# THREE THEOREMS IN KNOT THEORY

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ABSTRACT. In this paper, I will explore connections between the braid group and both free groups and fundamental groups. It is my goal to summarize and simplify the proofs of two important theorems about braid groups, as well as stating a third. Along the way, I hope to clarify the proofs by producing simple lemmas and exercises, as well as supplying examples of the material in the theorems.

## CONTENTS



## 1. INTRODUCTION

**Definition 1.1.** A braid on $n$-strings is a three dimensional object between two $xy$-planes in $\mathbb{R}^3$ given by the following information:

(1) $n$ points on a plane $z = a$, $P_1, \ldots, P_n$ so that if $i < j$, the $x-$coordinate of $P_i$ is less than that of $P_j$
(2) $n$ points on a plane $z = b$ where $b < a$, $Q_1, \ldots, Q_n$ so that if $i < j$, the $x-$coordinate of $Q_i$ is less than that of $Q_j$.
(3) For each $i$ there is a path joining $P_i$ to $Q_{i\mu}$ where $\mu$ is a permutation of $\{1, \ldots, n\}$ so that as one travels along this path, the $z-$coordinate strictly decreases. We say that the $i-$th string of the braid ends up at or goes to $Q_{i\mu}$.
(4) No two of these paths intersect.

We can look at projections of braids onto the $xz$-plane. If at some point, two of the paths have the same $z$-coordinate, the path with the greater $y-$coordinate at that point is said to go over the other. See Figures 1 and 2 for examples.

We denote a braid as $\sigma$. The simplest braids on $n$ strings we will call $\sigma_i$ for $1 \leq i \leq n-1$. The $j-th$ string of $\sigma_i$ goes from $P_j$ to $Q_j$ for $j \neq i, i+1$. Further,

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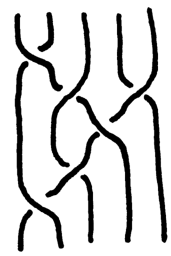

FIGURE 1. $\sigma \in B_5$

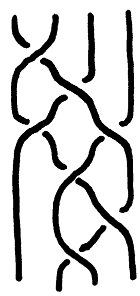

FIGURE 2. $\sigma' \in B_4$

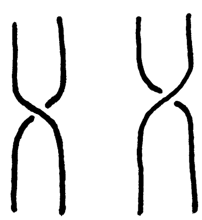

FIGURE 3. On the left is $\sigma_i$ and on the right $\sigma_i^{-1}$.

in $\sigma_i$, the path from $P_i$ ends up at $Q_{i+1}$ and the path from $P_{i+1}$ ends up at $Q_i$ with the path from $P_i$ going over the other. In, $\sigma_i^{-1}$, the paths go to the same places, but the path from $P_{i+1}$ goes over the other. Every braid on n-strings is a composition of $\sigma_i^{\pm 1}$ for $1 \leq i \leq n-1$. See Figure 3.

We can think of braids as continuous paths. Take a braid $\sigma$ on $n$ strings. For each $1 \leq i \leq n$, there is a continuous map $b_i : [0,1] \to \mathbb{R}^3$ so that $b_i(0) = P_i$ and $b_i(1) = Q_{i\mu}$. No two paths intersect. These $n$ paths or strings represent the braid.

**Definition 1.2.** Two braids are string isotopic if there is a continuous deformation from one braid into the other such that at each point in the deformation the 4 conditions are satisfied. By pulling strings to the left and right without breaking any and then straightening and pulling tight, it is sometimes easy to see that two braids are string isotopic. See Figure 5 to see that the two rightmost braids are string isotopic.

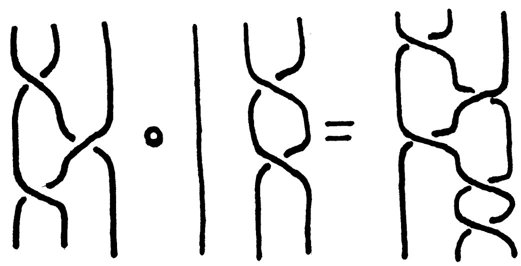

FIGURE 4. The product of two braids through concatenation.

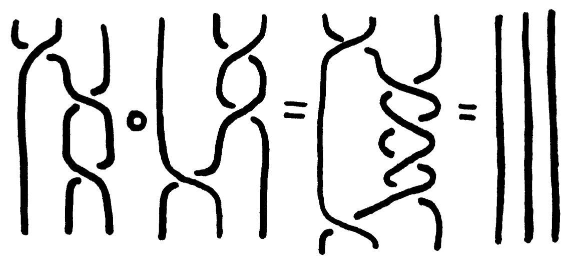

FIGURE 5. An example of two braids that are inverses of each other.

