Definitions and Assumptions

Define the following characteristics to be:
V(K) = the number of vertices within a graph K
E(K) = the number of edges within a graph K
F(K) = the number of faces a graph K separates the plane or surface into

Definition: A graph G is a tree if G is connected and has no simple cycles, i.e. a cycle with no repeated vertices besides the starting and ending vertex.

Assumptions:
Jordan Curve Theorem: Any simple closed curve separates the space it’s contained within into two distinct parts

1 Lemma: Euler’s Theorem in 2 Dimensions

Definition: The Euler characteristic for any graph K in 2 dimensions is defined by:

\[ \chi(K) = V(K) - E(K) \]

Claim 1: \( \forall \) connected graphs K \( \chi(K) \leq 1 \)
Claim 2: \( \chi(K) = 1 \) \( \iff \) K is a tree

Proof of Claim 1: Define \( G_k \) to be a connected graph with k edges. For \( G_0 \) there are two possible graphs:
The graph consisting of a single vertex satisfying:

\[ \chi(G_0) = 1 - 0 = 1 \]

And the graph consisting of no vertices satisfying:

\[ \chi(G_0) = 0 - 0 = 0 \]

Thus for k=0, \( \chi(G_k) \leq 1 \). Now we will induct on the number of edges in order to prove Claim 1.
Assume that \( \chi(G_n) \leq 1 \) with \( V(G_n) = l \) and \( E(G_n) = n \). In order to create any
graph $G_{n+1}$ there are only two possible methods for adding an additional edge:
(1) Add an edge between $v_i$ and $v_j$ with $v_i, v_j \in G_n$. Thus:

$$V(G_{n+1}) = V(G_n) = l$$
$$E(G_{n+1}) = E(G_n) + 1 = n + 1$$
$$\chi(G_{n+1}) = V(G_{n+1}) - E(G_{n+1}) = l - (n + 1) = (l - n) - 1 = \chi(G_n) - 1 < \chi(G_n)$$

(2) Add an edge between $v_i$ and $v_j$ with $v_i \in G_n$ and $v_j \notin G_n$. Thus:

$$V(G_{n+1}) = V(G_n) + 1 = l + 1$$
$$E(G_{n+1}) = E(G_n) + 1 = n + 1$$
$$\chi(G_{n+1}) = V(G_{n+1}) - E(G_{n+1}) = (l + 1) - (n + 1) = (l - n) + 1 - 1 = \chi(G_n)$$

Hence $\forall k, \chi(G_k) \leq 1$.

**Lemma**: Adding an edge to a connected graph $G$ using method (1) creates a simple cycle.

Proof: $G$ is connected $\Rightarrow \exists$ a path from $v_i$ to $v_j$ s.t. no vertex is repeated. If a new edge is added connecting $v_i$ to $v_j$, this edge will add allow the path to extend from $v_i$ to $v_i$ without crossing any other vertices twice.//

**Proof of claim 2**: We shall assume that $G_0$ with 0 vertices is not a tree, though it does make intuitive sense that the nonexistent vertices could have infinite cycles between nothingness. Thus our base case shall be $G_0$ with $V(G_0) = 1$, and hence as shown above $\chi(G_0) = 1$.

Assume $G_k$ is a tree. $\Rightarrow G_k$ is generated from $G_0$ solely by method(2) by the Lemma. Thus:

$$\chi(G_k) = \chi(G_{k-1}) = ... = \chi(G_1) = \chi(G_0) = 1$$

Since method (1) preserves the Euler Characteristic.

Now assume $\chi(G_k) = 1$. We shall also assume that $G_k$ is generated by at least one edge of type (1).

$$\Rightarrow \chi(G_k) = \chi(G_{k-1}) = ... \chi(G_{j+1}) < \chi(G_j) = ... = \chi(G') = \chi(G) = 1$$

$$\Rightarrow \chi(G_k) < 1$$

A contradiction of our initial assumption, thus all edges of $G_k$ must be added by method (2) $\Rightarrow G_k$ is a tree.//

2 **Proof of Euler’s Theorem in 3 Dimensions**

**Definition**: For any surface or solid $K$ in 3 Dimensions the euler characteristic $\chi$ of $K$ is denoted:

$$\chi(K) = V(K) - E(K) + F(K)$$
**Definition:** Let a net on a convex surface be defined as a graph of connected vertices and edges separating the surface into faces.

**Claim:** ∀ nets P on a convex surface χ(N)=2

**Proof:** Let P₀ be a net with k edges, and P₀ be a net on a surface consisting of a single vertex be the smallest possible net on a surface. Thus:

\[
\chi(P₀) = 1 - 0 + 1 = 2
\]

Thus \(\chi(P_k)=2\) for \(k=0\). We will now proceed by induction on the edges.

