* $\{u, v\} \in E$ is the graph $D_{uv}(G) = (V, E')$, where $E' = E \setminus \big\{\{u, v\}\big\}$.

Definition 1.4. The subdivision of an *edge* $\{u, v\} \in E$ is the graph $S_{uv}(G) = (V', E')$, where $V' = V \cup \{uv\}$ and $E' = E \setminus \big\{\{u, v\}\big\} \cup \big\{\{u, uv\}, \{v, uv\}\big\}$.

We now have three graph operations, namely, contraction, deletion, and subdivision. These yield three notions of a graph being "contained" in another.

Definition 1.5. A graph G has a subgraph G' (denoted "$G' \leq G$") if G' is the product of zero or more (vertex or edge) deletions.

Definition 1.6. A graph G has a topological subgraph G' (denoted "$G' \sqsubseteq G$") if there exists a product of zero or more subdivisions $G'' = S_{u_1 v_1} \circ \cdots \circ S_{u_k v_k}(G')$ such that $G'' \leq G$.

Definition 1.7. A graph G has a minor G' if G' (denoted "$G' \preceq G$") if there exists a product of zero or more contractions $G'' = C_{u_1 v_1} \circ \cdots \circ C_{u_k v_k}(G)$ such that $G' \leq G''$.

G' is a proper minor of G (denoted "$G' \prec G$") if $G' \preceq G$ and $G' \neq G$.

Example 1.8.

- $\forall G,\ G \preceq G$.
- $\forall G,\ \emptyset \preceq G$.
- $K_3 \preceq C_4$.
- $K_5 \preceq P$ (Figure 2).

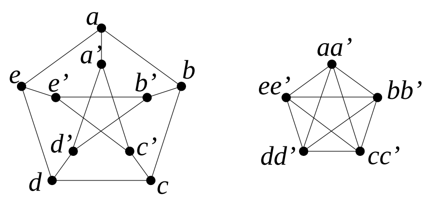

FIGURE 2. $K_5 \preceq P$, since $K_5 = C_{aa'} \circ \cdots \circ C_{ee'}(P)$.

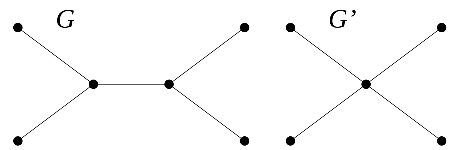

FIGURE 3. $G' \preceq G$ but $G' \not\sqsubseteq G$.

Remark 1.9. $G' \sqsubseteq G \implies G' \preceq G$, since a subdivision can be reversed by a contraction. However, the converse is not true; for instance, the contraction in Figure 3 cannot be reversed by subdivisions.

2. Kuratowski's Theorem and Wagner's Theorem

For our purposes, minors ("$\preceq$") will be much more interesting than topological subgraphs ("$\sqsubseteq$"); however, the theorem motivating the exploration of minor-closed properties was originally stated in topological subgraphs.

Definition 2.1. A graph is <u>embeddable</u> on a surface Σ if it can be drawn on that surface so that no two edges intersect.

A graph is <u>planar</u> if it is embeddable in $\mathbb{R}^2$ (or, equivalently, S^2).

Theorem 2.2 (Kuratowski)**.** *A graph G is planar $\iff$ $G \not\sqsupseteq K_5$ and $G \not\sqsupseteq K_{3,3}$.*

A similar (but not quite identical, due to Remark 1.9) result in graph minors is also true:

Theorem 2.3 (Wagner)**.** *A graph G is planar $\iff$ $G \not\succeq K_5$ and $G \not\succeq K_{3,3}$.*

We call K_5 and $K_{3,3}$ the <u>forbidden minors</u> for planar graphs.

What about embeddability on surfaces other than $\mathbb{R}^2$ (or equivalently, S^2)? Note that K_5 is embeddable on $\mathbb{R}P^2$ (Figure 4). However, there are other graphs not embeddable on $\mathbb{R}P^2$; for instance, notice that the edges in Figure 4 divide the projective plane into regions homeomorphic to the disk; thus, a second copy of K_5 is not embeddable.

In the 1970's, a set of 35 forbidden minors for embeddability on $\mathbb{R}P^2$ was found [8, 9]. In 1989, it was proved that such a finite set of forbidden minors exists for every non-orientable surface [3], and in 1990 for all surfaces [5].

