An Introduction to Homology

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Abstract

This paper explores the basic ideas of simplicial structures that lead to simplicial homology theory, and introduces singular homology in order to demonstrate the equivalence of homology groups of homeomorphic topological spaces. It concludes with a proof of the equivalence of simplicial and singular homology groups.

Contents

1 Simplices and Simplicial Complexes

Definition 1.1. The *n-simplex*, Δ^n, is the simplest geometric figure determined by a collection of $n+1$ points in Euclidean space $\mathbb{R}^n$. Geometrically, it can be thought of as the complete graph on $(n+1)$ vertices, which is solid in n dimensions.

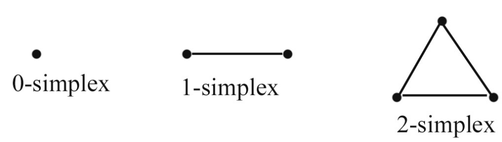

Figure 1: Some simplices

Extrapolating from Figure 1, we see that the 3-simplex is a tetrahedron.
Note: The n-simplex is topologically equivalent to D^n, the n-ball.

Definition 1.2. An *n-face* of a simplex is a subset of the set of vertices of the simplex with order $n+1$. The faces of an n-simplex with dimension less than n are called its *proper* faces.

Two simplices are said to be *properly situated* if their intersection is either empty or a face of both simplices (i.e., a simplex itself). By "gluing" (identifying) simplices along entire faces, we get what are known as *simplicial complexes.* More formally:

Definition 1.3. A *simplicial complex* K is a finite set of simplices satisfying the following conditions:

1 For all simplices $A \in K$ with α a face of A, we have $\alpha \in K$.

2 $A, B \in K \implies A, B$ are properly situated.

The *dimension* of a complex is the maximum dimension of the simplices contained in it.

More abstractly, a complex is a finite set of vertices $\{v_0, \ldots, v_k\}$ with certain subsets distinguished as abstract simplices, and the property that all faces of distinguished simplices are also distinguished.

We can assign to an abstact complex $\mathcal{R}$ a *geometric realization* which takes $v_0, v_1, \ldots, v_k$ to points in $\mathbb{R}^n$. For example, we get what is known as the *natural realization* if we take $n = k + 1$ and $v_0 = e_1, v_1 = e_2, \ldots v_k = e_{k+1}$, where the e_i are the standard basis vectors in $\mathbb{R}^n$. We note that although $\mathcal{R}$ may be n-dimensional, a realization K may not "fit" into $\mathbb{R}^n$.

By assigning $v_0, \ldots, v_k$ to different points, we can achieve geometrically different realizations of $\mathcal{R}$. Topologically speaking, however, we have a sense that these realizations should not be fundamentally different, and in fact all realizations of a complex $\mathcal{R}$ are homeomorphic.

2 Homology Groups

Given the set S of vertices of a simplex, we define an orientation on the simplex by selecting some particular ordering of S. Vertex orderings that differ from this by an odd permutation are then designated as reversed, while even permutations are regarded as unchanged. Any simplex, then, has only two possible orientations.

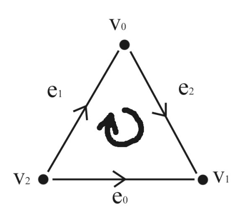

Figure 2: An oriented 2-simplex

An orientation on an n-simplex induces orientation on its $(n-1)$-faces; as illustrated by Figure 2, if the 2-simplex is given the orientation (v_0, v_1, v_2), then the orientation induced on its 1-faces is $e_2 = (v_0, v_1), e_0 = (v_1, v_2), e_1 = (v_2, v_0)$.

Formally, if $A^n = (v_0, v_1, \ldots, v_n)$ is an oriented n-simplex, then the orientation of the $(n-1)$-face of A^n with vertex set $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$ is given by: $F_i = (-1)^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

Definition 2.1. Given a set ${A^n}_1, \dots, {A^n}_k$ of arbitrarily oriented n-simplices of a complex K and an abelian group G, we define an *n-chain* x with coefficients in G as a formal sum:

$$x = g_1 {A^n}_1 + g_2 {A^n}_2 + \cdots + g_k {A^n}_k, \tag{1}$$

where $g_i \in G$.

Henceforth, we will assume that $G = \mathbb{Z}$.

The set of n-chains forms an abelian group over addition: for $x = \sum_{i=1}^k g_i {A^n}_i, y = \sum_{i=1}^k h_i {A^n}_i$, we have $x + y = \sum_{i=1}^k (g_i + h_i) {A^n}_i$.

We denote the group of n-chains by L_n.

Definition 2.2. Let A^n be an oriented n-simplex in a complex K. The *boundary* of A^n is defined as the $(n-1)$-chain of K over $\mathbb{Z}$ given by

$$\delta(A^n) = {A^{n-1}}_0 + {A^{n-1}}_1 + \cdots + {A^{n-1}}_n$$

where ${A^{n-1}}_i$ is an $(n-1)$-face of A^n. If $n = 0$, we define $\delta(\Delta^0) = 0$.

It is important to note that, since A^n was oriented, the ${A^{n-1}}_i$ have associated orientations as well.