If two braids are string isotopic, we can find continuous maps to represent the deformation that changes one braid into another. Let $\sigma$ and $\sigma'$ be two string isotopic braids in $B_n$. For each $1 \leq i \leq n$ there is a continuous map $c_i : [0,1] \times [0,1] \to \mathbb{R}^3$ so that $c_i(x, 0) = b_i(x)$, and $c_i(x, 1) = b_i'(x)$

**Definition 1.3.** The braids on $n$-strings form a group, $B_n$. The equivalence relation is string isotopy. The operation is concatenation, given as follows. Every braid $\sigma$ defined by planes $z = a, z = b$ is string isotopic to one defined by planes $z = a', z = b'$ where $b' < a'$. Let the braid $\sigma$ be defined by planes $z = a, z = b$ and points $\{P_i\}, \{Q_i\}$. Let $\sigma'$ be defined by planes $z = d, z = e$ and points $\{S_i\}, \{R_i\}$. To make the product $\sigma\sigma'$ we shift $\sigma'$ to the planes $z = b, z = c$ as allowed and associate the point $S_i$ with the point $Q_i$ for each $1 \leq i \leq n$. Now $\sigma\sigma'$ is defined by planes $z = a$ and $z = c$ and the points $\{P_i\}, \{R_i\}$. See Figure 4.

Is $B_n$ really a group? It is clear that concatenating two braids in this manner gives another braid, so that $B_n$ is closed under this operation. It is also clear that the operation obeys associativity since we are essentially gluing braids together end to end. The identity braid is the braid of $n$ straight lines. Since every braid is a finite composition of $\sigma_i$, we can just undo each $\sigma_i$ one at a time to unwrap the braid and get the identity braid, showing that every braid does in fact have an inverse. See Figure 5. These observations show us that $B_n$ is in fact a group.

Consider $\sigma_i\sigma_j \in B_n$. If $|i-j| \geq 2$, $\sigma_i\sigma_j$ is string isotopic to $\sigma_j\sigma_i$. This is because the braids $\sigma_i$ and $\sigma_j$ under such conditions do not affect each other or work with the same strings so we can slide them up and down with out changing the braid.

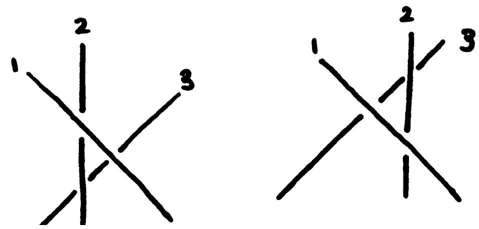


FIGURE 6

Thus, in $B_n$, we write $\sigma_i\sigma_j = \sigma_j\sigma_i$. Further, we can see that

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

The string isotopy between the two braids is shown in Figure 6. The isotopy is provided by sliding string 3 up and down behind the other two strings. We have already noted that every braid in $B_n$ is a finite composition of $\sigma_1, \dots, \sigma_n$ so that those $n$ braids generate the braid group. These equalities, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, now provide relations for the group. We will see later that these generators and relations suffice to describe the braid group.

**Definition 1.4.** A knot is a subset of points in $\mathbb{R}^3$ homeomorphic to a circle.

**Definition 1.5.** A link is a disjoint union of finitely many knots. Each of these knots is a component of the link. Thus a knot is a link with one component.

**Notation 1.6.** Throughout the paper, I will use $F_n$ to denote the free group on n elements, $x_1, \dots, x_n$ where $n$ is a fixed integer. So that

$$F_n =< x_1, \dots, x_n; - >$$

**Notation 1.7.** $\Pi(X, x)$ denotes the fundamental group of the space $X$ based at the point $x$. If it is clear in context, I may omit the $x$ and simply write $\Pi(X)$.

**Notation 1.8.** $C_X(Y)$ denotes the complement of $Y$ in $X$.

## 2. Theorems I Need

I will use the following theorems without proof.

**Theorem 2.1.** *Suppose that $f : X \to Y$ is a continuous mapping of topological spaces and $x_0 \in X$. Then $f$ gives rise in a natural way to group homomorphism*

$$f_\Pi : \Pi(X, x_0) \to \Pi(Y, f(x_0))$$

*which is defined by $f_\Pi(\ell) = f \circ \ell$ for all loops $\ell$ in $X$ at $x_0$. If $f$ is a homeomorphism, then $f_\Pi$ is an isomorphism.*

