SIMPLICIAL SETS AND VAN KAMPEN'S THEOREM

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Abstract. We present an abstract version of the van Kampen theorem involving fundamental groupoids, and we show that this result holds for both topological spaces and simplicial sets. The goal is to make the basic theory of simplicial sets accessible to undergraduates who have studied algebraic topology. This paper does not assume any familiarity with category theory.

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1. Introduction

A simplicial set is a construction in algebraic topology that models a well behaved topological space. The notion of a simplicial set arises from the notion of a simplicial complex and has some nice formal properties that make it ideal for studying topology. Simplicial sets are useful because they are entirely algebraic constructions and they make it possible to do topology indirectly, using only algebra. In this paper we illustrate the use of simplicial sets in algebraic topology by proving an abstract version of the van Kampen theorem, first for topological spaces and then for simplicial sets.

We assume that the reader has studied basic algebraic topology, including simplicial complexes, homotopy, the fundamental group, and van Kampen's theorem for fundamental groups. On the other hand, we do not assume any familiarity with category theory. This paper could well serve as an introduction to category theory in the context of algebraic topology.

In the next section we introduce some notions from category theory that we use throughout our discussion. Then we introduce simplicial sets as a generalization of simplicial complexes, and we rewrite the definition using category theory. We briefly

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discuss the concept of universality in mathematics, and then describe a method for constructing a topological space from a given simplicial set. Following this, we prove an abstract version of the van Kampen theorem for topological spaces. In the final two sections, we introduce the important idea of an adjunction and show that a version of van Kampen's theorem holds for simplicial sets.

2. Basic Category Theory

In this section we introduce some of the most important definitions from category theory, and we illustrate the definitions with examples from topology and algebra. Category theory is a valuable tool for studying abstract mathematical entities and the relationships between them. It is indispensable in any serious discussion of simplicial sets.

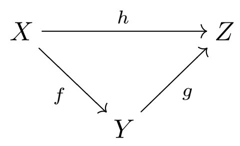
Definition 1. A *category* $\mathcal{C}$ consists of the following.

(1) A class $|\mathcal{C}|$, whose elements are called the *objects* of the category.
(2) For every pair X, Y of objects, a set $\mathcal{C}(X,Y)$, whose elements are called the *arrows* from X to Y in $\mathcal{C}$.
(3) For every triple X, Y, Z of objects, a binary operation
$$\mathcal{C}(X,Y)\times\mathcal{C}(Y,Z)\to\mathcal{C}(X,Z)$$
called *composition*, sending arrows f and g to their *composite* $g\circ f$.
(4) For every object X, an arrow $1_X\in\mathcal{C}(X,X)$ called the *identity arrow* on X.

These data must satisfy the following axioms.

(1) Composition of arrows is associative:
$$h\circ(g\circ f)=(h\circ g)\circ f$$
whenever either side is defined.
(2) An identity arrow is a two-sided unit for composition:
$$f\circ 1_X=f=1_Y\circ f$$
for $f\in(X,Y)$.

In this paper we usually use the notation $f:X\to Y$ in place of $f\in\mathcal{C}(X,Y)$. The object X is called the *domain* of f, and Y is called the *codomain* of f. We often display arrows in diagrams such as



for arrows $f:X\to Y$, $g:Y\to Z$, and $h:X\to Z$. We say that such a diagram *commutes* if any two composites sending one given object to another are equal. For example, we say that the above diagram commutes if $h=g\circ f$.

It is easy to see that the topological spaces form a category **Top** with arrows the continuous mappings between spaces. The composite of arrows f and g is defined in **Top** by $x\mapsto g(f(x))$ where x is any point in the domain of f. The identity arrow on a space X is simply the identity mapping $1_X:X\to X$, $x\mapsto x$. A similar example is the category $\mathbf{Top}_*$ of based topological spaces. The objects of this category are pairs (X, x) where X is a topological space and x is a point of X called the *basepoint* of X. An arrow from (X, x) to (Y, y) in $\mathbf{Top}_*$ is a continuous mapping $X \to Y$ that takes the basepoint x to the basepoint y. Composition and identities are the same as in **Top**.

There are also many examples of categories in algebra. The collection of all groups is a category called **Grp** with arrows the homomorphisms of groups. Composition of arrows is again just composition of functions, and the identity on a group G is simply the identity homomorphism $1_G : G \to G$, $g \mapsto g$.

The next definition describes mappings between categories.

Definition 2. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of the following.

(1) A map sending each object X of $\mathcal{C}$ to an object $F(X)$ of $\mathcal{D}$.
(2) A map sending each arrow $f : X \to Y$ in $\mathcal{C}$ to an arrow $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$.

These data must satisfy the following axioms.

(1) F preserves composition:
$$F(g \circ f) = F(g) \circ F(f)$$
whenever the composite arrow $g \circ f$ is defined.
(2) F preserves identities:
$$F(1_X) = 1_{F(X)}$$
for every object X of $\mathcal{C}$.

More precisely, this is the definition of a *covariant functor* F from $\mathcal{C}$ to $\mathcal{D}$. There are also *contravariant functors* which are defined in the same way, except that F sends each arrow $f : X \to Y$ in $\mathcal{C}$ to an arrow $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$, satisfying $F(g \circ f) = F(f) \circ F(g)$. Contravariant functors will be useful later on when we define simplicial sets.

The notion of a functor is especially useful in algebraic topology. For example, we can define a functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ sending each based topological space (X, x) to the fundamental group $\pi_1(X, x)$ of X at the basepoint $x \in X$, and sending each arrow $f : (X, x) \to (Y, y)$ of $\mathbf{Top}_*$ to the induced homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$. The functor concept thus allows us to map between categories, respecting their structure. Sometimes we also need to define maps between functors.

Definition 3. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \to G$ from F to G consists of an arrow $\alpha_X : F(X) \to G(X)$ for each object X of $\mathcal{C}$ such that the following diagram commutes for every arrow $f : X \to Y$ in $\mathcal{C}$.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

The arrows α_X are called the *components* of the natural transformation α.

If we are given two categories $\mathcal{C}$ and $\mathcal{D}$, this definition allows us to regard the collection of functors $\mathcal{C} \to \mathcal{D}$ as a category whose arrows are the natural transformations between functors. Constructions of this sort will appear throughout our discussion of simplicial sets.

3. Simplicial Sets

Recall that a *simplicial complex* is a set V, whose elements are called *vertices*, together with a set X of nonempty finite subsets of V, whose elements are called *simplices*, such that every vertex is an element of some simplex and every nonempty subset of a simplex is again a simplex. In topology it is often important to define an order on the vertices of each simplex in a simplicial complex. We define an *oriented n-simplex* to be an $(n+1)$-tuple $(x_0, \dots, x_n)$ whose coordinates form a simplex, and we write X_n for the set of all oriented n-simplices. Then there are functions $\partial_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, called the *face operators* on X_n, defined by

$$\partial_i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

For example, if (x_0, x_1, x_2) is an oriented 2-simplex, then we have $\partial_0(x_0, x_1, x_2) = (x_1, x_2)$, $\partial_1(x_0, x_1, x_2) = (x_0, x_2)$, and $\partial_2(x_0, x_1, x_2) = (x_0, x_1)$. In other words, the faces of a triangle are oriented 1-simplices. We also have the *degeneracy operators* $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$, defined by

$$s_i(x_0, \dots, x_n) = (x_0, \dots, x_i, x_i, x_{i+1}, \dots, x_n).$$

It is easy to check that these functions satisfy the following identities.

(1) $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ if $i < j$.
(2) $s_i \circ s_j = s_{j+1} \circ s_i$ if $i \leq j$.
(3) $\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \partial_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ \partial_{i-1} & \text{if } i > j+1. \end{cases}$

The notion of a simplicial set is a generalization of the notion of a complex with face and degeneracy operators.

