THE DYNAMICAL SYSTEM GENERATED BY THE 2D NAVIER-STOKES EQUATIONS

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Abstract. Herein I define the global attractor for the semidynamical system $(H, \{S(t)\}_{t\geq 0})$, where H is a Hilbert space and $S(t)$ is a semigroup. In particular, I will consider the semigroup $S(t)$ which acts on a relevant function space by $S(t)u_0 = u(t; u_0)$, where $u(t; u_0)$ is the solution of a given partial differential equation at time t with initial condition $u(0) = u_0$. By imposing regularity properties on the terms of the Navier-Stokes Equations (with periodic boundary conditions), we can find in H a maximal compact invariant set $\mathcal{A}$ that is also the minimal set that attracts all bounded sets $X \subset H$, and we call this set the 'global attractor'. On the global attractor, we can extend our semidynamical system to a true dynamical system $(\mathcal{A}, \{S(t)\}_{t\in\mathbb{R}})$. Finally, I will show that $\mathcal{A}$ is finite-dimensional by constructing an explicit bound on its fractal dimension, and I will discuss in what sense this implies that the dynamics of the attractor are determined by a finite number of degrees of freedom, by showing that we can parametrize the attractor with a finite set of coordinates. The reader should have some familiarity with the languages of Banach, Hilbert and Sobolev spaces, as well as with the basic notations of PDEs and Dynamical Systems. For conciseness, some well known inequalities and estimates will be utilized without proof. This exposition most closely follows the treatment given by Robinson [11].

Contents

1. The 2D Navier-Stokes Equations: Existence and Uniqueness of Strong Solutions

We will be interested in the semidynamical system $(H, \{S(t)\}_{t\geq 0})$, where H is an appropriate space containing the solutions of the 2D Navier-Stokes Equations, and $S(t)$ is the semigroup such that $S(t)u_0 = u(t; u_0)$, where $u(t; u_0)$ is the solution of the the equations at time t and for initial condition $u(0) = u_0$. Our final aim will be to show that the dynamics of fluid flow can be described with finitely many degrees of freedom, and this is accomplished in the final section. Before we can

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make sense of this concept, however, and before we can understand how to choose H appropriately, we must establish the existence and uniqueness of solutions to the Navier-Stokes Equations:

$$\rho\big(\vec{u_t} + (\vec{u}\cdot\nabla)\vec{u}\big) - \nu\Delta\vec{u} + \nabla p = f(\vec{x},t), \quad div\ \vec{u} = 0 \tag{1.1}$$

where $\nu \geq 0$ is the kinematic viscosity constant, $\rho > 0$ is the pressure constant, $u(\vec{x},t)$ is the vector-valued velocity function, $p(\vec{x},t)$ is the scalar pressure, and $f(\vec{x},t) \in L^2(\Omega,\mathbb{R})$, where Ω is our domain of interest, is the density of force per unit volume. [6] [12] We normalize the constant density as $\rho = 1$, which we can do since we are considering an incompressible fluid. Therefore, the condition $div\cdot\vec{u} = 0$ is just Newton's second law, $F = ma$, in the form $div\cdot\vec{u} = \nabla\cdot(\rho\vec{u}) = \nabla\cdot\vec{u} = 0$. [11] Henceforth, the vector arrows will be dropped.

For simplicity, we will consider a domain $Q = [0,L]^2$ and impose periodic boundary conditions. Namely, in this case we needn't worry about the boundary conditions, because our solutions will be L-periodic. However, this means that our solutions will not be unique, so to ensure uniqueness we must restrict our attention to functions $u \in L^2$ such that $\int_Q u = 0$. If we let $\lambda_1 = \frac{4\pi^2}{L^2}$, then by Poincaré's Inequality:

$$\|u\|_{L^2} \leq \lambda_1^{-\frac{1}{2}}\|\nabla u\|_{L^2} \tag{1.2}$$

In order to ensure that $\int_Q u = 0$, we will want to assume that $\int_Q u_0(x)dx = \int_Q f(x,t)dx = 0$ for all $t \geq 0$. This indeed gives us the desired result, if we compute:

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x,t)$$

after rearranging terms, gives

$$u_t = \nu\Delta u - (u\cdot\nabla)u - \nabla p + f(x,t)$$

Integration over Q now yields

$$\frac{d}{dt}\int_Q u_i dx = \int_Q \Big[\nu(D_1(D_1u_i) + D_2(D_2u_i)) - u_1D_1u_i - u_2D_2u_i - D_ip + f\Big]dx$$

which reduces to

$$\frac{d}{dt}\int_Q u_i dx = \int_Q ((D_1u_1)u_i + (D_2u_2)u_i)dx$$

after an integration by parts, since the integral over Q of a partial derivative of the L-periodic functions is zero, and hence the periodic terms drop out. But since $\nabla\cdot u = 0$, this becomes

$$\int_Q u(x,t)dx = 0$$

which is precisely the condition we wanted for u.

To simplify our treatment, we will want to reformulate the Navier-Stokes Equations in a weaker form, that will therefore be easier to solve. Then we will want to impose conditions on f and u_0 that will imply that a solution of this weaker form of the equation is actually a classical solution. We will want to reformulate the Navier-Stokes Equations in terms of a linear operator A and bilinear operator B as

$$\frac{du}{dt} + \nu Au + B(u,u) = f$$

To do this, we begin by defining a bilinear form $a(u,v) \equiv \int_Q \nabla u \cdot \nabla v dx$ and a trilinear form $b(u,v,w) \equiv \sum_{i,j=1}^2 \int_Q u_i \frac{\partial v_j}{\partial x_i} w_j dx$. We then define the 'Stokes operator' A to be the unique linear operator such that

$$\langle Au, v\rangle = a(u,v) \text{ for all } v \in V$$

with

$$V = \left\{u \in [H_p^1(Q)]^2 : \nabla \cdot u = 0 \text{ and } \int_Q u = 0\right\}$$

where we define the Sovolev space

$$H_p^k(\Omega) = \left\{u \text{ periodic} : D^\alpha u \in L^2(\Omega) \forall\, 0 \le |\alpha| \le k\right\}$$

We have incorporated the condition $\nabla \cdot u = 0$ into the definition of V, so that this condition need not concern us further. The domain of A is just

$$D(A) = \left\{u \in [H_p^2(Q)]^2 : \nabla \cdot u = 0 \text{ and } \int_Q u = 0\right\}$$