We can extend the definition of boundary linearly to all of L_n: for an n-chain $x = \sum_{i=1}^k g_i {A^n}_i$, define

$$\delta(x) = \sum_{i=1}^k g_i \delta({A^n}_i) \tag{2}$$

where ${A^n}_i$ are the n-simplexes of K. Therefore, the boundary operator δ is a homomorphism $\delta : L_n \to L_{n-1}$.

Example 2.3. Calculate $\delta(\delta(\Delta^2))$, where Δ^2 is the 2-simplex from Figure 2:

$$\begin{aligned}
\delta(\delta(\Delta^2)) = \delta(e_1 + e_2 + e_3) \\
&= \delta e_1 + \delta e_2 + \delta e_3 \\
&= \delta(v_0, v_1) + \delta(v_1, v_2) + \delta(v_2, v_0) \\
&= [(v_1) - (v_0)] + [(v_2) - (v_1)] + [(v_0) - (v_2)]
\end{aligned}$$

But L_0 is abelian, and oppositely oriented simplices cancel, so

$$\delta(\delta(\Delta^2)) = 0$$

This result generalizes to higher dimensions. Therefore, since δ is linear and the x is a sum of n-simplices, we conclude that $\delta^2(x) = 0$, for any n-chain x in L_n.

Definition 2.4. We call an n-chain a *cycle* if its boundary is zero, and denote the set of n-cycles of K over $\mathbb{Z}$ by Z_n. Z_n is a subgroup of L_n, and can also be written as $Z_n = Ker(\delta)$.

Example 2.3 shows that the boundary of any simplex is a cycle. Since $\delta^2 = 0$, we conclude that every boundary is a cycle.

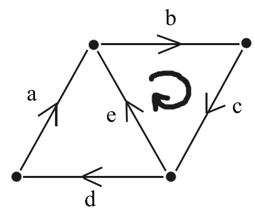

Figure 3: Boundaries?

Definition 2.5. We say that an n-cycle x of a k-complex K is *homologous to zero* if it is the boundary of an $(n+1)$-chain of $K, n = 0, 1, \ldots, k-1$. A boundary is then any cycle that is homologous to zero. This relation is written $x \sim 0$, and the subgroup of Z_n of boundaries is denoted B_n. We also write $B_n = Im(\delta)$.

Less formally, a cycle is a member of B_n if it "bounds" something contained in the complex K. For example, the chain $b + c + e$ in Figure 3 is a boundary, but $a + d + e$ is not. The relation $x \sim 0$ gives an equivalence relation: for two chains $x, y, (x - y) \sim 0 \implies x \sim y$, and we call x and y *homologous*.

Since B_n is a subgroup of Z_n, we may form the quotient group $H_n = Z_n / B_n$.

Definition 2.6. The group H_n is the n-dimensional *homology group* of the complex K over $\mathbb{Z}$.

H_n can also be written as $Ker(\delta)/Im(\delta)$.

Definition 2.7. A *subcomplex* is a subset S of the simplices of a complex K such that S is also a complex.

The set of all simplices in a complex K with dimension less than or equal to n is called the *n-skeleton* of K. From Definition 2.7, it is clear that the n-skeleton is a subcomplex.

Definition 2.8. A complex K is *connected* if it cannot be represented as the disjoint union of two or more non-empty subcomplexes. A geometric complex is *path-connected* if there exists a path made of 1-simplices from any vertex to any other.

Claim 2.9. *Path-connected $\iff$ connected.*

Proof. To prove the forward direction, suppose K is not connected. Then we can select two disjoint subcomplexes L and M such that $L \cup M = K$. Assume a path exists between some vertex $l_0 \in L$ and $m_0 \in M$. But then, if l_i is the last vertex in the path which is contained in L, the 1-simplex connecting l_i to the next vertex in the path cannot be contained in either L or M or they would have a nonempty intersection, contradicting the assumption that K is not connected.
For the other direction, assume there exist points l_0 and m_0 in K with no path between them. Then we define L as the path-connected subcomplex of K which contains l_0, and M as the path-connected subcomplex which contains m_0. If $v_0 \in L \cap M \neq \emptyset$, then there exists a path from l_0 to v_0 and a path from v_0 to m_0. Concatenating these paths gives a path from l_0 to m_0, contradicting the assumption for l_0 and m_0. Hence $L \cap M = \emptyset$, so K is not connected. $\square$

Theorem 2.10. *If $K_1, \ldots, K_p$ is the set of all connected components of a complex K, and H_n, H_{ni} are the homology groups of K and K_i, respectively, then H_n is isomorphic to the direct sum $H_{n1} \oplus \cdots \oplus H_{np}$.*

Proof. Let L_n be the group of n-chains of K, and K_i the ith component of K. Denote by L_{ni} the group of n-chains of K_i. It is clear that L_{n_i} is a subgroup of L_n and moreover, that

$$L_n = L_{n1} \oplus \cdots \oplus L_{np}$$

We wish to show that a similar componentwise decomposition holds for the groups B_n and Z_n. If we let $B_{ni} = \delta(L_{n+1_i})$be the image of δ restricted to the subgroup L_{ni}, then we can represent the group B_n by the direct sum of such restrictions:

$$B_n = B_{n1} \oplus \cdots \oplus B_{np},$$

so given an element $x \in L_{n+1}$, represented by

$$x = x_i + \cdots + x_p,$$

$$\delta x = \delta x_1 + \cdots + \delta x_p \in B_n,$$

where $x_i \in L_{n+1_i}$.