Definition 4. A *simplicial set* X is a collection of sets $X_0, X_1, X_2, \dots$ together with maps $\partial_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$, which satisfy the identities (1–3) above. An element of X_n is called an *n-simplex*, and the maps ∂_i and s_i are called *face operators* and *degeneracy operators*, respectively.

The notion of a simplicial set provides a purely algebraic model for studying topological spaces. Before we can use this definition, however, we must reformulate it in the language of category theory. Let $\mathbf{\Delta}$ be a category with objects the ordered sets $\mathbf{n} = \{0, 1, \dots, n\}$ and arrows the order-preserving maps between them. For example, the map $0 \mapsto 0$, $1 \mapsto 2$ and the map $0 \mapsto 1$, $1 \mapsto 1$ are both arrows in $\mathbf{\Delta}(\mathbf{1}, \mathbf{2})$. We consider the arrows $\delta_i : (\mathbf{n-1}) \to \mathbf{n}$ and $\sigma_i : (\mathbf{n+1}) \to \mathbf{n}$, $0 \leq i \leq n$, of $\mathbf{\Delta}$ defined by

$$\delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

and

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

These arrows have the property that if $f : \mathbf{m} \to \mathbf{n}$ is any arrow in $\mathbf{\Delta}$ then f can be written as a composite of the δ_i and the σ_i. To see this, suppose $i_1, \dots, i_s$, in decreasing order, are the elements of $\mathbf{n}$ not in the image $f(\mathbf{m})$ and suppose $j_1, \dots, j_t$, in increasing order, are the elements of $\mathbf{m}$ such that $f(j) = f(j+1)$. Then

$$f = \delta_{i_1} \circ \cdots \circ \delta_{i_s} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_t}$$

is a factorization of f. We express this property of the δ_i and σ_i by saying that these arrows *generate* the category $\mathbf{\Delta}$. One can also check that δ_i and σ_i satisfy the identities

(1) $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$ if $i < j$
(2) $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$ if $i \leq j$
(3) $\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \circ \sigma_j & \text{if } i > j+1 \end{cases}$

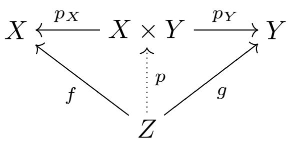
which are dual to the identities occurring in the definition of a simplicial set. Now, if X is a *contravariant* functor $\mathbf{\Delta} \to \mathbf{Set}$, where $\mathbf{Set}$ is the category of sets and functions between them, and if we define $\partial_i = X(\delta_i)$ and $s_i = X(\sigma_i)$ and write $X_n = X(\mathbf{n})$ for every object $\mathbf{n}$ of $\mathbf{\Delta}$, then we obtain a simplicial set by the previous definition. This discussion gives us the following useful characterization of a simplicial set.

Definition 5. A *simplicial set* is a contravariant functor $X : \mathbf{\Delta} \to \mathbf{Set}$.

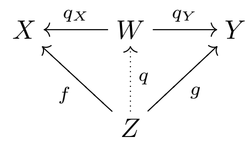
It is now clear that the simplicial sets form a category, which we denote **sSet**, with arrows the natural transformations between simplicial sets.

4. Universal Cones

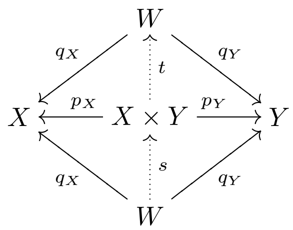
Many constructions in algebra and topology are "universal" in the sense that they construct something with a desirable property, and any other object with the property gets mapped to this one by a unique morphism. Consider the direct product of groups X and Y. We can describe it as a group $X \times Y$, whose elements are the pairs (x, y) with $x \in X$ and $y \in Y$, together with "projection" homomorphisms $p_X : X \times Y \to X$, $(x, y) \mapsto x$ and $p_Y : X \times Y \to Y$, $(x, y) \mapsto y$. If Z is any group equipped with homomorphisms $f : Z \to X$ and $g : Z \to Y$, then the map $p : Z \to X \times Y$ defined by the rule $p(x) = (f(x), g(x))$ is the unique homomorphism such that the following diagram commutes.



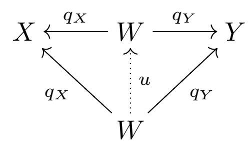
In fact, this description determines the direct product up to a unique isomorphism. To see this, let W be a group with projection homomorphisms $q_X : W \to X$ and $q_Y : W \to Y$ and suppose that if $f : Z \to X$ and $g : Z \to Y$ are any homomorphisms then there is a unique homomorphism q making



commute. Then, in particular, the diagram

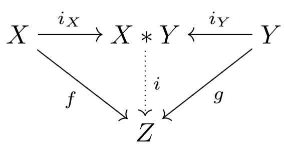


commutes for unique homomorphisms s and t. Now our hypothesis says that there is a unique homomorphism u such that



commutes, and the identity obviously has this property. Therefore we must have $t \circ s = u = 1_W$. Interchanging the roles of W and $X \times Y$ in this argument, we conclude also that $s \circ t = 1_{X \times Y}$. This proves that the homomorphism $s : W \to X \times Y$ has an inverse $t : X \times Y \to W$, and hence s is a unique isomorphism from W to $X \times Y$ commuting with the projections. This observation that the direct product is unique up to isomorphism means we can *define* the direct product of X and Y to be a group $X \times Y$ together with homomorphisms $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ satisfying the property that if $f : Z \to X$ and $g : Z \to Y$ are any homomorphisms then there is a unique homomorphism p making our original diagram commute.

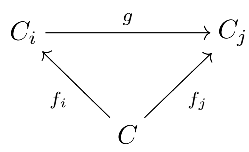
The free product of X and Y is another example of a "universal" construction in the category of groups. We can describe the free product of X and Y as a group $X * Y$, whose elements are formal combinations of the form $x_1 y_1 x_2 y_2 \cdots x_n y_n$ with $x_i \in X$ and $y_i \in Y$, together with "inclusion" homomorphisms $i_X : X \to X * Y$, $x \mapsto x 1_Y$ and $i_Y : Y \to X * Y$, $y \mapsto 1_X y$. If Z is any group equipped with homomorphisms $f : X \to Z$ and $g : Y \to Z$, then the map $i : X * Y \to Z$ defined by the rule $i(x_1 y_1 x_2 y_2 \cdots x_n y_n) = f(x_1) g(y_1) \ldots f(x_n) g(y_n)$ is the unique homomorphism such that the following diagram commutes.



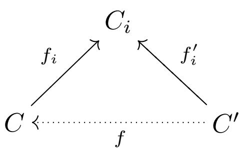
Using an argument like the one we used in our discussion of the direct product, one can show that this description determines the free product up to a unique isomorphism. Thus we can *define* the free product to be a group $X * Y$, together with homomorphisms $i_X : X \to X * Y$ and $i_Y : Y \to X * Y$ satisfying the above property. Notice that the diagram is the same as in the definition of a direct product, with the arrows reversed.

To make the notion of a universal construction more precise, we introduce the formal definitions of diagrams and cones. Intuitively, a diagram in a category $\mathcal{C}$ is exactly what one would expect: a collection of objects C_i of $\mathcal{C}$ together with arrows $g : C_i \to C_j$ between certain objects in the diagram. Formally, a *$\mathcal{D}$-shaped diagram* in $\mathcal{C}$ is a functor $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is a small category (a category for which $|\mathcal{D}|$ is a set, rather than a proper class). Thus, an object C_i in a $\mathcal{D}$-shaped diagram is the image of some object D_i of $\mathcal{D}$ under the functor F, and an arrow in the $\mathcal{D}$-shaped diagram is the image of some arrow in $\mathcal{D}$ under the functor F.