**Theorem 2.2.** *Let $(x_0, y_0)$ denote a point in the topological product $X \times Y$ of the topological spaces $X$ and $Y$. Let*

$$\Pi(X, x_0) \times \Pi(Y, y_0)$$

*denote the direct product of the fundamental groups. Then*

$$\Pi(X \times Y, (x_0, y_0)) \cong \Pi(X, x_0) \times \Pi(Y, y_0)$$

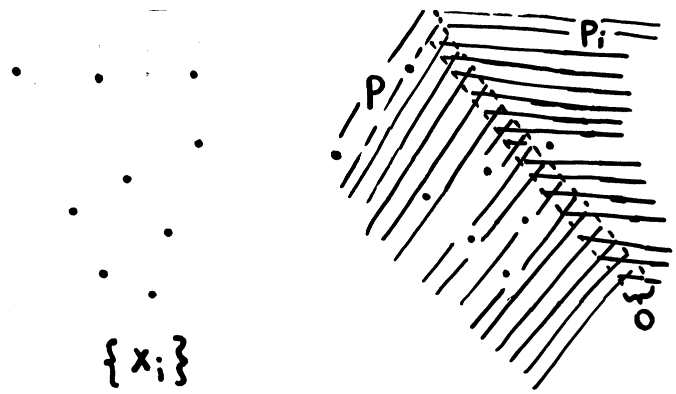


FIGURE 7

**Theorem 2.3.** *Theorem of Seifert and Van Kampen: Let $O_1$ and $O_2$ be pathwise connected open subspaces of a topological space $X$ such that*

$$X = O_1 \cup O_2$$
$$\text{and}$$
$$O = O_1 \cap O_2$$

*so that $O$ is pathwise connected, nonempty, and $x_0 \in O$. Then*

$$\Pi(X, x_0) \cong \Pi(O_1, x_0)^{(f_1)\Pi} \underset{\Pi(O,\ x_0)}{*}{}^{(f_2)\Pi}\ \Pi(O_2, x_0)$$

*where $f_1$ and $f_2$ are the natural injections of $O$ into $O_1$ and $O_2$ respectively.*

## 3. Artin Representation Theorem

**Lemma 3.1.** *The fundamental group of $\mathbb{R}^2$ with $n$ distinct points removed is isomorphic to the free group on $n$ elements, $F_n$.*

*Proof.* (Sketch.) By induction on $n$. Let $\mathbb{R}^2_n$ denote $\mathbb{R}^2$ with $n$ distinct points removed. We know that $\Pi(\mathbb{R}^2_0) =< e >= F_0$ and that $\Pi(\mathbb{R}^2_1) \cong \mathbb{Z} \cong F_1$.
Assumption: $\mathbb{R}^2_n \cong F_n$
Induction: Let $n+1$ points in $\mathbb{R}^2$ be given and remove them from the plane. We know that in $\mathbb{R}^2$ we can draw a line separating one point $x_i$ from the rest. This line divides $\mathbb{R}^2$ into two half-planes. Let the half-plane with the hole left by $x_i$ be $P_i$. Construct $P$ so that $C_{\mathbb{R}^2}(P_i) \subset P$ and the intersection of $P_i$ and $P$ is nonempty, pathwise connected, and has no holes. The idea would be that $P$ is the half-plane complement of $P_i$ shifted a little bit so that $P_i$ and $P$ overlap but so that $x_j$ is not in the intersection for any $1 \leq j \leq n+1$. See Figure 7 to get the picture. Now, let

$$O_1 = P_i, \quad O_2 = P, O = P' \cap P$$

so that $\mathbb{R}^2_{n+1} = O_1 \cup O_2$. All three regions are pathwise connected. Taking $x \in O$, they satisfy the conditions for Van Kampen's Theorem. We already know that $\Pi(O_1) = F_1$ and by assumption $\Pi(O_2) = F_n$. Since $O$ is an open subset of $\mathbb{R}^2$ with no holes, $\Pi(O) =< e >$. Then the homomorphisms from $O$ into each half-plane send $e$ to $e$, the identity loop, so that in the amalgamated free product no new relations are created. This gives us that

$$\Pi(\mathbb{R}^2_{n+1}) \cong F_n * F_1 = F_{n+1}$$

□

**Theorem 3.2.** *$B_n$ is isomorphic to the subgroup of right automorphisms $\beta$ of $F_n$ which satisfy the conditions*

1. *$(x_1x_2\dots x_n)\beta = x_1x_2\dots x_n$*
2. *$x_i\beta = A_i x_{i\mu} A_i^{-1}$ for $1 \le i \le n$*

*where $\mu$ is a permutation of $1, 2, \dots, n$ and every $A_i$ belongs to (i.e. is a word in) $F_n$. If the braid $\sigma$ goes to the automorphism $\beta$, $\mu$ is defined by the fact that $i-th$ string of sigma goes from $P_i$ to $Q_{i\mu}$ for all $i$. The braid $\sigma_i$ will be mapped to the automorphism $\beta$ of $F_n$ where*

$$\begin{aligned} x_i\beta &= x_i x_{i+1} x_i^{-1} \\ x_{i+1}\beta &= x_i \\ x_j\beta &= x_j && \textit{if } j \neq i, i+1 \end{aligned}$$

*Proof.* A braid $\sigma \in B_n$ is defined by two planes, $z = a$ and $z = b$ with points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ respectively. We take a point $P$ in the plane $z = a$ whose x-coordinate is less than x-coordinates of $P_i$ and $Q_i$ for all $1 \le i \le n$. Take the point $Q$ in the plane $z = b$ whose $x$ and $y$ coordinates are equal to the x and y coordinates of $P$.