Now let $Z_{ni} = Ker(\delta) \cap L_{ni}$. Then

$$Z_n = Z_{n1} \oplus \cdots \oplus Z_{np}$$

To verify this, we note that in order for $x \in L_n$ to be in Z_n, we need $\delta(x) = 0$.
But $\delta(x) = \delta(x_1) + \cdots + \delta(x_p)$, so $\delta(x) = 0 \implies \delta(x_i) = 0$, that is, that $x_i \in Z_{ni}$.

Since Z_n and B_n both break down componentwise,

$$Z_n/B_n = Z_{n1}/B_{n1} \oplus \cdots \oplus Z_{np}/B_{np},$$

and

$$H_n = H_{n1} \oplus \cdots \oplus H_{np}$$

□

Definition 2.11. The *index* of a chain $x = \sum_{i=1}^{k} g_i A_{ni}$ is defined as $I(x) = \sum_{i=1}^{k} g_i$.

Proposition 2.12. *If K is a connected complex, then for x a 0-chain, $I(x) = 0$ is equivalent to $x \sim 0$, and $H_0(K, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.*

Proof. We first show that $x \sim 0 \implies I(x) = 0$: Let $A^1 = (a_0, a_1)$ be a 1-simplex. Then

$$x = \delta(gA^1) = ga_1 - ga_0$$

but $x = \delta(gA^1) \implies x \sim 0$, and we can see that $I(x) = I(gA^1) = g - g = 0$.
Since $I(x + y) = I(x) + I(y)$, I is a homomorphism, and any $y \in L^1$ is of the form $\sum_{i=0}^{q} g_i {A^1}_i$, where ${A^1}_i = (a_i, a_{i+1})$, we have

$$x = \delta y \sim 0 \implies I(x) = I(\delta y) = 0$$

For the forward direction, we take v and w to be two vertices of K. K is connected, so there exists a path between them consisting of 1-simplices ${A^1}_i = (a_i, a_{i+1}), i = 0, \ldots, q-1$, where $a_0 = v$ and $a_q = w$. We consider the boundary of the chain $y = \sum_{i=0}^{q} g{A^1}_i$, given by

$$\delta y = \sum_{i=0}^{q} g\delta {A^1}_i = \sum_{i=0}^{q} g[(a_{i+1}) - (a_i)] = gw - gv$$

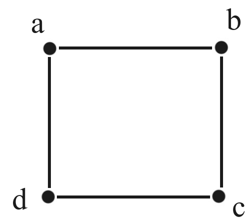

Figure 4: A simplicial structure on the circle

with $I(\delta y) = 0$. δy is a boundary, so $x = \delta y \sim 0 \implies (gw - gv) \sim 0 \implies gw \sim gv$, and this implies that any 0-chain x of K is homologous to the chain gv. As $x \sim 0 \implies I(x) = 0$, we see that homologous chains have equal indices. Thus $I(x) = I(gv) = g$. Then we have $x \sim gv \implies x \sim I(x)v$, but this shows that if $I(x) = 0, x \sim 0$, so $I(x) = 0 \iff x \sim 0$.

As noted, I is a homomorphism of $L_0 = Z_0$ into $\mathbb{Z}$. For x a 0-simplex and $g \in \mathbb{Z}$, $gx \in L_0$ is a cycle with $I(gx) = g$. Hence $I(Z_0) = \mathbb{Z}$. As $I(x) = 0 \iff x \sim 0$, we have $B_0 = ker(I)$, so that $H_0 = Z_0/B_0 \cong \mathbb{Z}$. □

Theorem 2.13. *The zero-dimensional homology group of a complex K over $\mathbb{Z}$ is isomorphic to $\mathbb{Z}^p = \oplus_p \mathbb{Z}$ where p is the number of connected components of K.*

Proof. This follows immediately from Theorem 2.9 and Proposition 2.11. □

Example 2.14. This means that the 0th homology group of the circle is isomorphic to $\mathbb{Z}$.

As in Figure 4, take a simplicial rendering of the circle as four 1-simplices. The group Z_0 consists of sums over the four 0-simplices a, b, c and d, with coefficients from $\mathbb{Z}$. Let x be a 0-chain with nonzero coefficients:

$$x = g_1 a + g_2 b + g_3 c + g_4 d$$

In order to reduce to an element of H_0, we subtract from this the chain $y = g_4 c - g_4 d \sim 0$ to get

$$x - y = g_1 a + g_2 b + (g_3 - g_4)c$$

and by repeating this process, we get a new chain

$$z = (g_1 - g_2 + g_3 - g_4)a$$

But $z \sim x$ and so represents an element in H_0, and since $g_i \in \mathbb{Z}, (g_1 - g_2 + g_3 - g_4) \in \mathbb{Z}$ we can write

$$z = ga,$$

where $g \in \mathbb{Z}$. Therefore, we can choose any such g, and this gives us $H_0 \cong \mathbb{Z}$.