This last discovery seemed amazingly general, but even it was superseded as a special case of what was, until its recent proof [6], known as Wagner's Conjecture:

Definition 2.4. A set of graphs $\mathcal{P}$ is said to be a <u>minor-closed graph property</u> if $\forall G \in \mathcal{P},\ G' \preceq G \implies G' \in \mathcal{P}$.

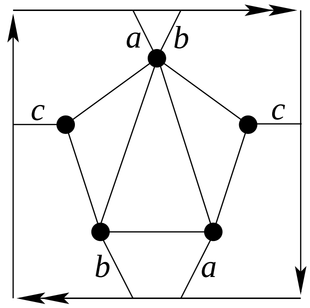

FIGURE 4. An embedding of K_5 on $\mathbb{R}P^2$.

Theorem 2.5 (Graph Minor Theorem)**.** *Any minor-closed graph property $\mathcal{P}$ is characterized by a finite set $\mathcal{F}(\mathcal{P})$ of forbidden minors.*

That is,

$$G \in \mathcal{P} \iff \forall F \in \mathcal{F}(\mathcal{P}),\ F \not\preceq G \tag{2.1}$$

and:

$$|\mathcal{F}(\mathcal{P})| < \infty \tag{2.2}$$

This theorem is a direct consequence of the Robertson-Seymour Theorem, which we will discuss before applying.

3. Robertson-Seymour Theorem

Theorem 3.1 (Robertson and Seymour)**.** *For every infinite sequence of graphs $G_1, G_2, \dots,$ $\exists i < j \ni G_i \preceq G_j$.*

The proof of this result is extremely long and difficult. The complete proof is found in [6], and a thorough summary is found in [2].

An easy corollary will be needed to infer the Graph Minor Theorem:

Corollary 3.2. There are no infinite descending chains of proper minors, that is, there exists no sequence $\{G_n\}$ of graphs such that $G_1 \succ G_2 \succ \dots$.

Proof. Assume we had such a chain $G_1 \succ G_2 \succ \dots$. Then $\forall i < j,\ G_i \succ G_j$, so $G_i \not\preceq G_j$, $\lightning$. □

4. Inference of the Graph Minor Theorem

We're given a minor-closed graph property $\mathcal{P}$. We must find $\mathcal{F}$ satisfying (2.1).

Claim 4.1. *$\mathcal{F}(\mathcal{P})$ is the set of minimal (under taking proper minors) elements of $\bar{\mathcal{P}}$; that is,*

$$\mathcal{F}(\mathcal{P}) = \{F : F \in \bar{\mathcal{P}},\ \forall H \prec F,\ H \in \mathcal{P}\} \subseteq \bar{\mathcal{P}} \tag{4.1}$$

Proof of $(2.1)_{\Rightarrow}$. We're given $G \in \mathcal{P}$.

Assume $\exists F \in \mathcal{F}(\mathcal{P}) \ni F \preceq G$. Since $\mathcal{P}$ is minor-closed, this implies $F \in \mathcal{P}$, but from (4.1), $F \in \bar{\mathcal{P}}$, $\lightning$. ✓

Proof of $(2.1)_{\Leftarrow}$. We're given $G \ni \forall F \in \mathcal{F}(\mathcal{P}),\ F \not\preceq G$.

Assume $G \notin \mathcal{P}$, that is, $G \in \bar{\mathcal{P}}$.

Case I (All of G's proper minors are in $\mathcal{P}$)**.** By (4.1), $G \in \mathcal{F}(\mathcal{P})$, $\lightning$.

Case II ($\exists G' : G \succ G' \in \bar{\mathcal{P}}$)**.** Then, again, G' will fall under one of these two cases; continue in this fashion for as long as we remain under Case II, building a chain of proper minors.