We define a *cone* for a $\mathcal{D}$-shaped diagram F to be an object C of $\mathcal{C}$ together with an arrow $f_i : C \to C_i$ for every object C_i in the $\mathcal{D}$-shaped diagram such that

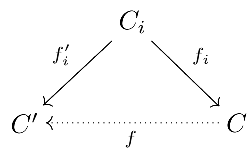


commutes whenever $g : C_i \to C_j$ is an arrow in the $\mathcal{D}$-shaped diagram. We denote such a cone by $\{f_i : C \to C_i\}$. A cone $\{f_i : C \to C_i\}$ is said to be *universal* if, given any other cone $\{f'_i : C' \to C_i\}$, there exists a unique arrow $f : C' \to C$ making



commute for every object C_i in the $\mathcal{D}$-shaped diagram. In this case, we call C a *limit* of the $\mathcal{D}$-shaped diagram F, and we write $C \cong \lim F$.

Dually, we can define a *cocone* $\{f_i : C_i \to C\}$ for a $\mathcal{D}$-shaped diagram F to be an object C of $\mathcal{C}$ together with an arrow $f_i : C_i \to C$ for every object C_i in the $\mathcal{D}$-shaped diagram such that all of the resulting triangles commute. A cocone is *universal* if, given any other cocone $\{f'_i : C_i \to C'\}$, there exists a unique arrow $f : C \to C'$ making



commute for every object C_i in the $\mathcal{D}$-shaped diagram. This time we call C a *colimit* of the $\mathcal{D}$-shaped diagram F, and we write $C \cong \operatorname{colim} F$.

If we now let $\mathcal{D}$ be a two-object category whose only arrows are identities and let $\mathcal{C}$ be the category of groups, then we can choose a functor $F : \mathcal{D} \to \mathcal{C}$ sending one of the the objects of $\mathcal{D}$ to a group X and sending the other object of $\mathcal{D}$ to a group Y. Then a limit of the $\mathcal{D}$-shaped diagram F is the group $X \times Y$, while a colimit of F is the group $X * Y$.

5. Geometric Realization

In Section 3 we began with the notion of a simplicial complex and by abstraction arrived at the purely algebraic notion of a simplicial set. In this section we use colimits to go the other direction, constructing topological spaces from given simplicial sets. Recall that the *standard topological n-simplex* is defined to be the subspace

$$\Delta_n = \{(t_0, \dots, t_n) : 0 \leq t_i \leq 1, \sum t_i = 1\}$$

of $\mathbb{R}^{n+1}$ with the usual subspace topology. For example, Δ_0 is the standard basis vector $\mathbf{e}_0$, Δ_1 is the line segment with endpoints $\mathbf{e}_0$ and $\mathbf{e}_1$, and Δ_2 a triangle with vertices $\mathbf{e}_0$, $\mathbf{e}_1$, and $\mathbf{e}_2$. Define maps

$$\delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

and

$$\sigma_i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1}).$$

for $0 \leq i \leq n$. We can think of the maps δ_i as "including" the standard topological $(n-1)$-simplex as faces of the standard n-simplex, and we can think of the maps σ_i as "collapsing" the standard $(n+1)$-simplex to give an n-simplex. The *geometric realization* of a simplicial set X is a topological space $|X|$ defined as follows. As a set we have

$$|X| = \left(\coprod_{n \geq 0} X_n \times \Delta_n \right) / \sim,$$

where $\sim$ is the equivalence relation generated by

$$(x, \delta_i(u)) \sim (\partial_i(x), u)$$

and

$$(x, \sigma_i(v)) \sim (s_i(x), v)$$

where u and v are points in Δ_{n-1} and Δ_{n+1}, respectively, and x is an n-simplex. We obtain a topology for $|X|$ by giving

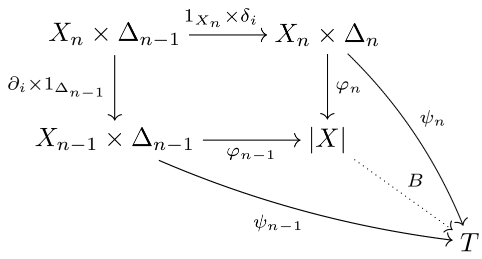
$$\left(\coprod_{0 \leq n \leq q} X_n \times \Delta_n \right) / \sim$$

the quotient topology for each q and taking the union of the resulting topologies. In other words, the geometric realization of a simplicial set X is the topological space obtained by associating, to each element of the set X_n, the standard topological n-simplex Δ_n, and then "gluing" these simplices at specific points determined by the face and degeneracy operators.

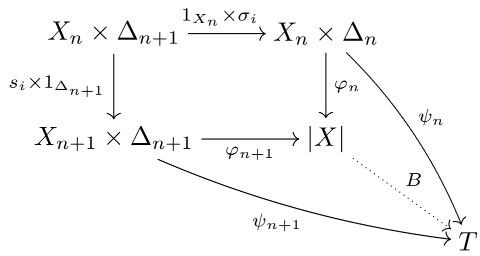
Now let φ_n be the map $X_n \times \Delta_n \to |X|$ induced by the inclusion of $X_n \times \Delta_n$ into $\coprod X_n \times \Delta_n$. Our definition of geometric realization says that the diagrams

$$\begin{array}{ccc} X_n \times \Delta_{n-1} & \xrightarrow{1_{X_n} \times \delta_i} & X_n \times \Delta_n \\ {\scriptstyle \partial_i \times 1_{\Delta_{n-1}}} \downarrow & & \downarrow {\scriptstyle \varphi_n} \\ X_{n-1} \times \Delta_{n-1} & \xrightarrow[\varphi_{n-1}]{} & |X| \end{array} \qquad \begin{array}{ccc} X_n \times \Delta_{n+1} & \xrightarrow{1_{X_n} \times \sigma_i} & X_n \times \Delta_n \\ {\scriptstyle s_i \times 1_{\Delta_{n+1}}} \downarrow & & \downarrow {\scriptstyle \varphi_n} \\ X_{n+1} \times \Delta_{n+1} & \xrightarrow[\varphi_{n+1}]{} & |X| \end{array}$$

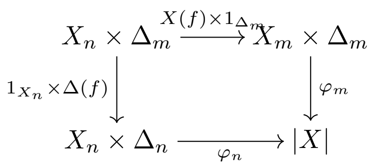
commute for every nonnegative integer n and $0 \leq i \leq n$, and they have the property that if $\{\psi_n : X_n \times \Delta_n \to T\}$ is any collection of maps to a topological space T such that the outer parts of the following diagrams commute, then there exists a unique continuous function $B : |X| \to T$ such that



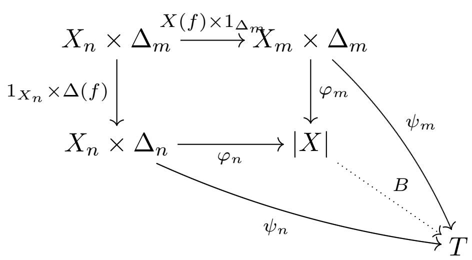
and



commute for all n. The function B simply maps each point (x, v) in $|X|$ with $x \in X_n$ and $v \in \Delta_n$ to the point $\psi_n(x, v)$ in T. This is well defined since the outer squares in the above diagrams commute. Now every arrow $f : \mathbf{m} \to \mathbf{n}$ is a product of the generators δ_i and σ_i described above for the category $\mathbf{\Delta}$ and thus induces a map $\Delta(f) : \Delta_m \to \Delta_n$ which is a product of the δ_i and the σ_i as defined in this section. Thus we could also say that $|X|$ is the unique topological space such that



commutes for every arrow $f : \mathbf{m} \to \mathbf{n}$, and, given maps $\psi_n : X_n \times \Delta_n \to T$ for each n making the outer parts of the following diagram commute, there is a unique continuous function B making



commute for every $f : \mathbf{m} \to \mathbf{n}$. In fact, this is equivalent to saying that $|X|$ is the colimit of the diagram which includes, for every $f : \mathbf{m} \to \mathbf{n}$,

$$\begin{array}{ccc} X_n \times \Delta_m & \xrightarrow{X(f)\times 1_{\Delta_m}} & X_m \times \Delta_m \\ {\scriptstyle 1_{X_n}\times\Delta(f)}\downarrow & & \\ X_n \times \Delta_n. & & \end{array}$$

To see this, recall that a cocone for this diagram consists of maps into $|X|$ such that the following diagram

$$\begin{array}{ccc} X_n \times \Delta_m & \xrightarrow{X(f)\times 1_{\Delta_m}} & X_m \times \Delta_m \\ {\scriptstyle 1_{X_n}\times\Delta(f)}\downarrow & \searrow & \downarrow \\ X_n \times \Delta_n & \longrightarrow & |X| \end{array}$$

commutes for every $f : \mathbf{m} \to \mathbf{n}$. Since the diagonal arrow is the composite of the sides of this square, we can ignore it and think of the cocone as consisting of just the vertical and horizontal arrows into $|X|$. Thus, the universal property just given for $|X|$ says that $|X|$ is a colimit. This characterization of the geometric realization will be useful in the next section.