We now calculate some more general homology groups:

Example 2.15. $H_n(S^n) \cong \mathbb{Z}$:

We recall the observation that the n-simplex Δ^n is topologically equivalent to the n-ball. Hence their boundaries, the collection of $n+1$ $(n-1)$-simplices, and the n-sphere, respectively, are also topologically equivalent. The logical simplicial structure to put on S^n, then, is that of the boundary of the $(n+1)$-simplex Δ^{n+1}.

Let $\{v_0, \ldots, v_{n-1}\}$ be the vertex set of Δ^{n+1}. Note that this set is *not* oriented; orientations of the $(n-1)$-simplices can be determined arbitrarily. We will use their numbering to do so. All n-chains on this structure, then, have the form:

$$x = \sum_{i=0}^{n+1} g_i(v_0, \ldots, v_{i-1}, v_{i+1}, ..., v_n), \tag{3}$$

where $g_i \in \mathbb{Z}$. Since Δ^{n+1} itself is not contained in the structure, there are no boundaries in Z_n, the group of cycles. Therefore $H_n = Z_n/B_n$ is the group of cycles.

If $x \in Z_n$, then $\delta x = 0$. Equation (3) gives:

$$\begin{aligned}\delta x = \delta(\sum_{i=0}^{n+1} g_i(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n))\\ &= \sum_{i=0}^{n+1} g_i(\sum_{j<i}^{n+1}(-1)^j(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)\\ &+ \sum_{j>i}^{n+1}(-1)^j(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))\end{aligned}$$

Algebraically, it is perhaps difficult to see that an expansion and redistribution of this sum gives us terms of the form

$$(g_k - g_l)(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \tag{4}$$

for all i, j. This is more intuitive geometrically (Figure 2 provides a low-dimensional example): Any two n-simplices of Δ^{n+1} intersect along an $(n-1)$-face. Hence we get terms of the form (4) for each such face.

From this we can see that if $\delta x = 0$, we must have $g_k = g_l$ for all k, l. That is, $g_0 = g_1 = \cdots = g_{n+1}$. Hence our original n-chain can be rewritten:

$$x = \sum_{i=0}^{n+1} g_0(v_0, \ldots, v_{i-1}, v_{i+1}, ..., v_n) \tag{5}$$

so we can choose g_0 freely from $\mathbb{Z}$. Hence $H_n(S^n) \cong \mathbb{Z}$.

Example 2.16. $H_n(D^n) = 0$:

We give D^n the easiest simplicial structure, that of the the n-simplex Δ^n. Then all n-chains are of the form:

$$x = g\Delta^n,$$

where $g \in \mathbb{Z}$. This is never a boundary, so $H_n = Z_n$. But $\delta x = 0$ only when $g = 0$. Hence $H_n(D^n) \cong 0$.

3 Singular Homology

Particularly in the lower dimensions, we have an intuitive idea of when two topological spaces are fundamentally "the same". We have some ways of generalizing and making rigorous this intuition, including the idea of homeomorphism. It would be nice to have some sort of relation between the homology groups of homeomorphic spaces, and, in fact, it turns out that if two topological spaces are homeomorphic, they have isomorphic homology groups.

We would like to verify this fact. To do so, we need some way of comparing homology groups. It is not immediately clear how we might do this at this point, and in fact this turns out to be rather a difficult problem using the machinery we have so far developed. To get around this difficulty, we introduce the idea of *singular homology.* The basic ideas are analogous to those already developed:

Definition 3.1. Given a topological space X, a *singular n-simplex* in X is a map $\sigma : \Delta^n \to X$, such that σ is continuous.

Definition 3.2. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices of X. Elements of $C_n(X)$ are called *singular n-chains* and are finite formal sums: $\sum_i g_i\sigma_i$, where $g_i \in \mathbb{Z}$.

We define a boundary map δ_n in the same manner as before:

Definition 3.3. The *boundary map* $\delta_n : C_n(X) \to C_{n-1}(X)$ is given by:

$$\delta_n(\sigma) = \sum_i (-1)^i \sigma|_{[\nu_0,\ldots,\nu_{i-1},\nu_{i+1},\ldots,\nu_n]},$$

where ν_i are the 0-simplices of σ, that is, the maps of the vertices of Δ^n: $\nu_i : \Delta^0 \to X$.

As before, given an n-chain $x, \delta^2 x = 0$. This suggests that we can define the singular homology groups in a similiar manner to the simplicial homology groups:

Definition 3.4. The *singular homology group* $H_n(X)$ is defined to be the quotient $H_n(X) = Ker(\delta_n)/Im(\delta_{n+1})$.

Note: We will now denote the simplicial homology group by $H^\Delta{}_n$ in order to distinguish it from the singular homology group H_n.

We shall see in the following section that with this definition of homology it is a simple matter that homeomorphic spaces have isomorphic homology groups, and indeed this fact is apparent already. This raises one item of concern. The definitions of H_n and $H^\Delta{}_n$ are analogous, and we have an intuitive sense these two groups should be the same. However, this is far from apparent: for one thing, $H^\Delta{}_n$ is finitely generated, while the chain group $C_n(X)$ from which we derived H_n is uncountable.