It turns out that for $u \in D(A)$ and $f \in [H^{-1}(Q)]^2$ (where $H^{-k}(\Omega)$ is the dual space of $H_0^k(\Omega)$, which is the completion of $C_c^\infty(\Omega)$ in $H^k(\Omega)$), $\nu Au = f$, so that A is a solution to Poisson's Equation, and therefore $Au = -\Delta u$ for all $u \in D(A)$. Returning to the Navier-Stokes Equations, if we take the inner product of (1.1) with an element $v \in \mathbb{V} = \left\{u \in [C_p^\infty(Q)]^2 : \nabla \cdot u = 0 \text{ and } \int_Q u = 0\right\}$, we get

$$\left(\frac{du}{dt}, v\right) - \nu \int_Q \Delta uv + \int_Q (u \cdot \nabla u)v + \int_Q (\nabla p)v = \int_Q fv$$

But if we integrate the pressure term by parts, we get

$$\int_Q (\nabla p)v = \int_Q p(\nabla \cdot v) = 0$$

since $\nabla \cdot v = 0$, so the pressure term drops out, and we are left with

$$\left(\frac{du}{dt}, v\right) - \nu \int_Q \Delta uv + \int_Q (u \cdot \nabla u)v = \int_Q fv$$

Integrating the second term by parts gives

$$-\nu \int_Q \Delta uv = \nu \sum_{j=1}^2 \int_Q \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j} = \nu a(u,v)$$

and we are left with

$$\left(\frac{du}{dt}, v\right) + \nu a(u,v) + \int_Q (u \cdot \nabla u)v = \int_Q fv$$

Actually, this can be written as

$$\left(\frac{du}{dt}, v\right) + \nu a(u,v) + b(u,u,v) = \langle f, v\rangle \text{ for all } v \in \mathbb{V}$$

but we can actually write this for all $v \in V$, since $\mathbb{V}$ is dense in V. If we define the bilinear operator $B(u, v)$ to be the unique operator such that $\langle B(u, v), w\rangle = b(u, v, w)$ for all $w \in V$, we are left with

$$\frac{du}{dt} + \nu Au + B(u, u) = f$$

with equality in $L^2(0, T; H)$ for strong solutions, where we have let $H \equiv \{u \in [L^2(Q)]^2 : \nabla \cdot u = 0 \text{ and } \int_Q u = 0\}$. This is the weak form of the Navier-Stokes Equations with periodic boundary conditions, as desired.

Proposition 1.3. *If $f \in C^0(\Omega)$, and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $u \in H_0^1(\Omega)$ and*

$$\sum_{i=1}^{2} \int_\Omega D_i u(x) D_i v(x) dx = \int_\Omega f(x) v(x) dx$$

for all $v \in C_c^1(\Omega)$, then u is a classical solution of

$$-\Delta u = f \textit{ with } u|_{\partial\Omega} = 0$$

Proof. We have

$$\int_\Omega \nabla u \cdot \nabla v dx = \int f(x) v(x) dx$$

which, after integration by parts and rearrangement becomes

$$\int_\Omega (\Delta u - f) v dx = 0$$

But $u \in C^2(\Omega)$ and $f \in C^0(\Omega)$, so $\varphi \equiv \Delta u - f \in C^0(\Omega)$. Suppose that there exists an $x \in \Omega$ such that $\varphi(x) \neq 0$. Since φ is continuous, there is a neighborhood U of x such that φ has constant sign on U. Let v be a positive function compactly supported in U. Then

$$\int_\Omega \varphi v dx = \int_U \varphi v dx \neq 0$$

which is a contradiction. □

Corollary 1.4. *If $u(t)$ is a strong solution of the weak form of the Navier-Stokes Equations, and $u \in [C_p^2(Q)]^2$, then u is a solution of the classical Navier-Stokes Equations.*

Proof. This works exactly the same as Proposition 1.3. Simply check that imposing this continuity condition on u will force an equality in $L^2(O, T; H)$ to be a true pointwise equality. □

We need only a couple of properties of the trilinear form $b(u, v, w)$, and we are ready to prove the existence and uniqness of strong solutions of the 2D Navier-Stokes Equations. I will state these without proof:

If $u \in H$ and $v, w \in V$, then

$$b(u, v, w) = -b(u, w, v) \tag{1.5}$$

and

$$b(u, u, Au) = 0 \; \forall \; u \in D(A) \tag{1.6}$$

Differentiating:

$$b(v,u,Au)+b(u,v,Au)+b(u,u,Av)=0\ \forall\, u,v\in D(A) \tag{1.7}$$

Finally,

(1.8)
$$|b(u,v,w)|\leq\begin{cases}\|u\|_\infty\|v\|_V\|w\|_H & \text{if } u\in L^\infty,\ v\in V,\ w\in H\\ k\|u\|_H^{\frac12}\|u\|_V^{\frac12}\|v\|_V\|w\|_H^{\frac12}\|w\|_V^{\frac12} & \text{if } u,v,w\in V\\ k\|u\|_H^{\frac12}\|u\|_V^{\frac12}\|v\|_V^{\frac12}\|Av\|_H^{\frac12}\|w\|_H & \text{if } u\in V,\ v\in D(A),\ w\in H\end{cases}$$

Theorem 1.9. *(Strong Solutions) If $u_0\in V$ and $f\in L^2_{loc}(0,\infty;H)$, then there is a unique solution of*

$$\frac{du}{dt}+\nu Au+B(u,u)=f\ \ (\text{as an equality in } L^2(0,T;H))$$

that satisfies

$$u\in L^\infty(0,T;V)\cap L^2(0,T;D(A))$$

and in fact $u\in C^0([0,T];V)$. The solutions depend continuously on u_0.

Proof. We will want to utilize a Galerkin approximation of the Navier-Stokes Equations, approximating the equation with a sequence of ODE's u_n. By obtaining uniform bounds on the u_n and their derivatives, we will then be able to establish that our sequence converges to some u in $L^\infty(0,T;V)\cap L^2(0,T;D(A))$, and we will show that this u is our desired solution.