Remove the strings of $\sigma$ from $\mathbb{R}^3$. Let p denote the plane $z = a$ with the points $P_1, \dots, P_n$ removed. Similarly, let q denote the plane $z = b$ with the points $Q_1, \dots, Q_n$ removed. Consider $\Pi(p; P)$ and $\Pi(q; Q)$. Let $x_i$ be a clockwise loop based at $P$ around the hole left by $P_i$ in the plane $p$ (or based at $Q$ around $Q_i$ in $q$.) (Clockwise is from the perspective of a point in $\mathbb{R}^3$ where $z > a$.)

By Lemma 3.1, we know that

$$\Pi(p; P) \cong \Pi(q; Q) \cong F_n = < x_1, \dots, x_n; - >$$

where $x_i$ is a generator of the free group and in our case a clockwise loop around the hole left by $P_i$ or $Q_i$ as stated. This relationship shows us that automorphisms of $F_n$ are equivalent to isomorphisms between $\Pi(p; P)$ and $\Pi(q; Q)$. Thus, we will talk about isomorphisms between the fundamental groups instead of automorphisms of $F_n$.

We want an isomorphism between $B_n$ and the group of automorphisms on $F_n$ (read: isomorphisms between the two fundamental groups). We will first define this isomorphism $\Phi$. We will write $\overline{\sigma}$ for $\Phi(\sigma)$. Let $\sigma \in B_n$. Given a loop $\ell \in \Pi(p; P)$, $\overline{\sigma}(\ell) \in \Pi(q; Q)$ is the loop obtained by pushing $\ell$ down the gaps in $\mathbb{R}^3$ left by the strings of $\sigma$.

Our first goal is to show that $\overline{\sigma}$ is in fact an automorphism of $F_n$ (read: isomorphism between the two fundamental groups).

1. Single-valued: If $\ell_1$ and $\ell_2$ are loops in p at P which are homotopic relative to P, there is a homotopy

$$h : [0, 1] \times [0, 1] \to \mathbb{R}^3$$

where

$$h(x,0) = \ell_1(x)$$
$$h(x,1) = \ell_2(x)$$
$$h(0,t) = p = h(1,t)$$

And $\overline{\sigma}$ can be thought of as a continuous map from a loop in $\Pi(p;P)$ to a loop in $\Pi(q;Q)$. Now define a new map $h' : [0,1]\times[0,1] \to \mathbb{R}^3$ that is going to send $\ell_1\overline{\sigma}$ to $\ell_2\overline{\sigma}$.

$$h'(x,t) = \begin{cases} \overline{\sigma}^{-1}(x,3t) & , \quad \text{for } 0 \le t \le 1/3 \\ h(x,3t-1) & , \quad \text{for } 1/3 \le t \le 2/3 \\ \overline{\sigma}(x,3t-1) & , \quad \text{for } 2/3 \le t \le 1 \end{cases}$$

where $\overline{\sigma}^{-1}$ is the inverse or pushing up process. From this we see

$$h'(x,0) = \overline{\sigma}(x,0) = \ell_1\overline{\sigma}(x)$$
$$h'(x,1/3) = \ell)1\overline{\sigma}\overline{\sigma}^{-1}(x) = \ell_1(x)$$
$$h'(x,2/3) = \ell_2(x)$$
$$h'(x,1) = \ell_2\overline{\sigma}(x)$$

So $h'$ is a homotopy from $\ell_1\overline{\sigma}$ to $\ell_2\overline{\sigma}$.
We conclude that if $\ell_1 \sim \ell_2 \in \Pi(P;P)$ then $\ell_1\overline{\sigma} \sim \ell_2\overline{\sigma} \in \Pi(q;Q)$. Since the equivalence relation in fundamental groups is homotopy, we know that $\overline{\sigma}$ is single-valued.

(2) Injectivity: Further, given two homotopic loops in $\Pi(q;Q)$, we can find a homotopy between the loops we get after pushing the loops up the braid in an exactly analogous way, implying that $\overline{\sigma}$ is also injective.
(3) Surjectivity: Since $\overline{\sigma}$ has the pushing up inverse process, given any loop in $\Pi(q;Q)$ we can find a corresponding loop in $\Pi(p;P)$, implying that $\overline{\sigma}$ is surjective.
(4) Homomorphism: Let $\ell_1, \ell_2 \in \Pi(p;P)$. It is clear that the product of $\ell_1\overline{\sigma}$ and $\ell_2\overline{\sigma}$ is the same as the loop obtained by pushing down the product of $\ell_1$ and $\ell_2$.
$$\Rightarrow \ell_1\overline{\sigma} \circ \ell_2\overline{\sigma} = (\ell_1 \circ \ell_2)\overline{\sigma}$$
so that $\overline{\sigma}$ is a homomorphism.
(5) As previously noted, the two fundamental groups we are looking at are isomorphic to $F_n$ and $\overline{\sigma}$ is a map from one fundamental group to the other.