In fact, for spaces on which both simplicial and singular homology groups can be calculated, the two are equivalent, and we will prove this later.

We present first some facts about singular homology that support the intuition $H_n \cong H^\Delta{}_n$:

Proposition 3.5. *Given a topological space X, $H_n(X)$ is isomorphic to the direct sum $H_n(X_1) \oplus H_n(X_2) \oplus \cdots \oplus H_n(X_p)$, where X_i are the path-connected components of X. This is the analogue of Theorem 2.10.*

Proof. Since the maps σ are continuous, a singular simplex always has a path-connected image in X. Hence $C_n(X)$ can be written as the direct sum of subgroups $C_n(X_1) \oplus \cdots \oplus C_n(X_p)$. The boundary map δ is a homomorphism, so it preserves this decomposition. Hence $Ker(\delta_n)$ and $Im(\delta_{n+1})$ also split, and we have $H_n(X) \cong H_n(X_1) \oplus H_n(X_2) \oplus \cdots \oplus H_n(X_p)$. $\square$

Proposition 3.6. *The zero-dimensional homology group of a space X is the direct sum of copies of $\mathbb{Z}$, one for each path-component of X. This is the analogue of Theorem 2.13.*

Proof. It suffices to show that, for X path-connected, $H_0(X) \cong \mathbb{Z}$. For x a 0-simplex, $\delta_0(x) = 0$ because the boundary of any 0-simplex vanishes. This means that $Ker(\delta_0) = C_0(X)$, so $H_0(X) = C_0(X)/Im(\delta_1)$ by definition. We recall Definition 2.11 of the *index*. Here, we have $I : C_0(X) \to \mathbb{Z}$, with $I(x) = \sum_i g_i$ for $x = \sum_i g_i \sigma_i \in C_0(X)$. We wish to show that $Ker(I) = Im(\delta_1)$, that is, that for any 0-chain x, $I(x) = 0 \iff x \sim 0$. The proof proceeds as in Theorem 2.12. $\square$

4 Chain Complexes, Exact Sequences, and Relative Homology Groups

We now introduce some ideas that will help us prove the equivalence of the groups H_n and ${H^{\Delta}}_n$:

Definition 4.1. A *chain complex* is a sequence of abelian groups connected by homomorphisms (called boundary operators) such that the composition of any two consecutive maps is 0.

Example 4.2. The groups $C_n(X)$ of singular n-chains form a chain complex with boundary operator δ_n:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0$$

Definition 4.3. As demonstrated by Example 4.2, the homology groups of a chain complex are given by $Ker(\delta_n)/Im(\delta_{n+1})$.

Definition 4.4. A *chain map* f between two chain complexes A, δ_A and B, δ_B is a collection of maps $f_n : A_n \to B_n$ such that f commutes with the operators δ_A and δ_B as in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\delta_A"] \arrow[d, "f"] & A_n \arrow[r, "\delta_A"] \arrow[d, "f"] & A_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\delta_B"] & B_n \arrow[r, "\delta_B"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Theorem 4.5. *A chain map between two chain complexes induces homomorphisms between homology groups.*

Proof. As illustrated by the preceding diagram, $f\delta_A = \delta_B f$. Then f maps cycles to cycles and boundaries to boundaries, so f induces a homomorphism $f_* : H_n(A) \to H_n(B)$. $\square$

We now apply Theorem 4.5 to the case of singular homology: Let X, Y be topological spaces. Then for any map $f : X \to Y$, we can easily define an induced homomorphism $f_a : C_n(X) \to C_n(Y)$

by composing singular n-simplices $\sigma : \Delta^n \to X$ with f to get $f_a \circ \sigma = f\sigma : \Delta^n \to Y$. We extend this definition by applying f_a to n-chains in $C_n(X)$. This gives us the commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\delta} & C_n(X) & \xrightarrow{\delta} & C_{n-1}(X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle f_a} & & \\ \cdots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\delta} & C_n(Y) & \xrightarrow{\delta} & C_{n-1}(Y) & \longrightarrow & \cdots \end{array}$$

The chain map f_a induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

It is now readily apparent that if X and Y are homeomorphic, that is, if $f : X \to Y$ is a homeomorphism, then the induced map f_* is an isomorphism.

In order to formalize the relationships between the homology groups of a topological space X, a subset $A \subset X$, and the quotient space X/A, we introduce the concept of *exact sequences*:

Definition 4.6. A sequence of the form:

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

where the A_i are abelian groups and the α_i are homomorphisms is called an *exact sequence* if $Ker(\alpha_n) = Im(\alpha_{n+1})$ for all n.

We note two things:

1 $Ker(\alpha_n) = Im(\alpha_{n+1}) \implies Im(\alpha_{n+1}) \subset Ker(\alpha_n) \iff \alpha_n\alpha_{n+1} = 0$, so an exact sequence is a chain complex.

2 As $Ker(\alpha_n) \subset Im(\alpha_{n+1})$, the homology groups of an exact sequence are trivial.

We can express some algebraic concepts using exact sequences:

1 $0 \longrightarrow A \xrightarrow{a} B$ is exact $\iff$ $Ker(a) = 0$, or a is injective.