6. Fundamental Groupoids

Although the fundamental group is extremely useful for studying topological spaces with chosen basepoints, it is often useful to forget about basepoints and work in the category **Top** of unbased topological spaces. Suppose we are given a topological space X. We can form the category $\Pi(X)$ whose objects are the points of X and whose arrows $[f] : x \to y$ are the homotopy classes of paths from x to y. Composition of arrows $[f]$ and $[g]$ in $\Pi(X)$ is given by $[g] \circ [f] = [g \cdot f]$, and the identity arrow on an object x of $\Pi(X)$ is the homotopy class $[c_x]$ of the constant path on x. To every arrow $[f]$ in the category $\Pi(X)$ there is an inverse arrow $[f^{-1}]$ such that the composites of these arrows are $[f^{-1}] \circ [f] = [c_x]$ and $[f] \circ [f^{-1}] = [c_y]$. In general, a category in which every arrow is invertible is called a *groupoid*. The category $\Pi(X)$ is called the *fundamental groupoid* of the space X. Notice that the set of arrows from x to x in $\Pi(X)$ is the fundamental group of X at the basepoint $x \in X$.

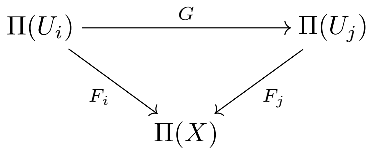
Now if **Gpd** denotes the category of groupoids and functors between them, then we can define a functor $\Pi : \mathbf{Top} \to \mathbf{Gpd}$ sending each space X to the fundamental groupoid $\Pi(X)$, and sending continuous maps $X \to Y$ to functors $\Pi(X) \to \Pi(Y)$. We shall call the functor Π the *fundamental groupoid functor.* This is clearly analogous to the fundamental group functor defined previously.

To illustrate how algebraic topologists use the fundamental groupoid we shall now state and prove a modern version of the classical van Kampen theorem. Let X be any topological space and consider a cover $\mathcal{O} = \{U_i\}$ of X in which each U_i is a path connected open subset of X. Suppose further that the intersection of finitely many of the U_i is again in $\mathcal{O}$. Then we can regard $\mathcal{O}$ as a category whose arrows are the inclusions of sets. We can also regard the fundamental groupoid functor, restricted to the spaces and maps in $\mathcal{O}$, as an $\mathcal{O}$-shaped diagram $\Pi|\mathcal{O} : \mathcal{O} \to \mathbf{Gpd}$.

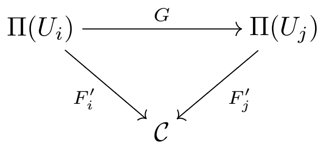
Whenever there is an arrow $U_i \hookrightarrow U_j$ in $\mathcal{O}$ we have exactly one functor $\Pi(U_i) \to \Pi(U_j)$ in the $\mathcal{O}$-shaped diagram $\Pi|\mathcal{O}$.

Theorem 1 (van Kampen)**.** *The fundamental groupoid* $\Pi(X)$ *is a colimit of the* $\mathcal{O}$*-shaped diagram* $\Pi|\mathcal{O}$*. That is,* $\Pi(X) \cong \operatorname{colim} \Pi|\mathcal{O}$*.*

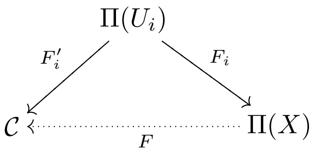
Proof. Observe that the functors $F_i : \Pi(U_i) \to \Pi(X)$ induced by the inclusions $U_i \hookrightarrow X$ form a cocone for the $\mathcal{O}$-shaped diagram $\Pi|\mathcal{O}$ since the following diagram commutes



for every functor $G : \Pi(U_i) \to \Pi(U_j)$ in the $\mathcal{O}$-shaped diagram. We need to show that this cocone is universal. Suppose $\{F'_i : \Pi(U_i) \to \mathcal{C}\}$ is any other cocone for the $\mathcal{O}$-shaped diagram. This means that

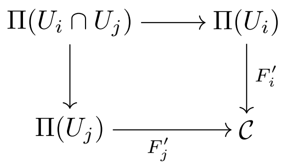


commutes for every functor $G : \Pi(U_i) \to \Pi(U_j)$ in the $\mathcal{O}$-shaped diagram. To say that $\{F_i : \Pi(U_i) \to \Pi(X)\}$ is universal means there is a unique functor $F : \Pi(X) \to \mathcal{C}$ such that



commutes for every i. In other words, F is the unique functor that restricts to F'_i on $\Pi(U_i)$. We need to show that such a functor exists.

For every object of $\Pi(X)$, that is, for every point x of X, define $F(x) = F'_i(x)$ where U_i is any of the open subsets containing x. This definition is independent of the choice of U_i. To see this, suppose $x \in U_i$ *and* $x \in U_j$. Then $x \in U_i \cap U_j \in \mathcal{O}$. Since $U_i \cap U_j \subseteq U_i$ and $U_i \cap U_j \subseteq U_j$, we then have induced maps $\Pi(U_i \cap U_j) \to \Pi(U_i)$ and $\Pi(U_i \cap U_j) \to \Pi(U_j)$ such that the following diagram commutes.



Then $F'_i(x) = F'_j(x)$, so F is well defined on objects. Next we must define F on arrows. If $f : x \to y$ is a path that lies entirely in some U_i, then we define $F([f]) = F'_i([f])$. It is easy to check that this definition is also independent of the choice of U_i if f lies entirely in more than one U_i. Since every path f is the composite of a finite number of paths f_i, each of which lies entirely in some U_i, we must define $F([f])$ to be the composite of all the $F'_i([f_i])$. This will give us a functor $F : \Pi(X) \to \mathcal{C}$ that restricts to F_i' on $\Pi(U_i)$, provided that F is well defined on arrows.

Suppose that f and g are two paths in the same homotopy class in X. Then there is a homotopy $I \times I \to X$ from f to g. We can subdivide the square $I \times I$ into rectangles $[a, b] \times [c, d]$, each of which gets mapped entirely into one of the U_i under the homotopy. Since the square $I \times I$ is compact, there is a finite collection of rectangles in this subdivision, such that every point of $I \times I$ is contained in some rectangle and each rectangle gets mapped entirely into one of the U_i. Moreover, we can choose this subdivision so that the resulting subdivision of $I \times \{0\}$ refines the subdivision used above to decompose f into paths f_i. We can also decompose g into paths g_i, each of which lies entirely in one of the U_i, and we can choose the subdivision of $I \times I$ so that that the resulting subdivision of $I \times \{1\}$ refines the subdivision used to compose g into the g_i. This shows that the relation $[f] = [g]$ is a consequence of a finite number of relations because each rectangle in the subdivision is a homotopy within one of the U_i. Therefore $F([f]) = F([g])$, and the functor F is well defined on arrows. This completes the proof. □

For a proof that the classical van Kampen theorem follows from this result, the reader should consult [3]. In this paper we are interested instead in formulating this more general result for simplicial sets. To do this, we must first define the fundamental groupoid of a simplicial set.