Let $\{w_j\}$ be the set of eigenfunctions $Aw_i=\lambda_i w_i$. By the Hilbert-Schmidt Theorem, $\{w_j\}$ is an orthonormal basis for H. It is also an orthogonal basis for V. Define

$$P_n u=\sum_{j=1}^n (u,w_j)w_j$$

to be the projection of an element of H onto the subspace spanned by $\{w_1,\dots,w_n\}$. Now take the n-th Galerkin approximation of the Navier-Stokes equation (by ODEs that we know we can solve using fixed point arguments)

$$\frac{du_n}{dt}+\nu Au_n+P_nB(u_n,u_n)=P_nf$$

If we take the inner product of this approximation with Au_n we get

$$\frac12\frac{d}{dt}\|u_n\|_V^2+\nu\|Au_n\|_H^2+(P_nB(u_n,u_n),Au_n)=(f,Au_n)$$

A commutes with P_n, since $Aw_j=\lambda_j w_j$, so

$$(P_nB(u_n,u_n),Au_n)=(B(u_n,u_n),Au_n)=b(u_n,u_n,Au_n)=0 \text{ by (1.7)}$$

so we have

$$\frac12\frac{d}{dt}\|u_n\|_V^2+\nu\|Au_n\|_H^2\leq\|f\|_H\|Au_n\|_H$$

Using Young's Inequality, we get

$$\frac12\frac{d}{dt}\|u_n\|_V^2+\nu\|Au_n\|_H^2\leq\frac{\|f\|_H^2}{\nu}$$

If we integrate from 0 to t, this becomes

$$\|u_n(t)\|_V^2 + \nu \int_0^t \|Au_n(s)\|_H^2 \, ds \le \|u_n(0)\|_V^2 + \frac{\|f\|_{L^2(0,t:H)}^2}{\nu}$$

But clearly, $\|u_n(0)\|_V \le \|u(0)\|_V$ for all n, so we have

$$\sup_{t\in[0,T]} \|u_n(t)\|_V^2 \le K = \|u_0\|_V^2 + \frac{\|f\|_{L^2(0,t:H)}^2}{\nu}$$

and

$$\int_0^T \|Au_n(s)\|_H^2 \, ds \le \frac{K}{\nu}$$

Therefore u_n is uniformly bounded in $L^\infty(0,T;V)$ and in $L^2(0,T;D(A))$, and by the Alaoglu weak-* compactness theorem we can take a subsequence $\{u_n\}$ such that

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;V)$$

and

$$u_n \rightharpoonup u \text{ in } L^2(0,T;D(A))$$

for some $u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$. We want to obtain a uniform bound for $\frac{du_n}{dt}$ in $L^2(0,T;H)$. We can actually show that $\frac{du_n}{dt}$ is uniformly bounded in $L^2(0,T;V^*)$ (where throughout X^* denotes the dual space of the Banach space X). For the equation

$$\frac{du_n}{dt} = -\nu Au_n - P_nB(u_n,u_n) + P_nf$$

we need to show that each term on the right is uniformly bounded in $L^2(0,T:V^*)$. Since u_n is uniformly bounded in $L^2(0,T:V)$ and $A : V \to V^*$ is a continuous linear operator, the term $-\nu Au_n$ is uniformly bounded. Since $f \in L^2(0,T:V^*)$, it is clear that the term P_nf is also uniformly bounded. Finally, by (1.8) we have

$$\|b(u,v,w)\|_H \le k \|u\|_H^{\frac{1}{2}} \|u\|_V^{\frac{1}{2}} \|v\|_H^{\frac{1}{2}} \|v\|_V^{\frac{1}{2}} \|w\|_V$$

But $b(u,v,w) = -b(u,w,v)$ by (1.5), so

$$\langle B(u,u),w\rangle \le k \|u\|_H \|u\|_V \|w\|_V \quad \text{for all } w \in V$$

and we therefore have

$$\|B(u,u)\|_{V^*} \le k \|u\|_H \|u\|_V$$

But clearly, $\|P_nB(u,v)\|_{V^*} \le \|B(u,v)\|_{V^*}$ for all $u,v \in V$, so

$$\|P_nB(u_n,u_n)\|_{L^2(0,T;V^*)}^2 \le \int_0^T \|B(u_n(s),u_n(s))\|_{V^*}^2 \, ds$$

and

$$\begin{aligned}\|P_nB(u_n,u_n)\|_{L^2(0,T;V^*)}^2 &\le k \int_0^T \|u_n(s)\|_H^2 \|u_n(s)\|_V^2 \, ds \\ &\le k \|u_n\|_{L^\infty(0,T;H)}^2 \|u_n\|_{L^2(0,T;V)}^2\end{aligned}$$

So $P_nB(u_n,u_n)$ is uniformly bounded in $L^2(0,T;V^*)$, and thus, $\frac{du_n}{dt}$ is uniformly bounded in $L^2(0,T;V^*)$, and hence in $L^2(0,T;H)$. Following Temam [12] we can take a further subsequence $\{\frac{du_{n_j}}{dt}\}$ such that

$$\frac{du_{n_j}}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;H)$$

Another result of Temam [12] confirms that $u \in C^0([0,T];V)$. We now have that $u_{n_j} \to u$ strongly in $L^2(0,T;V)$, so

$$\frac{du}{dt} + \nu Au + B(u,u) = f$$

holds as an equality in $L^2(0,T;H)$, and we have therefore proved the existence of strong solutions.

For uniqueness, let $w = u - v$ where u and v are strong solutions of the Navier-Stokes Equations, and consider

$$\frac{dw}{dt} + \nu Aw + B(u,u) - B(v,v) = f$$

Taking the inner product with Aw we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_V^2 + \nu\|Aw\|_H^2 &= b(v,v,Aw) - b(u,u,Aw)\\ &\leq k\bigg[\|w\|_H^{\frac{1}{2}}\|w\|_V^{\frac{1}{2}}\|u\|_V^{\frac{1}{2}}\|Au\|_H^{\frac{1}{2}}\|Aw\|_H\\ &\qquad + \|v\|_H^{\frac{1}{2}}\|v\|_V^{\frac{1}{2}}\|w\|_V^{\frac{1}{2}}\|Aw\|_H^{\frac{3}{2}}\bigg]\end{aligned}$$

Applying Young's Inequality and neglecting the terms in $\|Aw\|_H^2$, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|_V^2 \leq C\bigg[\|u\|_V\|Au\|_H + \|v\|_V^4\bigg]\|w\|_V^2$$

so that

$$\|w(t)\|_V^2 \leq exp\bigg(C\int_0^t\Big(\|u(s)\|_V\|Au(s)\|_H + \|v(s)\|_V^4\Big)ds\bigg)\|w(0)\|^2$$

Since u and v are strong solutions, they are bounded in $L^\infty(0,T;V)$ and in $L^2(0,T;D(A))$, and we have continuous dependence on initial conditions. Furthermore, if $w(0) = 0$, since the integral expression in the exponential function is finite this implies that $w(t) = 0$ for all t. In other words, if $u(0) = v(0) = u_0$, then $u(t) = v(t)$ for all t, so the strong solutions of the Navier-Stokes Equations are unique. □

Having proved the existence and uniqueness of strong solutions of the 2D Navier-Stokes equations with periodic boundary conditions, we can now make sense of our semidynamical system. If $f \in V^*$, we can define $(H,\{S(t)\}_{t\geq 0})$ as above sensibly. However, we encounter difficulties for $t < 0$, since we could only confirm unique solutions $u(t,u_0)$ for positive time. [11] We will see, however, that by considering the asymptotic behavior of $(H,\{S(t)\}_{t\geq 0})$, we can define a structure $\mathcal{A}$ called the global attractor, on which we can sensibly define a dynamical system for all time.