Finally, we see that $\overline{\sigma}$ is a single-valued, bijective homomorphism from $F_n$ to $F_n$. We now know that $\Phi$ does in fact send braids to automorphisms.

$$\Phi : B_n \to \mathrm{Aut}(F_n)$$

Now to show that $\Phi$ is an isomorphism.

(1) Single-valued: Let $\sigma$ and $\sigma'$ be two string isotopic braids in $B_n$. Then for each $1 \le i \le n$ there is a map $c_i : [0,1]\times[0,1] \to \mathbb{R}^3$ that brings $\sigma$ to $\sigma'$. Let $\ell \in \Pi(p;P)$. Consider $\ell\overline{\sigma}, \ell\overline{\sigma'} \in \Pi(q;Q)$. The string isotopy maps $c_i$ give a natural homotopy from $\ell\overline{\sigma}$ to $\ell\overline{\sigma'}$. So
$$\sigma \sim \sigma' \Rightarrow \ell\overline{\sigma} \sim \ell\overline{\sigma'} \quad \forall \ell \in \Pi(p;P)$$

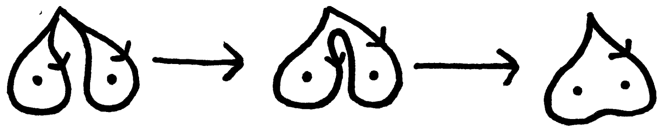

FIGURE 8. Here we can see that $x_1x_2$ is homotopic to a loop around both of them. This can be extended to $x_1 \dots x_n$.

(2) Injectivity and Surjectivity: Given by Lemma 3.3
(3) Homomorphism: Let $\sigma_1, \sigma_2 \in B_n$. Take any $\ell \in \Pi(p; P)$. It is clear that the loop obtained by pushing $\ell$ down $\sigma_1\sigma_2$ is the same as the loop obtained by pushing $\ell$ down $\sigma_1$ and then again down $\sigma_2$. So for any $\ell$, $\ell\overline{(\sigma_1 \circ \sigma_2)} = \ell\overline{\sigma_1} \circ \ell\overline{\sigma_2}$ and thus $\overline{\sigma_1 \circ \sigma_2} = \overline{\sigma_1} \circ \overline{\sigma_2}$
$$\Rightarrow \Phi(\sigma_1 \circ \sigma_2) = \Phi(\sigma_1)\Phi(\sigma_2)$$
so that $\Phi$ is a homomorphism.

We see that $\Phi$ is a single-valued, bijective homomorphism and thus an isomorphism.

From the statement of theorem, we want that $B_n$ is isomorphic to the subgroup of $\mathrm{Aut}(F_n)$ satisfying (1) and (2). We now need to show that the automorphisms in the set $\{\Phi(\sigma) \mid \sigma \in B_n\} \subset \mathrm{Aut}(F_n)$ do in fact satisfy these conditions.

(1) $x_1 \dots x_n \in \Pi(p; P)$ is the product of loops going once around each of the points $P_1, \dots, P_n$ in the clockwise direction. These n loops are homotopic to one big loop circling all n points once in the clockwise direction. In Figure 8 we can see the homotopy between loops going once around two points and one loop around both of them. This is then clearly extended to loops around $n$ points. If we push this big loop down any braid $\sigma$, we still end up with one big loop around all n points because the loop will always encircle the whole braid. In other words, if we take $x_1 \dots x_n \in \Pi(p; P)$ and let $\sigma$ be any braid in $B_n$, then $(x_1 \dots x_n)\overline{\sigma}$ is the same as $x_1 \dots x_n \in \Pi(q; Q)$. So we see that
$$(x_1 \dots x_n)\overline{\sigma} = x_1 \dots x_n$$
for any $\sigma \in B_n$.
(2) Since $\Phi$ is a homomorphism and $B_n$ is generated by $\{\sigma_i, 1 \leq i \leq n-1\}$ it suffices to consider the automorphisms created by each $\sigma_i$. It's easy to see that pushing down a loop around the j-th hole is not affected by the braid $\sigma_i$ if $j \neq i, i+1$ because the j-th string isn't affected by such a braid. It makes sense that if we push a loop around $P_{i+1}$ down $\sigma_i$ it becomes a loop around $P_i$, thus $x_i$. However, it is not immediately clear why $x_i\overline{\sigma_i} \neq x_{i+1}$. If $x_{i+1}\overline{\sigma_i} = x_i$ then