Recall that each object $\mathbf{n}$ of the category $\mathbf{\Delta}$ is an ordered set $\{0, 1, \ldots, n\}$. We can also think of $\mathbf{n}$ as a category consisting of objects $0, 1, \ldots, n$ with arrows

$$0 \to 1 \to \cdots \to n$$

between consecutive objects, plus identities and composites. In Section 5 we defined the geometric realization of a space X to be the colimit of the diagram which includes, for every $f : \mathbf{m} \to \mathbf{n}$,

$$\begin{array}{ccc} X_n \times \Delta_m & \xrightarrow{X(f)\times 1_{\Delta_m}} & X_m \times \Delta_m \\ {\scriptstyle 1_{X_n}\times\Delta(f)}\downarrow & & \\ X_n \times \Delta_n. & & \end{array}$$

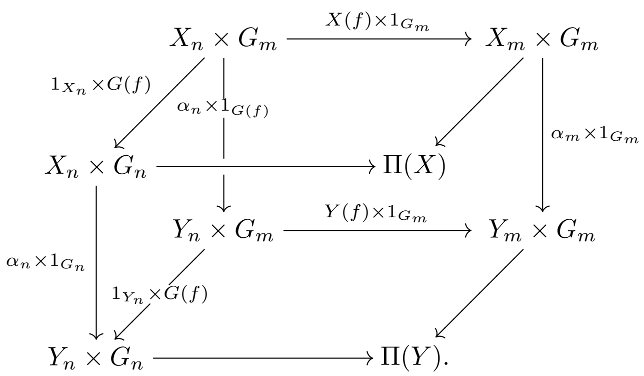
For the fundamental groupoid of a simplicial set, we need something like the geometric realization with invertible arrows. We define a (covariant) functor $G : \mathbf{\Delta} \to \mathbf{Gpd}$ which sends $\mathbf{n}$ to the groupoid G_n with the same objects as $\mathbf{n}$ and exactly one arrow and its inverse between each object i and $i + 1$. We can display this groupoid schematically as

$$0 \rightleftarrows 1 \rightleftarrows \cdots \rightleftarrows n.$$

The functor G is defined in the obvious way on arrows in $\mathbf{\Delta}$. Let $X_n \times G_n$ denote the groupoid whose objects are pairs (x, i) with $x \in X_n$ and $i \in |G_n|$ and whose arrows are the products $1_x \times p$ where 1_x is the function $x \mapsto x$ and p is an arrow in G_n. Then we can define the *fundamental groupoid* $\Pi(X)$ of a simplicial set X to be the colimit of the diagram which includes, for every $f : \mathbf{m} \to \mathbf{n}$,

$$\begin{array}{ccc} X_n \times G_m & \xrightarrow{X(f)\times 1_{G_m}} & X_m \times G_m \\ {\scriptstyle 1_{X_n}\times G(f)}\downarrow & & \\ X_n \times G_n. & & \end{array}$$

This is well defined because all colimits exist in the category of groupoids. In fact, $\Pi(X)$ is the image of the simplicial set X under a functor $\Pi : \mathbf{sSet} \to \mathbf{Gpd}$. To see this, suppose $\alpha : X \to Y$ is an arrow between simplicial sets X and Y and consider the three-dimensional diagram



The top and bottom faces of this diagram commute for all f by the definition of the fundamental groupoid. The left side face commutes for all f trivially, and the back face commutes for all f by naturality (recall that an arrow α between simplicial sets is a natural transformation). It follows that any two composites from $X_n \times G_m$ to $\Pi(Y)$ are equal. By the universal property of the fundamental groupoid $\Pi(X)$, there is a unique functor $\Pi(X) \to \Pi(Y)$ such that the above diagram commutes for all f with this additional arrow. If we call this arrow $\Pi(\alpha)$, then we have completely specified the functor $\Pi : \mathbf{sSet} \to \mathbf{Gpd}$. We call this functor the *fundamental groupoid functor* on simplicial sets.

We can now prove that the fundamental groupoid of the geometric realization of a simplicial set X is equivalent (as a category) to the fundamental groupoid $\Pi(X)$ just defined. This is one reason it makes sense to use the term "fundamental groupoid functor" for both functors Π.

Definition 6. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be

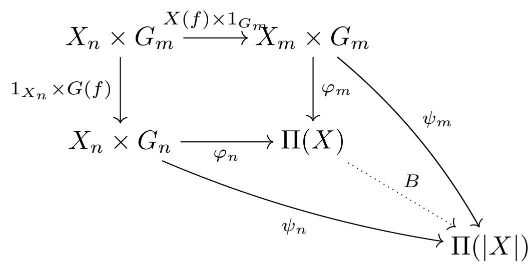
(1) *full* if the map $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ induced by F is surjective for all objects X and Y of $\mathcal{C}$.
(2) *faithful* if $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ is injective for all objects X and Y of $\mathcal{C}$.
(3) *essentially surjective* if each object X' of $\mathcal{D}$ is isomorphic to an object of the form $F(X)$ for some object X of $\mathcal{C}$.

If there is a is full, faithful, and essentially surjective functor $F : \mathcal{C} \to \mathcal{D}$, then every set $\mathcal{D}(X', Y')$ is in bijection with some set $\mathcal{C}(X, Y)$ where X' is isomorphic to $F(X)$ and Y' is isomorphic to $F(Y)$, so we say that the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*.

Theorem 2. *Let X be a simplicial set. Then the categories $\Pi(X)$ and $\Pi(|X|)$ are equivalent.*

Proof. When we construct the geometric realization of X, we map each object $(x, \mathbf{e}_i)$ of $X_n \times \Delta_n$ to a point of $|X|$. Regard this point as an object of $\Pi(|X|)$ and call it $\psi_n(x, i)$. Next, suppose $1_{X_n} \times p$ is an arrow in $X_n \times G_n$ where $p : i \to j$. There is a path $(1-t)\mathbf{e}_i + t\mathbf{e}_j$ in Δ_n, parameterized by t, and this corresponds to a path in $|X|$. Then the homotopy class of this path is an arrow in $\Pi(|X|)$, which we can call $\psi_n(1_{X_n} \times p)$. This specifies a functor $\psi_n : X_n \times G_n \to \Pi(|X|)$ of groupoids for each nonnegative integer n.

By the universal property of $\Pi(X)$, there is a unique functor $B : \Pi(X) \to \Pi(|X|)$ such that



commutes for all $f : \mathbf{m} \to \mathbf{n}$. Now, each standard topological simplex Δ_n is simply connected, so any path between vertices $\psi_n(x, i)$ and $\psi_n(x, j)$ in the space $|X|$ is homotopic to one of the induced paths $\psi_n(1_x \times p)$. This shows that B is full. Since there is exactly one arrow between any two objects of G_n, there is exactly one arrow between any two objects of $X_n \times G_n$. It follows that there is at most one arrow between any two objects $\varphi_n(x, i)$ and $\varphi_n(x, j)$ of $\Pi(X)$. Thus, if two paths between the points $\psi_n(x, i)$ and $\psi_n(x, j)$ of $|X|$ are homotopic, then they are in the image of only one arrow under the functor B. This shows that B is faithful. Finally, since each Δ_n is path-connected, there is a path from any point of $|X|$ to some $\psi_n(x, i)$. Since every path in $|X|$ corresponds to an invertible arrow in $\Pi(|X|)$, this shows that B is essentially surjective. This completes the proof. □

7. Adjunctions

Before we can prove the van Kampen theorem for simplicial sets, we need to introduce the notion of an adjunction. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and there are functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$. Given an object X of $\mathcal{C}$ and an object Y of $\mathcal{D}$, we can consider the arrows $X \to R(Y)$ in $\mathcal{C}$ and $L(X) \to Y$ in $\mathcal{D}$. The term "adjunction" refers to a situation in which each arrow $X \to R(Y)$ corresponds exactly to an arrow $L(X) \to Y$ and this correspondence preserves categorical structure as X and Y vary.