2. Existence of the Global Attractor

To begin, we should define some important concepts:

Definition 2.1. A set $B \subset H$ is called an absorbing set if for each bounded set $X \subset H$ there exists a time $t_0(X)$ such that $S(t)X \subset B$ for all $t \geq t_0(X)$.

Definition 2.2. A semigroup $S(t)$ (here the semigroup relevant to us) is dissipative if it possesses a compact absorbing set B.

Definition 2.3. For any set $X \subset H$ we define the ω-limit set of X as

$$\omega(X) = \{y : \exists t_n \to \infty, x_n \in X \text{ with } S(t_n)x_n \to y\}$$

The ω-limit is an important concept. Here is a useful characterisation:

Proposition 2.4. *For a bounded set* X, $\omega(X) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)X}$

Proof. Let X be a bounded set. Let $\omega_1(X) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)X}$. If $y \in \omega(X)$, we clearly have $y \in \overline{\bigcup_{s\geq t} S(s)X}$ for all $t \geq 0$, $y \in \omega_1$, and we now know that $\omega \subset \omega_1$. Conversely, if $y \in \omega_1$, then for all $t \geq 0$ we have $y \in \overline{\bigcup_{s\geq t} S(s)X}$, so there are sequences $\{\tau_m^{(t)}\}$ and $\{x_m^{(t)}\}$, with $\tau_m^{(t)} \geq t$, $x_m^{(t)} \in X$, such that $S(\tau_m^{(t)})x_m^{(t)} \to y$. Now we can construct new sequences $\{t_n\}$ and $\{x_n\}$, with t_n chosen from $\{\tau_m^{(n)}\}$ and x_n chosen from $\{x_m^{(n)}\}$ such that

$$\left\| S(\tau_m^{(n)})x_m^{(n)} - y \right\|_H \leq \frac{1}{n}$$

Then $S(t_n)x_n \to y$ as $t_n \to \infty$, so $y \in \omega \Rightarrow \omega_1 \subset \omega \Rightarrow \omega = \omega_1$. $\square$

Definition 2.5. A set X is called invariant if $S(t)X = X$ for all $t \geq 0$.

We are now ready to define the global attractor:

Definition 2.6. (The Global Attractor) The global attractor $\mathcal{A}$ is the maximal compact invariant set, and the minimal set that attracts all bounded sets $X \subset H$ (i.e. $dist(S(t)X, \mathcal{A}) \to 0$ as $t \to \infty$, where $dist$ is the semidistance between sets: $dist(X, Y) = \sup_{x\in X} \inf_{y\in Y} |x - y|$).

Therefore, if $\mathcal{A}$ exists for the 2D Navier-Stokes Equations, then at least asymptotically, all the dynamics of $(H, \{S(t)\}_{t\geq 0})$ will occur in $\mathcal{A}$. Furthermore, since $\mathcal{A}$ is invariant, no part of $\mathcal{A}$ can be neglected. The condition that $\mathcal{A}$ be compact is what prevents us from simply choosing $\mathcal{A} = H$, and, as we will see, $\mathcal{A}$ will turn out to be a much smaller space than H. First, however, we must establish the existence of $\mathcal{A}$ for $(H, \{S(t)\}_{t\geq 0})$. The following lemma is key:

Lemma 2.7. *If $S(t)$ is dissipative, and B is a compact absorbing set, then the global attractor exists, and $\mathcal{A} = \omega(B)$.*

Proof. We first want to show that $\omega(B)$ is nonempty, compact, and invariant. We want to use the characterisation from Proposition 2.4. Since for $t \geq t_0$ (where t_o is defined in the definition of an absorbing set), the sets $\overline{\bigcup_{s\geq t} S(s)B}$ are nonempty compact sets decreasing with t, their intersection $\omega(B)$ is nonempty and compact. To show that $\omega(B)$ is invariant, suppose $x \in \omega(B)$. Then there exists sequences $\{t_n\}$ and $\{x_n\}$, $t_n \to \infty$ and $x_n \in B$, such that $S(t_n)x_n \to x$. Therefore,

$$S(t)S(t_n)x_n = S(t + t_n)x_n \to S(t)x$$

since $S(t)$ is continuous. So $S(t)\omega(B) \subset \omega(B)$. Conversely, for $t_n \geq t + t_0$, we have

$$S(t_n - t)x_n \in \overline{\bigcup_{s \geq t} S(s)B}$$

so that there is a convergent subsequences $\{t_{n_j}\}$ and $\{x_{n_j}\}$ such that $S(t_{n_j} - t)x_{n_j} \to y$ for some y, and so $y \in \omega(B)$. But since $S(t)$ is continuous,

$$x = \lim_{j \to \infty} S(t)S(t_{n_j} - t)x_{n_j} = S(t)y$$

so $\omega(B) \subset S(t)\omega(B) \Rightarrow S(t)\omega(B) = \omega(B)\ \forall\, t \geq 0$, and $\omega(B)$ is invariant. We know now that, for $t \geq t_0(B)$, $\overline{\bigcup_{s \geq t} S(s)B}$ is a compact subset of $\omega(B)$. It is clear that $\omega(B)$ is the maximal compact invariant set, since if Y is compact and invariant, we know that Y is bounded, and hence that there exists a $t_0(Y)$ such that for all $t \geq t_0(Y)$, we have $Y = S(t)Y \subset \omega(B)$.