$$\begin{aligned}
(x_1 \dots x_n)\overline{\sigma_i} &= x_1 \dots x_n && \text{by (1)} \\
x_1\overline{\sigma_i} \dots x_i\overline{\sigma_i}x_{i+1}\overline{\sigma_i} \dots x_n\overline{\sigma_i} &= x_1 \dots x_i x_{i+1} \dots x_n && \text{since } \overline{\sigma_i} \text{ is a homomorphism} \\
x_1 \dots x_i\overline{\sigma_i}x_i \dots x_n &= x_1 \dots x_i x_{i+1} \dots x_n && \text{by above} \\
x_i\overline{\sigma_i}x_i &= x_i x_{i+1} \\
x_i\overline{\sigma_i} &= x_i x_{i+1} x_i^{-1}
\end{aligned}$$

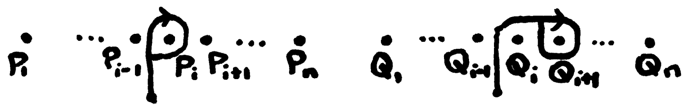


FIGURE 9. On the left is $x_i$, where the base point has been changed from $P$ to $P'$. When this loop is pushed down $\sigma_i$, we get the loop on the right.

We can also see this topologically. Since $\mathbb{R}^2$ with the points $P_1, \ldots, P_n$ removed is path connected, it does not really matter where we choose the base point of our fundamental group. For present purposes, we move the base point from $P$ to a point $P'$ whose $x$ coordinate is between that of $P_{i-1}$ and that of $P_i$, whose $y$ coordinate is less than the $y$ coordinate of all $P_i$ and whose $z$ coordinate is $a$. Pictorially, this is a point in the plane p between $P_{i-1}$ and $P_i$ and below them. Then the loop $x_i$ still goes clockwise around $P_i$, but it now goes in front of or to the left of $P_i$ before it loops the point. When we push this loop down $\sigma_i$, the resulting loop goes clockwise around $Q_{i+1}$, and still goes in front of $Q_i$. This is illustrated in Figure 9. The loop $x_{i+1}$ loops clockwise around $Q_{i+1}$, but it should not go in front of $Q_i$. To make up for this, first we loop $Q_i$, then $Q_{i+1}$, and then undo the loop $Q_i$, resulting in $x_i x_{i+1} x_i^{-1}$, as found above. This is actually arbitrary. We could decide to make $x_i\overline{\sigma_i} = x_{i+1}$, but then by analogous logic, $x_{i+1}\overline{\sigma_i} = x_{i+1}^{-1} x_i x_{i+1}$. So for our purposes, we will choose

$$x_j\overline{\sigma_i} = \begin{cases} x_j & , \text{ if } j \neq i, i+1 \\ x_i x_{i+1} x_i^{-1} & , \text{ if } j = i \\ x_i & , \text{ if } j = i+1 \end{cases}$$

So we find that if we want to have that $x_i\beta = A_i x_{i\mu} A_i^{-1}$, for all $1 \leq i \leq n$, where $A_i \in F_n$ and $\mu$ is some permutation, as stated in the theorem,

$$j\mu = \begin{cases} j & , \text{ if } j \neq i, i+1 \\ i+1 & , \text{ if } j = i \\ i & , \text{ if } j = i+1 \end{cases}$$

where we can see that $j\mu$ denotes where the j-th string ends up in $\sigma_i$. It will be important to remember later that this permutation $\mu$ associated with the homomorphism $\overline{\sigma}$ and thus with $\sigma$ is exactly the physical permutation that tells where each of the strings of the braid goes to. And

$$A_i = \begin{cases} e & , \text{ if } j \neq i, i+1, \text{ where } e \text{ is the empty word} \\ x_i & , \text{ if } j = i \\ e & , \text{ if } j = i+1 \end{cases}$$

Let $B = \{\beta \in \text{Aut}(F_n) \mid \beta \text{ satisfies } (1), (2)\}$. I would like to verify that $B$ is a subgroup of $\text{Aut}(F_n)$. Let $e$ be the identity automorphism. Clearly,

(1) $(x_1 \ldots x_n)e = x_1 \ldots x_n$
(2) $x_i e = x_i$ so that $i\mu = i$ and $A_i = e$ for all $1 \leq i \leq n$

Let $\beta_1, \beta_2 \in B$. Then

(1) $(x_1 \ldots x_n)\beta_1\beta_2 = (x_1 \ldots x_n)\beta_2 = x_1 \ldots x_n$

(2) $x_i\beta_1\beta_2 = (A_{i_1}x_{i\mu_1}A_{i_1}^{-1})\beta_2 = A_{i_1}\beta_2 x_{i\mu_1\mu_2}(A_{i_1}\beta_2)^{-1}$ so that $i\mu = i\mu_1\mu_2$ and $A_i = A_{i\mu_1}\beta_2$. It might happen that cancellation occurs either between $A_{i\mu_1}\beta_2$ or its inverse and $x_{i\mu_1\mu_2}$. If this happens, either $x_{i\mu_1\mu_2}^{-1}$ is the last letter of $A_{i\mu_1}\beta_2$ or the first letter of $(A_{i\mu_1}\beta_2)^{-1}$. But then $x_{i\mu_1\mu_2}$ would be the first letter of $(A_{i\mu_1}\beta_2)^{-1}$ or the last letter of $A_{i\mu_1}\beta_2$ respectively. So that $x_{i\mu_1\mu_2}$ is still the center letter of the word $x_i\beta_1\beta_2$.