Definition 7. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ consists of functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ together with a bijection

$$\theta_{X,Y} : \mathcal{D}(L(X), Y) \cong \mathcal{C}(X, R(Y))$$

of sets, defined for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$, and *natural* in these arguments X and Y. This last requirement means that the diagram

$$\begin{array}{ccc} L(X) & \xrightarrow{f} & Y \\ {\scriptstyle L(g)}\downarrow & & \downarrow{\scriptstyle h} \\ L(X') & \xrightarrow[f']{} & Y' \end{array}$$

commutes if and only if

$$\begin{array}{ccc} X & \xrightarrow{\theta f} & R(Y) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle R(h)} \\ X' & \xrightarrow[\theta f']{} & R(Y') \end{array}$$

commutes. Given such an adjunction, we say that L is a *left adjoint* of R, and R is a *right adjoint* of L.

Adjunctions are ubiquitous in mathematics. In topology the functor $L : \mathbf{Set} \to \mathbf{Top}$ which sends each set X to the topological space X with the discrete topology is a left adjoint of the functor $R : \mathbf{Top} \to \mathbf{Set}$, $(X, \mathcal{T}) \mapsto X$ which "forgets" the topology $\mathcal{T}$. In algebra the functor $L : \mathbf{Set} \to \mathbf{Grp}$ which sends each set X to the free group with generators $x \in X$ is a left adjoint of the functor $R : \mathbf{Grp} \to \mathbf{Set}$ which sends each group to its underlying set.

The following technical lemma will give us a very interesting example of adjoint functors involving simplicial sets.

Lemma 1. *Suppose $\mathcal{G}$ and $\mathcal{H}$ are groupoids and X is a set. Then there is a bijection $\phi : \mathbf{Gpd}(X \times \mathcal{G}, \mathcal{H}) \cong \mathbf{Set}(X, \mathbf{Gpd}(\mathcal{G}, \mathcal{H}))$ which is natural in $\mathcal{H}$ and X.*

Proof. Let $S : X \times \mathcal{G} \to \mathcal{H}$ be a functor in $\mathbf{Gpd}(X \times \mathcal{G}, \mathcal{H})$. For each element x of X, define a functor $(\phi S)(x)$ in $\mathbf{Gpd}(\mathcal{G}, \mathcal{H})$ sending each object G of $\mathcal{G}$ to the object $S(x, G)$ of $\mathcal{H}$ and sending each arrow $f : G_1 \to G_2$ in $\mathcal{G}$ to the arrow $S(1_x \times f)$ in $\mathcal{H}$. Then there is a function $\phi : \mathbf{Gpd}(X \times \mathcal{G}, \mathcal{H}) \to \mathbf{Set}(X, \mathbf{Gpd}(\mathcal{G}, \mathcal{H}))$ defined by $S \mapsto \phi S$. Next, suppose t is a function in $\mathbf{Set}(X, \mathbf{Gpd}(\mathcal{G}, \mathcal{H}))$. Then t assigns a functor $\mathcal{G} \to \mathcal{H}$ to every element of X. Define a functor $\phi^{-1}t$ in $\mathbf{Gpd}(X \times \mathcal{G}, \mathcal{H})$ sending each object (x, G) of $X \times \mathcal{G}$ to the object $t(x)(G)$ of $\mathcal{H}$ and sending each arrow $1_x \times f : (x, G_1) \to (x, G_2)$ to the arrow $t(x)(f)$ in $\mathcal{H}$. Then the function $\phi^{-1} : t \mapsto \phi^{-1}t$ is an inverse for ϕ since $\phi^{-1}(\phi S) : (x, G) \mapsto (\phi S)(x)(G) = S(x, G)$ and $\phi^{-1}(\phi S) : 1_x \times f \mapsto (\phi S)(x)(f) = S(1_x \times f)$ and also $\phi(\phi^{-1}t) : x \mapsto \phi(\phi^{-1}t)(x) = t(x)$. Hence ϕ is a bijection.

We must now show that ϕ is natural in $\mathcal{H}$ and X. Consider the following diagram

$$\begin{array}{ccc} X \times \mathcal{G} & \xrightarrow{S} & \mathcal{H} \\ {\scriptstyle u \times 1_{\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle V} \\ X' \times \mathcal{G} & \xrightarrow[S']{} & \mathcal{H}' \end{array}$$

where u is a set function and S, S', and V are functors of groupoids. The composite functor $V \circ S$ takes each object (x, G) of $X \times \mathcal{G}$ to an object $V(S(x, G))$ of $\mathcal{H}'$ and takes each arrow $1_x \times f$ of $X \times \mathcal{G}$ to an arrow $V(S(1_x \times f))$ of $\mathcal{H}'$. On the other hand, the composite $S' \circ (u \times 1_{\mathcal{G}})$ takes the object (x, G) to an object $S'(u(x), G)$ and the arrow $1_x \times f$ to an arrow $S'(1_{u(x)} \times f)$. Thus the diagram commutes precisely when $V(S(x, G)) = S'(u(x), G)$ and $V(S(1_x \times f)) = S'(1_{u(x)} \times f)$. Next, consider the following diagram.

$$\begin{CD} X @>\phi S>> \mathbf{Gpd}(\mathcal{G}, \mathcal{H}) \\ @VuVV @VV\_\circ VV \\ X' @>>\phi S'> \mathbf{Gpd}(\mathbf{G}, \mathcal{H}') \end{CD}$$

If we start with an element x in X, compute the functor $(\phi S)(x)$, and then post-compose the result with the functor V, we obtain a functor from $\mathcal{G}$ to $\mathcal{H}'$ which sends each object G of $\mathcal{G}$ to the object $V(S(x, G))$ of $\mathcal{H}'$ and sends each arrow f in $\mathcal{G}$ to an arrow $V(S(1_{\mathcal{G}} \times f))$ in $\mathcal{H}'$. On the other hand, the composite $(\phi S')(u)$ takes x to a functor which sends each object G of $\mathcal{G}$ to the object $S'(u(x), G)$ of $\mathcal{H}'$ and sends each arrow f in $\mathcal{H}$ to an arrow $S'(1_{u(x)} \times f)$ in $\mathcal{H}'$. Again, the diagram commutes precisely when $V(S(x, G)) = S'(x', G)$ and $V(S(1_{\mathcal{G}} \times f)) = S'(1_{u(x)} \times f)$. This proves that the bijection is natural in X and $\mathcal{H}$. ☐

Let the *nerve* $N\mathcal{C}$ of a small category $\mathcal{C}$ be a simplicial set, that is, a contravariant functor $\mathbf{\Delta} \to \mathbf{Set}$, defined as follows. On objects we have $\mathbf{n} \mapsto \mathbf{Cat}(\mathbf{n}, \mathcal{C})$ where $\mathbf{Cat}$ is the category of small categories and functors between them. In other words, the nerve $N\mathcal{C}$ sends the category $\mathbf{n}$ to the set of functors $\mathbf{n} \to \mathbf{C}$. It sends each arrow $f : \mathbf{m} \to \mathbf{n}$ to the arrow $\mathbf{Cat}(\mathbf{n}, \mathcal{C}) \to \mathbf{Cat}(\mathbf{m}, \mathcal{C})$ defined by $t \mapsto t \circ f$.

Our next result involves the functor $N : \mathbf{Gpd} \to \mathbf{sSet}$ which sends each groupoid $\mathcal{G}$ to the nerve $N\mathcal{G}$. If $F : \mathcal{G} \to \mathcal{H}$ is a functor of groupoids, this functor N gives us a natural transformation $NF : N\mathcal{G} \to N\mathcal{H}$ whose nth component NF_n sends the functor $T : \mathbf{n} \to \mathcal{G}$ in $N\mathcal{G}_n = \mathbf{Cat}(\mathbf{n}, \mathcal{G})$ to the functor $TF : \mathbf{n} \to \mathcal{H}$ in $N\mathcal{H}_n = \mathbf{Cat}(\mathbf{n}, \mathcal{H})$.