It remains only to show that $\omega(B)$ attracts all bounded sets. Suppose for a contradiction that it does not. Then there is a bounded set X, a $\delta > 0$, and a sequence $\{t_n\}$, $t_n \to \infty$, such that

$$dist(S(t_n)X, \omega(B)) \geq \delta \text{ for all } n$$

Then there are $x_n \in X$ such that

$$dist(S(t_n)x_n, \omega(B)) \geq \frac{\delta}{2} \text{ for all } n$$

Since X is bounded, $S(t_n)x_n \in B$ for large n. But B is compact, so there is a subsequence with $S(t_{n_j})x_{n_j} \to \beta \in B$ and $dist(\beta, \omega(B)) \geq \frac{\delta}{2}$. But

$$\beta = \lim_{j \to \infty} S(t_{n_j})x_{n_j} = \lim_{j \to \infty} S(t_{n_j} - t_0(X))S(t_0(X))x_{n_j}$$

Set $\beta_j = S(t_0(X))x_{n_j}$. Then $\beta_j \in B \Rightarrow \beta \in \omega(B)$, which is a contradiction. Therefore, $\omega(B)$ attracts all bounded sets. It also minimal, since it must attract itself, and it is invariant. Therefore, $\omega(B) = \mathcal{A}$. □

We are now ready to prove the existence of the global attractor for the 2D Navier-Stokes Equations with periodic boundary conditions:

Theorem 2.8. *If $f \in H$, then there exists a global attractor $\mathcal{A}$ for the system $(H, \{S(t)\}_{t \geq 0})$.*

Proof. Since $V \subset\subset H$, it suffices to show that there is an absorbing set in V. In other words, there exists a time $t_1(\|u_0\|_H)$, a ρ_V, and an I_A such that

$$\|u(t)\|_V \leq \rho_V \quad \text{and} \quad \int_t^{t+1} \|Au(s)\|_H^2 \, ds \leq I_A$$

for all $t \geq t_1(\|u_0\|_H)$. Indeed, by taking the inner product of the weak form of the Navier-Stokes equation with Au (noting that $b(u, u, Au) = 0$), we get

$$\frac{d}{dt} \|u\|_V^2 + \nu \|Au\|_H^2 \leq \frac{\|f\|_H^2}{\nu}$$

Neglecting the second term

$$\frac{d}{dt} \|u\|_V^2 \leq \frac{\|f\|_H^2}{\nu}$$

Integrating both sides between s and t, with $t-1 \leq s \leq t$, gives us

$$\|u(t)\|_V^2 \leq \|u(s)\|_V^2 + \frac{\|f\|_H^2}{\nu}$$

We now integrate from $s = t-1$ to $s = t$ to obtain

$$\|u(t)\|_V^2 \leq \int_{t-1}^{t} \|u(s)\|_V^2 \, ds + \frac{\|f\|_H^2}{\nu}$$

We now want to bound

$$\int_t^{t+1} \|u(s)\|_V^2 \, ds$$

To do this, take the inner product of the weak form of the Navier-Stokes equation with u to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 + \nu \|u\|_V^2 + b(u,u,u) = \langle f, u \rangle$$

But $b(u,u,u) = 0$, so we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 + \nu \|u\|_V^2 \leq \|f\|_{V^*} \|u\|_V$$

Applying Young's inequality, we get

$$\frac{d}{dt}\|u\|_H^2 + \nu \|u\|_V^2 \leq \frac{\|f\|_{V^*}^2}{\nu} \tag{2.9}$$

Finally, integrating between t and $t+1$ gives

$$\nu \int_t^{t+1} \|u(s)\|_V^2 \, ds \leq \frac{\|f\|_{V^*}^2}{\nu} + \|u(t)\|_H^2$$

So we now have to bound $\|u(t)\|_H^2$. To do this, use Poincaré's inequality to obtain $\|u\|_V^2 \geq \lambda_1 \|u\|_H^2$, (where λ_1 is defined as in Section 1). Applying this to (2.9), we get

$$\frac{d}{dt}\|u\|_H^2 + \nu\lambda_1 \|u\|_H^2 \leq \frac{\|f\|_{V^*}^2}{\nu}$$

Applying Gronwall's lemma, we get

$$\|u(t)\|_H^2 \leq \|u_0\|_H^2 \, e^{-\nu\lambda_1 t} + \frac{\|f\|_{V^*}^2}{\nu^2\lambda_1}(1 - e^{-\nu\lambda_1 t)}$$

So if

$$t_0(\|u_0\|_H) \equiv \max\Bigg(-\frac{1}{\nu\lambda_1} ln\bigg(\frac{\|f\|_{V^*}^2}{\nu^2\lambda_1 \|u_0\|_H}\bigg), 0\Bigg)$$

then for all $t \geq t_0$, we have

$$\|u(t)\|_H^2 \leq 2\frac{\|f\|_{V^*}^2}{\nu^2\lambda_1} \tag{2.10}$$

Which therefore implies that

$$\int_t^{t+1} \|u(s)\|_V^2 \, ds \leq \frac{\|f\|_{V^*}^2}{\nu^2} + 2\frac{\|f\|_{V^*}^2}{\nu^3\lambda_1} \tag{2.11}$$

Therefore if $t \geq t_1(\|u_0\|_H) \equiv t_0(\|u_0\|_H) + 1$, then we have

$$\|u(t)\|_V^2 \leq \rho_V \equiv \frac{\|f\|_{V^*}^2}{\nu^2} + 2\frac{\|f\|_{V^*}^2}{\nu^3\lambda_1} + \frac{\|f\|_H^2}{\nu}$$

To obtain the integral bound, integrate the expression

$$\frac{d}{dt}\|u\|_V^2 + \nu\|Au\|_H^2 \leq \frac{\|f\|_H^2}{\nu}$$

between t and $t+1$ to get

$$\int_t^{t+1}\|Au(s)\|_H^2\,ds \leq I_A \equiv \frac{\|f\|_H^2}{\nu^2} + \frac{\rho_V^2}{\nu}$$

So V has a bounded absorbing set. But since $V \subset\subset H$, this means that H has a compact absorbing set, and that therefore $(H, \{S(t)_{t\geq 0})$ has a global attractor $\mathcal{A}$. $\square$

Furthermore, one can show that the 2D Navier-Stokes equations have the injective property, so that if $u_0, v_0 \in \mathcal{A}$, and $S(T)u_0 = S(T)v_0$ for some $T > 0$, then we must have $u_o = v_0$. The proof is straightforward from the estimates we have made thus far, and the interested reader can refer to Robinson [11]. A corollary of this is that, on $\mathcal{A}$, solutions of the Navier-Stokes Equations are unique backwards in time as well as forwards in time, so that we can generate a dynamical system $(\mathcal{A}, \{S(t)\}_{t\in\mathbb{R}})$. Surprisingly, $\mathcal{A}$ is finite dimensional (which we will prove in the next section), and as we will see in Section 4, this has profound implications for the dynamics of $(\mathcal{A}, \{S(t)\}_{t\in\mathbb{R}})$

3. Dimension of the Global Attractor

We proceed by showing that the global attractor $\mathcal{A}$, which lies in the infinite dimensional phase space $H = \{u \in [L^2(Q)]^2 : \nabla \cdot u = 0 \text{ and } \int_Q u = 0\}$, is nonetheless finite dimensional. Before we can prove this, however, we will need to define some relevant concepts.