2 $A \xrightarrow{a} B \longrightarrow 0$ is exact $\iff$ $Im(a) = B$, or a is surjective.

3 $0 \longrightarrow A \xrightarrow{a} B \longrightarrow 0$ is exact iff a is an isomorphism.

4 $0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$ is exact iff a injective, b surjective, and $Ker(b) = Im(a)$, in which case b gives an isomorphism $C \cong B/Im(a)$. If $a : A \hookrightarrow B$ is an inclusion, we have $C \cong B/A$. This type of exact sequence is called a *short exact sequence.*

The next concept we will need is that of *relative homology groups*: Given a space X and a subspace $A \subset X$, define $C_n(X, A)$ to be the quotient group $C_n(X)/C_n(A)$. This means that chains in A get identified with the trivial chains in $C_n(X)$. Since the operator $\delta : C_n(X) \to C_{n-1}(X)$ also takes $C_n(A) \to C_{n-1}(A)$, we get a natural boundary map on the quotient group $\delta : C_n(X, A) \to C_{n-1}(X, A)$.

This gives us the sequence:

$$\cdots \longrightarrow C_{n+1}(X, A) \xrightarrow{\delta_{n+1}} C_n(X, A) \xrightarrow{\delta_n} C_{n-1}(X, A) \longrightarrow \cdots$$

which is a chain complex because $\delta_{n+1}\delta_n = 0$. We can then define *relative homology groups* $H_n(X, A)$ to be the homology groups of this chain complex.

Two important facts about $H_n(X, A)$ are:

1 elements in $H_n(X, A)$ are represented by *relative cycles*, or n-chains x in $C_n(X)$ such that $\delta_n x = C_{n-1}(A)$.

2 A relative cycle x is trivial iff it is a *relative boundary*, i.e. x is the sum of a chain in $C_n(A)$ and the boundary of a chain in $C_{n+1}(X)$.

We next show that the relative homology groups $H_n(X, A)$ fit into the long exact sequence:

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots \longrightarrow H_0(X, A) \longrightarrow 0$$

and we can prove this algebraically.

Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\delta"] & C_n(X) \arrow[r, "j"] \arrow[d, "\delta"] & C_n(X, A) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

where i is the inclusion map $C_n(A) \hookrightarrow C_n(X)$, and j is the quotient map $C_n(X) \to C_n(X, A)$.

The diagram is commutative, and we turn it by 90 degrees to get one of the form:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\delta"] \arrow[d, "i"] & A_n \arrow[r, "\delta"] \arrow[d, "i"] & A_{n-1} \arrow[r, "\delta"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\delta"] \arrow[d, "j"] & B_n \arrow[r, "\delta"] \arrow[d, "j"] & B_{n-1} \arrow[r, "\delta"] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta"] \arrow[d] & C_n \arrow[r, "\delta"] \arrow[d] & C_{n-1} \arrow[r, "\delta"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns are short exact sequences and the rows are chain complexes of the abelian groups A_i, B_i, and C_i.

Writing the diagram in this form indicates that i and j are chain maps, and so induce maps i_* and j_* on homology, as in Theorem 4.5. We take some $c \in C_n$ to be a cycle. j is surjective, so $c = j(b)$ for some $b \in B_n$. For $\delta b \in B_{n-1}$, $j(\delta b) = \delta j(b)$ by commutativity. Then $\delta j(b) = \delta c = 0$ since c is a cycle. Thus $\delta b \in Ker(j)$.

Since the columns are exact, we have $Ker(j) = Im(i)$, and this means that $\delta(b) = i(a)$ for some $a \in A_{n-1}$. Commutativity gives $i(\delta(a)) = \delta i(a) = \delta\delta b = 0$, and so i injective $\implies \delta(a) = 0$. Therefore a is a cycle and represents an element $[a] \in H_{n-1}(A)$ of homology. We can now define $\delta : H_n(C) \to H_{n-1}(A)$ by sending the homology class of $[c]$ to the homology class of $[a], \delta[c] = [a]$. This is well-defined for the following reasons:

1 i is injective, so a is uniquely determined by δb.

2 Choosing b' instead of b gives $j(b') = j(b) \implies j(b') - j(b) = 0 \implies j(b' - b) = 0 \implies b - b' \in Ker(j) = Im(i)$. So $b - b' = i(a')$, or $b' = b + i(a')$, and $\delta(b + i(a')) = \delta(b) + \delta i(a') = i(a) + i\delta(a') = i(a + \delta a')$. But $\delta a' \sim 0$, so $a + \delta a' \sim a$

3 Choosing c_* from the coset of c implies $c_* = c + \delta c'$. $c' = j(b')$ for some b', so $c + \delta c' = c + \delta j(b') = j(b) + j\delta(b') = j(b + \delta b')$. Thus changing c has the effect of changing b to a homologous element, which does not affect a at all.

Claim 4.7. *The map $\delta : H_n(C) \to H_{n-1}(A)$ defined above is a homomorphism:*

Proof. If $\delta[c_1] = [a_1]$ and $\delta[c_2] = [a_2]$ via b_1 and b_2, as above, then we have $j(b_1+b_2) = j(b_1)+j(b_2) = c_1 + c_2$, and $i(a_1 + a_2) = i(a_1 + a_2) = \delta b_1 + \delta b_2 = \delta(b_1 + b_2)$, so $\delta([c_1] + [c_2]) = [a_1] + [a_2]$. □

Proposition 4.8. *The sequence*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \cdots$$

is exact.