So $B$ is closed under composition. It is not hard to see that if $\beta \in B$, $\beta^{-1}$ is as well. So $B$ is a subgroup of $\mathrm{Aut}(F_n)$. Since $B$ is closed under composition, and any $\sigma \in B_n$ is a finite composition of $\sigma_i$, then the automorphisms associated with any braid $\sigma$ will also be in $B$, or will satisfy (1) and (2).

There is one last note about the permutation $\mu$. We saw above that for $\overline{\sigma_i}$, $i\mu$ does have the physical significance desired, i.e. that the permutation that comes with the automorphism $\overline{\sigma_i}$ is the same permutation that denotes where each of the strings of $\sigma_i$ ends up. Since the $\overline{\sigma_i}$ maps are homomorphisms, when we compose two we can say that

$$x_k\overline{\sigma_i\sigma_j} = x_k\overline{\sigma_i} \circ \overline{\sigma_j} = (A_{k_i}x_{k\mu_i}A_{k_i}^{-1})\overline{\sigma_j} = ((A_{k_i})\overline{\sigma_j})x_{k\mu_i\mu_j}((A_{k_i}^{-1})\overline{\sigma_j})^{-1}$$

We know that $k\mu_i$ denotes where the $k$-th string of $\sigma_i$ ends up and that $k\mu_i\mu_j$ denotes where the $k\mu_i$-th string of $\sigma_j$ ends up. So that $k\mu_i\mu_j$ must denote where the $k$-th string of $\sigma_i\sigma_j$ ends up. Then we can continue this a finite number of times to show that the permutation $\mu$ associated with the automorphism of any braid $\sigma$ is the permutation that denotes where the strings of the braid go. In so doing, we use that any braid $\sigma \in B_n$ is a finite composition of $\sigma_i$.

The theorem states that $B_n$ is isomorphic to the subgroup of $\mathrm{Aut}(F_n)$ of automorphisms satisfying (1) and (2). The homomorphism is $\Phi$, which sends a braid $\sigma$ to the pushing-down map $\overline{\sigma}$ which is an automorphism of $F_n$ because of the relationship between the fundamental groups of the planes with $n$ points removed and $F_n$. We saw that $\overline{\sigma_i}$ does act on $x_j$ as specified for all $1 \leq j \leq n$. We also saw that the permutation $\mu$ in condition (2) has the physical significance we desired. The only thing that remains left to be shown is that $\Phi$ is injective and surjective, and thus an isomorphism. For this we have the following lemma. □

Before we move on, some calculations are in order. We need to know how $\overline{\sigma_i}$ and $\overline{\sigma_i}^{-1} = \overline{\sigma^{-1}}$ act on all possible loops. By using the fact that $\overline{\sigma_i}$ is a homomorphism we can deduce the following computations. $(* = \pm 1)$

$$x_j^*\overline{\sigma_i} = \begin{cases} x_j^* & , \text{ if } j \neq i, i+1 \\ x_i x_{i+1}^* x_i^{-1} & , \text{ if } j = i \\ x_i^* & , \text{ if } j = i+1 \end{cases}$$

$$x_j^*\overline{\sigma_i}^{-1} = \begin{cases} x_j^* & , \text{ if } j \neq i, i+1 \\ x_{i+1}^* & , \text{ if } j = i \\ x_{i+1}^{-1}x_i^*x_{i+1} & , \text{ if } j = i+1 \end{cases}$$

**Lemma 3.3.** *Suppose that $\beta$ is an endomorphism of $F_n$ satisfying (1) and (2) from the previous theorem. Then there exists an $n$-braid $\beta_0$ such that $\overline{\beta_0} = \beta$. From this it follows that $\beta$ is an automorphism of $F_n$ since $\Phi$ maps braids to automorphisms of $F_n$. It also follows that since every such automorphism can be found by $\Phi(\sigma)$ for some $\sigma$ that $\Phi$ is bijective.*

*Proof.* If $\beta$ satisfies (2), for each $1 \leq i \leq n$, $x_i\beta = A_i x_{i\mu} A_i^{-1}$ where $A_i \in F_n$. So we can assign a number value to $\beta$ depending on the length of these $A_i$'s. (The length of an element of the free group $F_n$ is the number of elements concatenated or multiplied to achieve it. Let this be denoted $\ell(A)$ if $A \in F_n$.) Let

$$\ell(\beta) = \sum_{i=1}^{n} \ell(A_i)$$

We will proceed by induction on this integer $\ell(\beta)$.