If this process terminates, we have $G \succ G' \succ G'' \succ \ldots \succ G^{(r)}$, with $G^{(r)} \in \mathcal{F}(\mathcal{P})$ falling under Case I, $\lightning$. Otherwise, $G \succ G' \succ G'' \succ \ldots$, which is an infinite descending chain of proper minors, contradicting Corollary 3.2, $\lightning$. ✓

Proof of (2.2). By Theorem 3.1, if $\mathcal{F}(\mathcal{P})$ is infinite, then $\exists G_1, G_2 \in \mathcal{F}(\mathcal{P})$, $G_1 \neq G_2 \ni G_1 \preceq G_2$, $\therefore G_1 \prec G_2$. But (4.1) says each element of $\mathcal{F}(\mathcal{P})$ has no proper minors in $\bar{\mathcal{P}}$ and thus certainly no proper minors in $\mathcal{F}(\mathcal{P})$. $\lightning$ ✓□

5. Consequences of the Graph Minor Theorem

The Graph Minor Theorem gives the existence of solutions for the entire class of forbidden-minor problems; some of these have explicit lists of forbidden minors, while others have little known about them other than what the Graph Minor Theorem gives.

5.1. Minor-closed properties with known forbidden minors.

Cycle-free: K_3.
Embeddability on $\mathbb{R}^2$: $K_5, K_{3,3}$.
Linklessness in $\mathbb{R}^3$: There are 7 forbidden minors, including P and K_6.

Definition 5.1. A graph is <u>linklessly embeddable</u> if it can be embedded in $\mathbb{R}^3$ so that no two cycles $C, C' \leq G$ pass through each other (as in Figure 5).

Embeddability on $\mathbb{R}P^2$: There are 35 forbidden minors, including $K_5 \amalg K_5$.
Hadwiger number $\leq k$: K_{k+1} (This is the very definition of Hadwiger number).

5.2. Minor-closed properties with unknown forbidden minors.

Tree-width $\leq w$: Tree-width is a key concept in Robertson and Seymour's proof [2,5,6]. It quantifies how "tree-like" a graph is (trees have tree-width 1), and one of its key properties is that every minor $G' \preceq G$ has tree-width $\leq$ the tree-width of G.

Definition 5.2. A <u>tree decomposition</u> of a graph $G = (V, E)$ is a tree $T = (V', E')$, where each $V_i \in V'$ is a subset $V_i \subseteq V$, each edge $e \in E$ is a subset $e \subseteq V_i$ for some $V_i \in V'$, and whenever V_c lies on the path between V_a and V_b in T, $V_a \cap V_b \subseteq V_c$.

The <u>width</u> of a tree decomposition T is $\max_{V_i \in V}\{|V_i|\}$.

The <u>tree-width</u> of a graph G is $\min_T\{$ width of $T\}$, where T ranges over all tree decompositions of G.

Forbidden minors are known for $w = 1$ (K_3), $w \leq 2$ (K_4), and $w \leq 3$ (there are four, including K_5 and $K_{2,2,2}$ [12]).

Embeddability on $\Sigma_g^{\sim}$: Embeddability on non-orientable surfaces was known to satisfy Wagner's Conjecture years before Robertson and Seymour proved the Graph Minor Theorem, due to the work of, among others, Archdeacon and Huneke [3]. However, the explicit lists of forbidden minors remain unknown except for $\mathbb{R}P^2$.

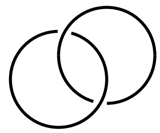

FIGURE 5. Linked cycles.

Embeddability on Σ_g: As in the non-orientable case, Wagner's Conjecture for this special case was proven (by the very same Robertson and Seymour) before Robertson and Seymour completed their proof.

Knotlessness in $\mathbb{R}^3$:

Definition 5.3. A graph is knotlessly embeddable in $\mathbb{R}^3$ if it can be embedded in $\mathbb{R}^3$ so that no cycle forms a nontrivial topological knot.

5.3. **An algorithmic implication.** Previously, it was not known whether or not linkless embeddability (Definition 5.1) was even *decidable*, meaning no algorithm guaranteed to terminate in *any amount of time* was known. However, *a cubic-time ($O(n^3)$) algorithm checking for minors in a graph is known* [1, 2, 10]*!*

We can simply apply this algorithm seven times, checking for each of the forbidden minors, thus deciding linklessness in $7 \cdot O(n^3) = O(n^3)$ time.

So the Graph Minor Theorem gives us not only decidability for linkless embeddability (and all minor-closed properties), but a theoretically "fast" $c \cdot O(n^3) = O(n^3)$ time algorithm. Unfortunately, the constant swallowed by the "O" notation is large enough to make the algorithm completely impractical.

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