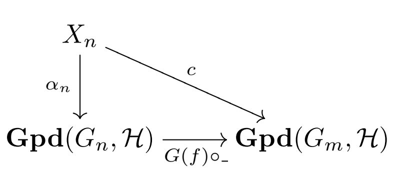
Theorem 3. *The fundamental groupoid functor* $\Pi : \mathbf{sSet} \to \mathbf{Gpd}$ *is a left adjoint of the nerve functor* $N : \mathbf{Gpd} \to \mathbf{sSet}$.

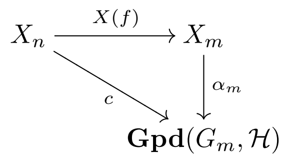
Proof. There is an obvious bijection $\mathbf{Cat}(\mathbf{n}, \mathcal{H}) \cong \mathbf{Gpd}(G_n, \mathcal{H})$, where G_n is one of the groupoids that we defined above to be the image of $\mathbf{n}$ under the functor $G : \mathbf{\Delta} \to \mathbf{Gpd}$. Then the nerve functors $N\mathcal{H} : \mathbf{n} \mapsto \mathbf{Cat}(\mathbf{n}, \mathcal{H})$ are in bijective correspondence with the contravariant functors $N'\mathcal{H} : \mathbf{n} \mapsto \mathbf{Gpd}(G_n, \mathcal{H})$, and this gives us a bijection $\mathbf{sSet}(X, N\mathcal{H}) \cong \mathbf{sSet}(X, N'\mathcal{H})$. This bijection is natural in X and $\mathcal{H}$, so it is enough to show that there exists a bijection $\mathbf{sSet}(X, N'\mathcal{H}) \cong \mathbf{Gpd}(\Pi(X), \mathcal{H})$ which is natural in X and $\mathcal{H}$.

Let α be an element of $\mathbf{sSet}(X, N'\mathcal{H})$. This is a mapping $\alpha : X \to N'\mathcal{H}$ of simplicial sets. It consists of a function $\alpha_n : X_n \to \mathbf{Gpd}(\mathbf{n}, \mathcal{H})$ for each nonnegative integer n such that

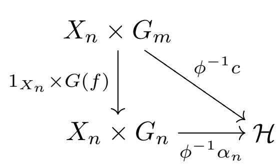
$$\begin{CD} X_n @>X(f)>> X_m \\ @V\alpha_nVV @VV\alpha_mV \\ \mathbf{Gpd}(G_n, \mathcal{H}) @>>G(f)\circ\_> \mathbf{Gpd}(G_m, \mathcal{H}) \end{CD}$$

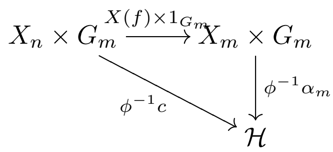
commutes where $f : \mathbf{m} \to \mathbf{n}$ is any arrow in the category $\mathbf{\Delta}$. If we let c be the composite map $X_n \to \mathbf{Gpd}(G_m, \mathcal{H})$ then we can divide this commutative square diagram into the following two commutative triangular diagrams



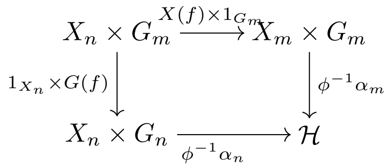


where $f : \mathbf{m} \to \mathbf{n}$. If we apply the inverse of the bijection $\phi : \mathbf{Gpd}(X_i \times G_i, \mathcal{H}) \cong \mathbf{Set}(X_i, \mathbf{Gpd}(G_i, \mathcal{H}))$ constructed in Lemma 1, we find that the following diagrams commute

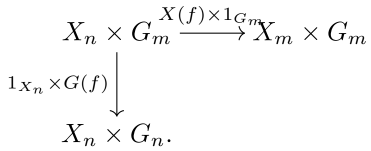




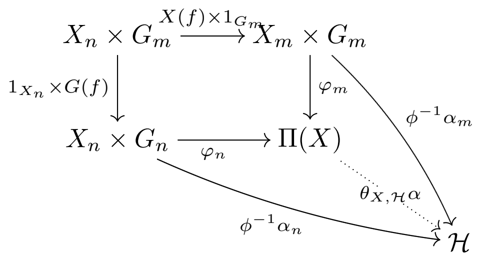
for all $f : \mathbf{m} \to \mathbf{n}$. But the diagonal arrows are the same for these two diagrams, so we conclude that



commutes for all $f : \mathbf{m} \to \mathbf{n}$. This proves that $\{\phi^{-1}\alpha_i : X_i \times G_i \to \mathcal{H}\}$ is a cocone for the diagram which includes, for every arrow $f : \mathbf{m} \to \mathbf{n}$,



But $\Pi(X)$ is the colimit for this diagram, so there is a unique functor of groupoids $\Pi(X) \to \mathcal{H}$, which we shall call $\theta_{X,\mathcal{H}}\alpha$, such that the following diagram commutes



for all arrows $f : \mathbf{m} \to \mathbf{n}$. Define a function $\theta_{X,\mathcal{H}} : \mathbf{sSet}(X, N'\mathcal{H}) \to \mathbf{Gpd}(\Pi(X), \mathcal{H})$ by $\alpha \mapsto \theta_{X,\mathcal{H}}\alpha$. Given any functor $B : \Pi(X) \to \mathcal{H}$, let $\theta^{-1}_{X,\mathcal{H}}B$ be the natural transformation whose nth component is $\phi(B \circ \varphi_n)$. Then the function $\theta^{-1}_{X,\mathcal{H}}$ defined by $B \mapsto \theta^{-1}_{X,\mathcal{H}}B$ is an inverse for $\theta_{X,\mathcal{H}}$ since the nth component of $\theta^{-1}_{X,\mathcal{H}}(\theta_{X,\mathcal{H}}\alpha)$ is

$\phi((\theta_{X,\mathcal{H}}\alpha)\circ\varphi_n)=\alpha_n$ and since $\theta_{X,\mathcal{H}}(\theta^{-1}_{X,\mathcal{H}}B)=B$. This proves that $\theta_{X,\mathcal{H}}$ is a bijection. It remains to show that $\theta_{X,\mathcal{H}}$ is natural in in X and $\mathcal{H}$.

Let T be any functor from $\mathcal{H}$ to $\mathcal{H}'$ and let σ be a mapping from a simplicial set X to a simplicial set X'. Consider the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\alpha_n} & \mathbf{Gpd}(G_n,\mathcal{H}) \\ {\scriptstyle\sigma_n}\downarrow & & \downarrow{\scriptstyle \_\circ T} \\ X'_n & \xrightarrow[\alpha'_n]{} & \mathbf{Gpd}(G_n,\mathcal{H}'). \end{array}$$

Since the bijection $\phi:\mathbf{Gpd}(X_n\times G_n,\mathcal{H})\cong\mathbf{Set}(X_n,\mathbf{Gpd}(G_n,\mathcal{H}))$ constructed in Lemma 1 is natural in X_n and $\mathcal{H}$, it follows that this diagram commutes for all n if and only if

$$\begin{array}{ccc} X_n\times G_n & \xrightarrow{\phi^{-1}\alpha_n} & \mathcal{H} \\ {\scriptstyle\sigma_n\times 1_{G_n}}\downarrow & & \downarrow{\scriptstyle T} \\ X'_n\times G_n & \xrightarrow[\phi^{-1}\alpha'_n]{} & \mathcal{H}' \end{array}$$