Proof. There are six inclusions to be verified:

1 $Im(i_*) \subset Ker(j_*)$: $ji = 0 \implies j_*i_* = 0$.

2 $Im(j_*) \subset Ker(\delta)$: $\delta b = 0$ by definition, so $\delta j_* = 0$.

3 $Im(\delta) \subset Ker(i_*)$: $i_*\delta = 0$ since $i_*\delta[c] = [\delta b] = 0$.

4 $Ker(j_*) \subset Im(i_*)$: A homology class in $Ker(j_*)$ can be represented by a cycle $b \in B_n$ such that $j(b) = \delta c'$ is a boundary for some $c' \in C_{n+1}$. Surjectivity of j gives $c' = j(b')$ for some $b' \in B_{n+1}$. But then $j(b) = \delta c' = \delta j b'$, so $j(b - \delta b') = 0$, and $b - \delta b' = i(a)$ for some $a \in A_n$. a is a cycle, since $i\delta a = \delta i a = \delta(b - \delta b') = \delta b = 0$ because b is a cycle and i injective. Therefore, $i_*[a] = [b]$, and the two inclusions give us: $Im(i_*) = Ker(j_*)$

5 $Ker(\delta) \subset Im(j_*)$: we take c a representative of a homology class in $Ker(\delta)$. Then we have $a = \delta a'$ for some $a' \in A_n$. $b - i(a')$ is a cycle because $\delta(b - i(a')) = \delta(b) - \delta i(a') = \delta b - i(\delta a') = \delta b - i(a) = 0$. We also have $j(b - i(a')) = j(b) - ji(a') = j(b) = c$, so $Ker(\delta) \subset Im(j_*)$.

6 $Ker(i_*) \subset Im(\delta)$: We take a cycle $a \in A_{n-1}$ such that $i(a) = \delta b$ for some $b \in B_n$. $j(b)$ is a cycle, because $\delta(j(b)) = j(\delta b) = ji(a) = 0$. Thus $\delta[j(b)] = [a]$, and $Ker(i_*) \subset Im(\delta)$.

Thus we have: $Im(i_*) = Ker(j_*), Im(j_*) = Ker(\delta)$, and $Im(\delta) = Ker(i_*)$, and so the sequence is exact. □

Proposition 4.9. *The sequence:*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \longrightarrow \cdots \longrightarrow H_0(X, A) \longrightarrow 0$$

is exact.

This follows from the previous proposition, with the note that for a relative cycle x in $H_n(X, A)$, $\delta[x]$ is the class of the cycle $[\delta x] \in H_{n-1}(A)$.

We also cite here the following theorem, known as the *Excision Theorem*

Theorem 4.10. *Given $Y \subset A \subset X$, with the closure of Y contained in the interior of A, then $(X - Y, A - Y) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Y, A - Y) \to H_n(X, A)$ for all n.*

Although the statement of this theorem is straightforward and seems intuitive, it is nevertheless rather complicated to prove, and so we merely state it here.

5 The Equivalence of $H^{\Delta}{}_n$ and H_n

We wish to prove that the groups $H_n(X)$ and $H^{\Delta}{}_n(X)$ are equivalent. Immediately we recall that simplicial homology groups only have meaning and can only be calculated for simplicial structures. This is not a serious problem, as we can calculate singular homology groups on any topological space, including a simplicial complex. Moreover, the fact that homeomorphic spaces have isomorphic singular homology groups lends itself to the idea that we can put some sort of simplicial structure on a topological space. Therefore, in order to prove the equivalence of $H_n(X)$ and $H^{\Delta}{}_n(X)$, we take an arbitrary simplicial complex as our topological space X. It should be noted, however, that not all topological spaces are homeomorphic to a simplicial complex; we will ignore such spaces for the purposes of this paper.

In order to show the equivalence of $H_n(X)$ and $H^{\Delta}{}_n(X)$, we need to show the existence of an isomorphism between the two groups for all n. It is easy enough to see the existence of a homomorphism: we already have a map $L_n(X) \to C_n(X)$ from the simplicial chain group to the singular chain group which sends each simplex of X to $\sigma : \Delta^n \to X$. This induces a map $H^{\Delta}{}_n(X) \to H_n(X)$.

Theorem 5.1. *For all n, the homomorphisms $H^{\Delta}{}_n(X) \to H_n(X)$ are isomorphisms. Thus the singular and simplicial homology groups are equivalent.*

Proof. We take X to be a simplicial complex. For X^k the k-skeleton of X, we get the following commutative diagram of exact sequences, since $X^{k-1} \subset X^k$.

$$\begin{array}{ccccccccc}
H^{\Delta}{}_{n+1}(X^k, X^{k-1}) & \longrightarrow & H^{\Delta}{}_n(X^{k-1}) & \longrightarrow & H^{\Delta}{}_n(X^k) & \longrightarrow & H^{\Delta}{}_n(X^k, X^{k-1}) & \longrightarrow & H^{\Delta}{}_{n-1}(X^{k-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

The space X^k/X^{k-1} contains only simplices of dimension k. Hence for $n \neq k$, the group $L_n(X^k, X^{k-1})$ is equal to zero. When $n = k$, $L_n(X^k, X^{k-1})$ is a free abelian group with basis consisting of the k-simplices of X. Since the cycles Z_n form a subgroup in L_n, and the boundary group B_n is empty, $H^{\Delta}{}_n(X^k, X^{k-1})$ has the same description as L_n, with the caveat that when $n = k$, the basis of Z_n consists of k-cycles.