Definition 3.1. The semigroup $S(t)$ is uniformly differentiable on $\mathcal{A}$ if for each $u \in \mathcal{A}$, there exists a linear operator $\Lambda(t,u)$ such that, for all $t \geq 0$,

$$\sup_{u,v\in\mathcal{A};0<\|u-v\|_H\leq\epsilon} \frac{\|S(t)v - S(t)u - \Lambda(t,u)(v-u)\|_H}{\|v-u\|_H} \to 0 \text{ as } \epsilon \to 0$$

and

$$\sup_{u\in\mathcal{A}}\|\Lambda(t,u)\|_{op} < \infty \text{ for all } t \geq 0$$

If we know that our semigroup is uniformly differentiable, we can linearise our differential equation to obtain

$$\frac{dU}{dt} = L(t;u_0)U(t) \;\; ; \; U(0) = \xi$$

where $L(t,u_0) = L(u(t))$ where in the case of the 2D Navier-Stokes Equations we will see that L is the linear operator such that

$$L(u)w = \nu Aw - B(w,u) - B(u,w)$$

for all $w \in H$. We can define another important concept:

Definition 3.2. If $\{\delta x^{(i)}\}$ is an orthogonal set of infintesimal displacements from an initial position $u_0 \in \mathcal{A}$ and $\{\phi^{(i)}\}$ a set of orthonormal vectors in the directions of the $\delta x^{(i)}$,

$$\mathcal{TR}_n(\mathcal{A}) \equiv \sup_{x_0 \in \mathcal{A}} \sup_{P^{(n)}(0)} \limsup_{t \to \infty} \frac{1}{t} \int_0^t Tr\big(L(s; u_0) P^{(n)}(s)\big) ds$$

where Tr denotes the trace of a matrix and $P^{(n)}(t)$ denotes the projection of the parallelepiped formed by $\{S(t)\delta x^{(1)}, \dots, S(t)\delta x^{(n)}\}$ onto the space spanned by $\{\phi^{(1)}, \dots, \phi^{(n)}\}$

If we write $\langle \cdot \rangle$ for the time-average operation, this becomes:

$$\mathcal{TR}_n(\mathcal{A}) = \sup_{x_0 \in \mathcal{A}} \sup_{P^{(n)}(0)} \big\langle Tr(L(t; u_0) P^{(n)}(t) \big\rangle$$

Thus, $\mathcal{TR}_n(\mathcal{A})$ represents the maximum possible asymptotic growth of an infintesimal n-volume over all initial positions $u_0 \in \mathcal{A}$. Therefore, if $\mathcal{TR}_n(\mathcal{A}) < 0$, then our infintesimal n-volume necessarily decays to nothing in $\mathcal{A}$. [11] It is then reasonable to conjecture that $\mathcal{A}$ contains no n-dimensional subsets and thus that the fractal dimension of $\mathcal{A}$ must be no larger than n. This is indeed true, but the proof (attributable to Hunt) is quite involved, and a proof will not be given.

Lemma 3.3. *(Hunt) If $S(t)$ is uniformly differentiable on $\mathcal{A}$ and there exists a t_0 such that $\Lambda(t, u_0)$ is compact for all $t \geq t_0$, and also $\mathcal{TR}_n(\mathcal{A}) < 0$, then $d_f(\mathcal{A}) \leq n$.*

Before proceeding to bound the dimension of $\mathcal{A}$, we will need two additional lemmas:

Lemma 3.4. *Let P_n be a rank n orthogonal projection in $L^2(Q)$ (Q the periodic domain $[0, L]^2 \in \mathbb{R}^2$). Then $Tr(-\Delta P_n) \geq Cn^2$.*

Proof. Write $A = -\Delta$, and let w_j be its orthonormal eigenfunctions, with eigenvalues λ_j ordered as $\lambda_{j+1} \geq \lambda_j$. If P_n is the projection onto the space spanned by the orthonormal vectors $\{\phi_1, \dots, \phi_n\}$, then

$$Tr(AP_n) = \sum_{j=1}^{n} (\phi_j, A\phi_j)$$

so that

$$\begin{aligned} Tr(AP_n) &= \sum_{j=1}^{n} \sum_{k=1}^{\infty} \lambda_k \left\| (\phi_j, w_k) \right\|_H^2 \\ &= \sum_{k=1}^{\infty} \lambda_k \left(\sum_{j=1}^{n} \left\| (\phi_j, w_k) \right\|_H^2 \right) \end{aligned}$$

But the ϕ_j are normal, so

$$\sum_{j=1}^{n} \sum_{k=1}^{\infty} \left\| (w_k, \phi_j) \right\|_H^2 = n$$

But ϕ_j do not span H, so

$$\sum_{j=1}^{n} \left\| (w_k, \phi_j) \right\|_H^2 \leq 1$$

so that

$$Tr(AP_n) \geq \sum_{j=1}^{n} \lambda_j$$

We now want to bound the eigenvalues λ_j by

$$cj \le \lambda_j \le Cj$$

To do this, note that since the eigenvalues are proportional to the sums of squares of two integers (since we have periodic boundary conditions), we will have reached the eigenvalue $2k^2$ once we have taken k^2 combinations of integers, so $\lambda_{k^2} = Ck^2$, and so if $k^2 < n, (k+1)^2$, then

$$Ck^2 \le \lambda_n \le C(k+1)^2$$

so $k < n^{\frac{1}{2}} < (k+1)$ so that

$$\frac{1}{2}n^{\frac{1}{2}} < k < k+1 < 2n^{\frac{1}{2}}$$

which gives

$$cn \le \lambda_n \le Cn$$

as wanted. But then

$$\sum_{j=1}^{n} \lambda_j \ge c\sum_{j=1}^{n} j \ge cn^2$$

which implies that

$$Tr(-\Delta P_n) \ge cn^2$$

as desired. □

Lemma 3.5. *$S(t)$ for the 2D Navier-Stokes Equations is uniformly differentiable, where $\Lambda(t; u_0)\xi$ is the solution of the equation*

$$\frac{dU}{dt} + \nu AU + B(u, U) + B(U, u) = 0 \;\; ; \; U(0) = \xi \tag{3.6}$$

Furthermore, $\Lambda(t, u_0)$ is compact for all $t > 0$.