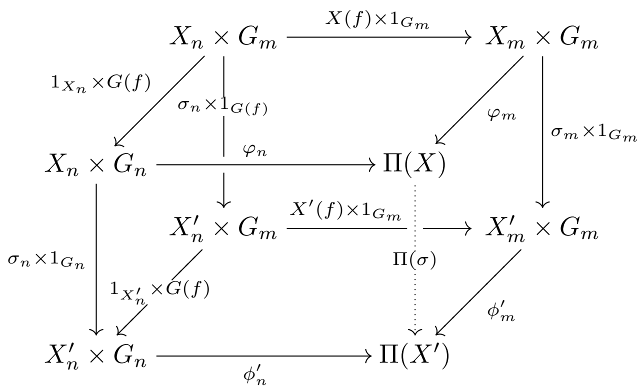
commutes for all n. Now the fundamental groupoids $\Pi(X)$ and $\Pi(X')$ have the property that $\theta_{X,\mathcal{H}}\alpha$ and $\theta_{X,\mathcal{H}}\alpha'$ are the unique functors making the diagrams

$$\begin{array}{ccc} X_n\times G_n & \xrightarrow{\varphi_n} & \Pi(X) \\ & {\scriptstyle\phi^{-1}\alpha_n}\searrow & \downarrow{\scriptstyle\theta_{X,\mathcal{H}}\alpha} \\ & & \mathcal{H} \end{array} \qquad \begin{array}{ccc} X'_n\times G_n & \xrightarrow{\varphi'_n} & \Pi(X') \\ & {\scriptstyle\phi^{-1}\alpha'_n}\searrow & \downarrow{\scriptstyle\theta_{X',\mathcal{H}'}\alpha'} \\ & & \mathcal{H}' \end{array}$$

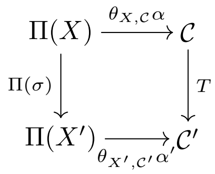
commute for each nonnegative integer n. Thus the last square diagram commutes for all n precisely when

$$\begin{array}{ccccc} X_n\times G_n & \xrightarrow{\varphi_n} & \Pi(X) & \xrightarrow{\theta_{X,\mathcal{H}}\alpha} & \mathcal{H} \\ {\scriptstyle\sigma_n\times 1_{G_n}}\downarrow & & & & \downarrow{\scriptstyle T} \\ X'_n\times G_n & \xrightarrow[\varphi'_n]{} & \Pi(X') & \xrightarrow[\theta_{X',\mathcal{H}'}\alpha']{} & \mathcal{H}' \end{array}$$

commutes for all n. Now $\Pi(\sigma)$ is defined to be the unique functor such that the following diagram commutes for all $f : \mathbf{m} \to \mathbf{n}$.



In particular, the front face of this diagram commutes for all f. It follows that our last rectangular diagram commutes for all n precisely when

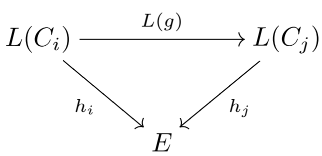


commutes for all n. This is the desired commutative diagram, so the proof is complete. □

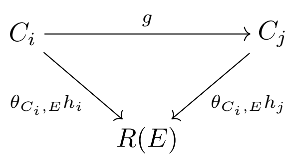
Theorem 4. *Left adjoints preserve colimits. That is, if $F : \mathcal{D} \to \mathcal{C}$ is a $\mathcal{D}$-shaped diagram and $L : \mathcal{C} \to \mathcal{E}$ is a left adjoint of some functor $R : \mathcal{E} \to \mathcal{C}$, then $L(\mathrm{colim}\ F) \cong \mathrm{colim}\ LF$.*

Proof. First put $C = \mathrm{colim}\ F$. Our definition of the colimit says that there is a universal cocone $\{f_i : C_i \to C\}$ for the $\mathcal{D}$-shaped diagram F. Since L is a functor, we know that $\{L(f_i) : L(C_i) \to L(C)\}$ is a cocone for the $\mathcal{D}$-shaped diagram $LF : \mathcal{D} \to \mathcal{E}$. We want to show that $L(C)$ is the colimit of LF; hence we must show that the cocone $\{L(f_i) : L(C_i) \to L(C)\}$ is universal.

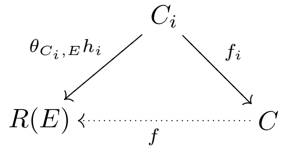
Let $\{h_i : L(C_i) \to E\}$ be any other cocone for LF. By our definition this means that



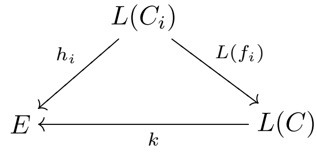
commutes whenever $g : C_i \to C_j$ is an arrow in the $\mathcal{D}$-shaped diagram F. Now $L : \mathcal{C} \to \mathcal{E}$ is a left adjoint of $R : \mathcal{E} \to \mathcal{C}$, so there is a bijection of sets $\theta_{C_i,E} : \mathcal{E}(L(C_i), E) \cong \mathcal{C}(C_i, R(E))$ which is natural in C_i. Therefore



commutes whenever $g : C_i \to C_j$ is an arrow in the $\mathcal{D}$-shaped diagram F. This shows that $\{\theta_{C_i,E} h_i : C_i \to R(E)\}$ is a cocone for F. Since $\{f_i : C_i \to C\}$ is a universal cocone, it follows that there is a unique arrow $f : C \to R(E)$ such that



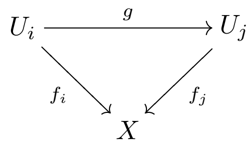
commutes for every object C_i in the $\mathcal{D}$-shaped diagram F. Since L is a left adjoint of R, there is a bijection of sets $\theta_{C,E} : \mathcal{E}(L(C), E) \cong \mathcal{C}(C, R(E))$, and we can define $k = \theta_{C,E}^{-1} f : L(C) \to E$. Then



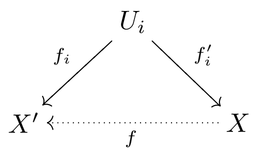
commutes for every object C_i in the $\mathcal{D}$-shaped diagram F. Since the arrow f is unique and $\theta_{C,E}^{-1}$ is injective, we conclude that k is the unique map making the above diagram commute. This proves that the cocone $\{L(f_i) : L(C_i) \to L(C)\}$ is universal. □

8. Van Kampen's Theorem

Now that we have discussed adjunctions, we can finally return to our discussion of van Kampen's theorem and show that a version of the theorem holds for simplicial sets. First, let us observe that van Kampen's theorem is really a statement about the preservation of colimits. If $\mathcal{O} = \{U_i\}$ is a cover of X satisfying the hypotheses of van Kampen's theorem, then



commutes where g is any arrow in $\mathcal{O}$ and the functions f_i are the inclusions of the U_i into X. This shows that $\{f_i : U_i \to X\}$ is a cocone for the diagram F consisting of the sets U_i and the inclusion maps between them. If $\{f'_i : U_i \to X'\}$ is any other cocone for this diagram, then the function $f : X \to X'$ sending each point $x \in U_i \subseteq X$ to $f'_i(x)$ is the unique function making



commute for all U_i. Hence X is a colimit of F, and van Kampen's theorem says

$$\Pi(\text{colim } F) \cong \text{colim } \Pi F.$$

On the other hand, if F is any diagram in the category of simplicial sets, then this equation still holds by Theorems 3 and 4. This shows the fundamental groupoid functor $\Pi : \mathbf{sSet} \to \mathbf{Gpd}$ preserves *all* colimits and thus satisfies what could be viewed as a stronger version of the van Kampen theorem. The fact that versions of van Kampen's theorem hold for both topological spaces and simplicial sets is a manifestation of the fact that topological spaces and simplicial sets are, in some sense, the "same up to homotopy." It is possible to develop an entire homotopy theory for simplicial sets, but readers wishing to understand these statements will have to consult the more advanced texts listed below.

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