We observe that the characteristic maps $\Delta^k \to X$ for all the k-simplices of X give us a map $\Phi : \sqcup_i(\Delta^k{}_i, \Delta^{k-1}{}_i) \to (X^k, X^{k-1})$. It is then fairly clear that this map induces a homeomorphism $\Phi_* : \sqcup_i \Delta^k{}_i / \sqcup_i \Delta^{k-1}{}_i \to X^k/X^{k-1}$. But then, $H_n(\sqcup_i \Delta^k{}_i / \sqcup_i \Delta^{k-1}{}_i) \cong H_n(X^k/X^{k-1})$.

It is a consequence of the Excision Theorem (Theorem 4.10) that there exists an isomorphism $H_n(X, A) \to H_n(X/A)$ for all good pairs (X, A). Then we have $H_n \sqcup_i (\Delta^k{}_i, \Delta^{k-1}{}_i) \cong H_n(\sqcup_i \Delta^k{}_i / \sqcup_i \Delta^{k-1}{}_i)$ and $H_n(X^k, X^{k-1}) \cong H_n(X^k/X^{k-1})$. By transitivity, this gives us $H_n \sqcup_i (\Delta^k{}_i, \Delta^{k-1}{}_i) \cong H_n(X^k, X^{k-1})$.

This gives us that $H_n(X^k, X^{k-1})$ is zero for $n \neq k$ and a free abelian group with basis the relative cycles given by the maps $\Delta^k \to X$. Therefore the map $H^{\Delta}{}_n(X^k, X^{k-1}) \to H_n(X^k, X^{k-1})$ is an isomorphism. That is, the first and fourth vertical maps in the diagram are isomorphisms.

We use induction to complete the argument, and therefore assume that the second and fifth arrows are isomorphisms. We can then prove that the third arrow is an isomorphism using the following lemma, known as the *Five Lemma*:

Lemma 5.2. *In a commutative diagram of the form:*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

if $\alpha, \beta, \delta,$ and ϵ are all isomorphisms, and the two rows are exact, then γ is also an isomorphism.

Proof. Commutativity of the diagram gives us that γ must be a homomorphism. Therefore, it suffices to show that γ is a bijection.

Take $c' \in C'$. Since δ is surjective, $k'(c') = \delta(d)$ for some $d \in D$. Injectivity of ϵ gives us that $\epsilon l(d) = l'\delta(d) = l'k'(c') = 0 \implies l(d) = 0$. But the rows are exact, so we have $d = k(c)$ for some $c \in C$.
$k'(c') - k'(\gamma(c)) = k'(c') - \delta k(c) = k'(c') - \delta(d) = 0$, so $k'(c' - \gamma(c)) = 0$, and, by exactness, $c' - \gamma(c) = j'(b')$ for some $b' \in B'$. Surjectivity of β gives $b' = \beta(b)$ for some $b \in B$, so $\gamma(c + j(b)) = \gamma(c) + \gamma(j(b)) = \gamma(c) + j'\beta(b) = \gamma(c) - j'(b') = c'$, so γ is surjective.

For injectivity, suppose $\gamma(c) = 0$. δ is injective, so $\delta(k(c)) = k'(\gamma(c)) = 0 \implies k(c) = 0$. Then $c = j(b)$ for some $b \in B$. $\gamma(c) = \gamma(j(b)) = j'(\beta(b))$, so we have $\beta(b) = i'(a')$ for some $a' \in A'$. Surjectivity of α gives $a' = \alpha(a)$ for some $a \in A$. β is injective, so $\beta(i(a) - b) = \beta(i(a)) - \beta(b) = i'(\alpha(a)) - \beta(b) = i'(a') - \beta(b) = 0 \implies i(a) - b = 0$. That is, $b = i(a)$, so $c = j(b) = j(i(a)) = 0$ by exactness of rows. Hence, γ has trivial kernel and is therefore injective. □

Applying this to the earlier diagram, we note that the arrow $H^{\Delta}{}_n(X^k) \to H_n(X^k)$ must therefore be an isomorphism, and so the theorem is proven.

□

Although simplicial and singular homology express the same ideas, the two constructions have different uses. For instance, as perhaps demonstrated by some of the examples and theorems in this paper, it is much simpler to calculate homology groups using the ideas of simplicial homology, but in many cases is easier and more straightforward to prove theorems using singular homology, as the notion of continuous maps is much more compatible with the second theory. The equivalence of singular and simplicial homology groups, then, gives us a powerful tool with which to attack both types of problems and is an important result in algebraic topology.

References

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