Proof. For a proof, see Robinson [11] page 347. □

With these results in hand, we can finally bound the dimension of the global attractor:

Theorem 3.7. *The fractal dimension of the global attractor $\mathcal{A}$ for the 2D Navier-Stokes Equations is finite-dimensional, and in fact*

$$d_f(\mathcal{A}) \le \alpha\Big(\frac{\rho_V}{\nu}\Big)^2 \tag{3.8}$$

where ρ_V is as defined above and α is a constant.

Proof. We have

$$L(u)w = \nu Aw - B(w, u) - B(u, w)$$

by Lemma 3.5, so that

$$\langle P_n L(u)\rangle = \Big\langle \sum_{j=1}^{n}(L(u)\phi_j, \phi_j)\Big\rangle$$

$$-\Big\langle \sum_{j=1}^{n}(-\nu\Delta\phi_j, \phi_j)\Big\rangle - \Big\langle \sum_{j=1}^{n} b(\phi_j, u, \phi_j)\Big\rangle$$

But by (1.8) we therefore have

since ϕ_j are normal in H. Applying Young's inequality to the last term, we get

$$\begin{aligned}\langle P_n L(u)\rangle &\leq -\nu \sum_{j=1}^{n} \langle \|\phi_j\|_V^2 \rangle + \sum_{j=1}^{n} \left\langle \frac{\nu}{2} \|\phi_j\|_V^2 + \frac{k^2}{2\nu} \|u\|_V^2 \right\rangle \\ &= -\frac{\nu}{2} \sum_{j=1}^{n} \langle \|\phi_j\|_V^2 \rangle + \frac{k^2}{2\nu} \sum_{j=1}^{n} \langle \|u\|_V^2 \rangle \\ &= -\frac{\nu}{2} \langle Tr(-\Delta P_n)\rangle + \frac{k^2 n}{2\nu} \langle \|u\|_V^2 \rangle\end{aligned}$$

By Lemma 3.4, we have

$$\langle P_n L(u)\rangle \leq -\frac{c\nu}{2} n^2 + \frac{k^2 n}{2\nu} \langle \|u\|_V^2 \rangle$$

so that the trace is negative for $n > \alpha\left(\frac{\langle \|u\|_V^2 \rangle}{\nu^2}\right)$. But $\langle \|u\|_V \rangle \leq \rho_V$ on $\mathcal{A}$, so our result follows by Lemma 3.3. □

Now that we have bounded the dimension of the global attractor, we can study the implications of this finite-dimensionality for the dynamics of fluid flow. In particular, by parametrizing the global attractor with a finite number of parameters, we can show that, at least asymptotically, the dynamics of fluid flow are determined by a finite number of degrees of freedom.

4. Parametrizing the Global Attractor

Ideally, if we have $d_f(\mathcal{A}) < d$, we would hope to find a d-dimensional invariant smooth manifold that contains $\mathcal{A}$. The desired notion of a manifold, termed inertial manifold, is attributable to Foias:

Definition 4.1. (Inertial Manifold) An inertial manifold $\mathcal{M}$ for the semidynamical system $(H, \{S(t)\}_{t\geq 0})$ is a finite-dimensional Lipschitz manifold that is positively invariant and attracts all trajectories exponentially:

$$dist(S(t)u_0, \mathcal{M}) \leq C(\|u_0\|_H) e^{-kt} \text{ for all } u_0 \in H$$

If we know that there is an inertial manifold for our system, then we can determine the dynamics of the attractor with a finite system of ordinary differential equations. Unfortunately, though the concept of an inertial manifold was created specifically in order to be applied to the Navier-Stokes Equations, the existence of an inertial manifold even for the 2D Navier-Stokes Equations with periodic boundary conditions remains open.

We are not without hope, however. There is no a priori reason to believe that $\mathcal{A}$ should be smooth enough to be imbedded on an inertial manifold, and it should at least be clear that this is a deep problem, but that does not mean that we cannot imbed $\mathcal{A}$ into some appropriate Euclidean space. It is a familiar result that a compact n-manifold can be imbedded in $\mathbb{R}^{2n+1}$. A first guess, then, would be that if $d_f(\mathcal{A}) < d$ for an integer d, then we can inject $\mathcal{A}$ into $\mathbb{R}^{2d+1}$. This turns out to be correct, and will follow from a more general result:

Theorem 4.2. *Let X be a compact subset of a Hilbert space H, $d_f(X) < d$ with d an integer, and let $k \geq 2d+1$. If L_0 is a bounded linear map into $\mathbb{R}^k$, then for any*

$\epsilon > 0$ *there exists a bounded linear map into* $\mathbb{R}^k$*,* $L = L(\epsilon)$*, such that* L *is injective on* X *and* $\|L - L_0\|_{op} \leq \epsilon$*.*

Proof. We will construct an appropriate collection of countable dense subsets of $L(H, \mathbb{R}^k)$, and then take a countable intersection and utilize the Baire Category Theorem. Let $Y \equiv \{v - w : v, w \in X\}$. Y is the image of $X \times X$ under the Lipschitz map that takes $(v, w) \mapsto v - w$, so since the fractal dimension does not increase under the operation of a Lipschitz map, we have $d_f(Y) \leq d_f(X \times X) = 2d_f(X) < 2d$. Now define

$$A_r \equiv \big\{v - w : v, w \in X \text{ and } \|v - w\|_H \geq \frac{1}{r}\big\}$$

and let

$$A_{r,j,n} \equiv \big\{u \in A_r : |(e_j, u)| \geq \frac{1}{n}\big\}$$

where $\{e_j\}$ are an orthonormal basis for H. $A_{r,j,n}$ is compact and $0 \notin A_r$, so we have

$$A_r = \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{r,j,n}$$

Define

$$\mathbb{L}_{r,j,n} \equiv \{L \in L(H, \mathbb{R}^k) : L^{-1}(0) \cap A_{r,j,n} = \emptyset\}$$

Then

$$\bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \mathbb{L}_{r,j,n}$$

consists of maps L for which $L^{-1}(0) \cap A_r = \emptyset$, or for which $diam(L^{-1}(x) \cap X) < \frac{1}{r}$ for all $x \in \mathbb{R}^k$. Therefore,

$$\bigcap_{r=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \mathbb{L}_{r,j,n}$$

consists of maps that are injective on X.