Assume \(P_n\) is a net with \(V(P_n)=l, E(P_n)=n,\) and \(F(P_n)=m\) and that \(\chi(P_n)=2\).

As in the previous section, in order to create \(P_{n+1}\) from \(P_n\) there are two methods for adding an additional edge, with the same consequences for the vertices and edges as above, with the addition that for method (1) creates a simple closed curve, and thus separates 1 of the existing faces into 2 distinct faces, though method (2) creates no additional faces. Thus the following is obtained by each method:

**Method (1):**

\[
V(P_{n+1}) = V(P_n) = l
\]
\[
E(P_{n+1}) = E(P_n) + 1 = n + 1
\]
\[
F(P_{n+1}) = F(P_n) + 1 = m + 1
\]
\[
\chi(P_{n+1}) = V(P_{n+1}) - E(P_{n+1}) + F(P_{n+1}) = l - (n+1) + (n+1) = (l-n+m) - 1 + 1 = \chi(P_n)
\]

**Method (2):**

\[
V(G_{n+1}) = V(F_n) + 1 = l + 1
\]
\[
E(P_{n+1}) = E(P_n) + 1 = n + 1
\]
\[
F(P_{n+1}) = F(P_n) = m
\]
\[
\chi(P_{n+1}) = V(P_{n+1}) - E(P_{n+1}) + F(P_{n+1}) = (l+1) - (n+1) + m = (l-m+n) + 1 - 1 = \chi(P_n)
\]

Using our inductive assumption we find that \(\chi(P_{n+1})=\chi(P_n)=2\).

Thus \(\chi(P)=2\) for all nets on convex surfaces, and hence \(\chi(P)=2\) for all polyhedra, a subset of those nets. //

### 3 Euler Characteristic an Homeotopy to the Sphere

For this section we will be showing the equivalency of 3 statements:

1. \(\chi(K)=2\)
2. Any embedded loop in \(K\) separates \(K\) into two distinct parts
3. \(K\) is homeotopic to the sphere

**Proof:** \(3 \Rightarrow 2\): First by the Jordan Curve Theorem we can say already that \(3 \Rightarrow 2\) since the sphere has one surface, any closed curve would have to separate it.
Now we will show that (2)⇒(1)⇒(3) to complete the proof. In doing this we will use the lemma stating that the euler characteristic of any tree is equal to 1.
To do this we will consider a closed combinatorial surface K, essentially a triangulation of the surface, or a net composed solely of triangles.

(2)⇒(1): First create a maximal tree T, in K. Since the tree is maximal ⇒

\[ V(T) = V(K) \]

Then create a dual graph Γ with each face of K being a vertex of Γ, and adding an edge between every pair of vertices \( v_i, v_j \in Γ \) s.t. the faces \( f_i, f_j \in K \) corresponding to \( v_i, v_j \) satisfy:

\[ f_i \cap f_j \notin T \]

Thus Γ will have the property

\[ F(K) = V(Γ) \]

and the two graphs together will satisfy:

\[ E(K) = E(T) + E(Γ) \]

Thus combining these results we obtain:

\[ \chi(K) = V(K) - E(K) + F(K) = V(T) - [E(T) + E(Γ)] + V(Γ) = \]

\[ [V(T) - E(T)] + [V(Γ) - E(Γ)] = \chi(T) + \chi(Γ) \]

By the Lemma we know that \( \chi(T) = 1 \) since T is a tree, so to show that \( \chi(K) = 2 \), we simply need to show that \( \chi(Γ) = 1 \), or that Γ is a tree.
Suppose Γ is a tree
⇒ Γ has an embedded loop
⇒ Γ separates K, and hence that T isn’t connected, but T is connected, a contradiction. Thus Γ must be a tree, and \( \chi(Γ) = 1 \).

(1)⇒(3): Now K itself is composed of a neighborhood around T and a neighborhood around Γ, which we have just shown are both trees, and the neighborhood around a tree is homeomorphic to a disk. If we take the neighborhood around T and Γ and combine them together we obtain an area homeomorphic to a sphere.

4 All Convex Polyhedra are Homeomorphic to the sphere

Proof: In the previous two sections we found that:
1) \( \chi(K) = 2 \) \( \forall \) convex polyhedra
2) \( \chi(K) = 2 \) \( \Leftrightarrow \) All embedded loops in K separate \( \Leftrightarrow \) K is homeotopic to the sphere.

Thus we can now say that all convex polyhedra are homeomorphic to the sphere//