We now want to show that $\mathbb{L}_{r,j,n}$ are open and dense in $L(H, \mathbb{R}^k)$. To show that they are open, let $L \in \mathbb{L}_{r,j,n}$ We want to find an $\epsilon > 0$ such that $\|L - \tilde{L}\|_{op} \leq \epsilon$ implies $\tilde{L} \in \mathbb{L}_{r,j,n}$. But from the definition of $\mathbb{L}_{r,j,n}$, this is just to say that if $Lx \neq 0$ for all $x \in A_{r,j,n}$, then $\tilde{L}x \neq 0$ for all $x \in A_{r,j,n}$. But since $A_{r,j,n}$ is compact, if $Lx \neq 0$ for all $x \in A_{r,j,n}$, then $\min_{x \in A_{r,j,n}} |Lx| = \eta > 0$ and $\max_{x \in A_{r,j,n}} \|x\|_H \leq R$ for some $0 < \eta \leq R$. Then we have

$$\begin{aligned} |\tilde{L}x| &= |Lx - (L - \tilde{L})x| \\ &\geq |Lx| - |(L - \tilde{L})x| \\ &\geq \eta - \|L - \tilde{L}\|_{op} R \end{aligned}$$

so that if we choose $0 < \epsilon \leq \frac{\eta}{2R}$, then $|\tilde{L}x| \neq 0$ for all $x \in A_{r,j,n}$, and it follows that $\mathbb{L}_{r,j,n}$ is open.

To show that $\mathbb{L}_{r,j,n}$ are dense in $L(H, \mathbb{R}^k)$, fix $L_0 \in L(H, \mathbb{R}^k)$ and $\epsilon > 0$. Define a map $\phi : \mathbb{R}^k \to S^{k-1}$, where S^{k-1} is the $k-1$-sphere, as

$$\phi(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ p & \text{if } x = 0 \end{cases}$$

where p is just some point on S^{k-1}. ϕ is Lipschitz on $\mathcal{O}_\rho \equiv \{x : |x| \geq \rho\}$ for all $\rho > 0$. Letting $W \subset \mathbb{R}^k$, we can write

$$\phi(W) = \left\{ \bigcup_{k=1}^{\infty} \phi(W \cap \mathcal{O}_{\frac{1}{k}}) \right\} \underbrace{\bigcup p}_{\text{if } 0 \in W}$$

If we let d_H denote Hausdorff dimension, we know that $d_H(p) = 0$, and so by the countable additivity of Hausdorff dimension,

$$d_H(\phi(W)) \leq \sup_k d_H(\phi(W \cap \mathcal{O}_{\frac{1}{k}}))$$

and since ϕ is Lipschitz on $\mathcal{O}_{\frac{1}{k}}$, we have $d_H(\phi(W)) \leq d_H(W)$. In particular,

$$d_H(\phi(L_0 A_r)) \leq d_H(L_0 A_r) \leq d_H(A_r) \leq d_f(A_r) \leq d_f(Y) < 2d$$

since Hausdorff dimension is bounded above by fractal dimension and the fractal dimension of a subset of Y is no greater than the fractal dimension of Y. But $d_H(S^{k-1}) = k - 1 \geq 2d$, so it follows that $\phi|_{L_0 A_r}$ is not surjective. Choose $z \in S^d$ such that $z \notin \phi(L_0 A_r)$, and set $L \equiv L_0 + \epsilon z e_j^*$, where e_j^* is the linear functional that takes $u \mapsto (u, e_j)$. $L \in L(H, \mathbb{R}^k)$, and also $\|L - L_0\|_{op} \leq \epsilon$. It remains only to show that $L \in \mathbb{L}_{r,j,n}$. Suppose for a contradiction that it is not. Then there exists a $u \in A_{r,j,n}$ with $Lu = 0$. Then

$$L_0 u = -(e_j, u)\epsilon z$$

and since $u \in A_{r,j,n}$ implies that $|(e_j, u)| \geq \frac{1}{r} > 0$, we can write

$$z = -((e_j, u)\epsilon)^{-1} L_0 u$$

and hence

$$z = \phi(z) = \phi(L_0 u) \in \phi(L_0 A_r)$$

which is a contradiction. Therefore, $\mathbb{L}_{r,j,n}$ are dense open subsets of $L(H, \mathbb{R}^k)$, and so by the Baire Category Theorem,

$$\bigcap_{r=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \mathbb{L}_{r,j,n}$$

is a dense open subset of $L(H, \mathbb{R}^k)$, but since elements of this set are injective on X the result follows. $\square$

Corollary 4.3. *If $d \in \mathbb{Z}$, $d > \alpha\left(\frac{\rho_V}{\nu}\right)^2$, then there exists a parametrisation of $\mathcal{A}$ using $2d + 1$ coordinates.*

Proof. We have bounded the dimension of $\mathcal{A}$ as above, and we know $\mathcal{A} \subset H$, so it remains only to show that $L^{-1}|_{L\mathcal{A}}$ is continuous for $L \in \bigcap_{r=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \mathbb{L}_{r,j,n}$. Suppose for a contradiction that it is not. Then there exists $\epsilon > 0$ and a sequence $\{x_n\} \in L\mathcal{A}$ with $x_n \to y \in L\mathcal{A}$ with $\left\|L^{-1}x_n - L^{-1}y\right\|_H \geq \epsilon$. But $L^{-1}x_n \in \mathcal{A}$, and since $\mathcal{A}$ is compact there exists a subsequence $\{x_{n_j}\}$ such that $L^{-1}x_{n_j} \to z$ for some $z \in \mathcal{A}$. Since L is continuous, it follows that $x_{n_j} \to Lz = y$. But since L is injective on $\mathcal{A}$, we therefore have $z = L^{-1}y$, which is a contradiction. Therefore, L^{-1} is continuous on $\mathcal{A}$ and the result follows. $\square$

We therefore know that, at least asymptotically (i.e. on the global attractor), we can describe the dynamics of fluid flow (at least in a 2D periodic domain, with $2d+1$ parameters. This is also true for the 3D Navier-Stokes Equations with Dirichlet boundary condition, but the arguments necessary are much more involved, and the ambitious reader is referred to Constantin [2] for details. Unfortunately, these $2d+1$ parameters need not correspond to any known observable physical conditions, so modeling turbulent flows remains a difficult task. This result, however, provides hope that such work is not fruitless.

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