Base case: If $\ell(\beta) = 0$ $A_i$ is the empty word for every $1 \leq i \leq n$. So that

$$x_{1\mu} \dots x_{i\mu} \dots x_{n\mu} = x_1 \dots x_i \dots x_n$$

since $\beta$ must satisfy (1). But then $\mu$ must be the identity permutation since the free group is not commutative. So we deduce that $\beta$ is the identity endomorphism. Then let $\beta_0$ be the identity braid of $n$ strings.

Assumption: If $\ell(\beta) < m$ for some integer $m$ then we can find a braid $\beta_0$ such that $\overline{\beta_0} = \beta$.

Induction: Take $\beta$ such that $\ell(\beta) = m$. Then since $\beta$ satisfies (1) and (2) we see that

$$A_1 x_{1\mu} A_1^{-1} \dots A_n x_{n\mu} A_n^{-1} = (x_1 \dots x_n)\beta = x_1 \dots x_n$$

and more directly that

$$A_1 x_{1\mu} A_1^{-1} \dots A_n x_{n\mu} A_n^{-1} = x_1 \dots x_n$$

The two sides must have equal length. Since the right hand side has length n, so must the left. Then some elements must cancel so that we can be left with n letters in $F_n$. The cancellation will occur at some lowest i. This can happen in one of two ways:

(1) $A_i x_{i\mu} A_i^{-1} \cdot A_{i+1} x_{(i+1)\mu} A_{i+1}^{-1} = A_i \cdot B_{i+1} \cdot x_{(i+1)\mu} A_{i+1}^{-1}$ where $B_{i+1} = x_{i\mu} A_i^{-1} A_{i+1}$ so that $A_{i+1} = A_i x_{i\mu}^{-1} B_{i+1}$. Then

$$\ell(A_{i+1}) = \ell(A_i) + 1 + \ell(B_{i+1})$$

In this case, we compose the endomorphism $\overline{\sigma_i}$ with $\beta$ where $\overline{\sigma_i}$ is as defined in the preceding theorem. This only affects $x_i$ and $x_{i+1}$. So

$$\begin{aligned} x_i \overline{\sigma_i} \beta &= (x_i x_{i+1} x_i^{-1})\beta \\ &= x_i \beta x_{i+1} \beta x_i^{-1} \beta \\ &= A_i x_{i\mu} A_i^{-1} \cdot A_{i+1} x_{(i+1)\mu} A_{i+1}^{-1} \cdot A_i x_{i\mu} A_i^{-1} \\ &= A_i B_{i+1} x_{(i+1)\mu} B_{i+1}^{-1} A_i^{-1} \end{aligned}$$

and

$$x_{i+1} \overline{\sigma_i} \beta = x_i \beta = A_i x_{i\mu} A_i^{-1}$$

Consider

$$x_j \overline{\sigma_i} \beta = \begin{cases} x_j \beta = A_j x_{j\mu} A_j^{-1} & , \text{ if } j \neq i, i+1 \\ A_i B_{i+1} x_{(i+1)\mu} B_{i+1}^{-1} A_i^{-1} & , \text{ if } j = i \\ A_i x_{i\mu} A_i^{-1} & , \text{ if } j = i+1 \end{cases}$$

Then

$$\begin{aligned}\ell(\overline{\sigma_i}\beta) &= \sum_{j=1}^{j-1} A_j + \ell(A_iB_{i+1}) + \ell(A_i) + \sum_{j=i+2}^{n} A_j \\ &< \sum_{j=1}^{i-1} A_j + \ell(A_{i+1}) + \ell(A_i) + \sum_{j=i+2}^{n} A_j \\ &= \ell(\beta)\end{aligned}$$

Since $\ell(\overline{\sigma}\beta)$ is less than $\ell(\beta) = m$, we know that we can find a braid $(\overline{\sigma_i}\beta)_0$ such that $\overline{(\overline{\sigma_i}\beta)_0} = \overline{\sigma_i}\beta$. Then define $\beta_0$ to be $\sigma_i^{-1}(\overline{\sigma_i}\beta)_0$. Then

$$\overline{\beta_0} = \overline{\sigma_i^{-1}(\overline{\sigma_i}\beta)_0} = \overline{\sigma_i^{-1}} \circ \overline{(\overline{\sigma_i}\beta)_0} = \overline{\sigma_i^{-1}} \circ \overline{\sigma_i} \circ \beta = \beta$$

in which we use that the map $\Phi$ is a homomorphism. So that $\beta_0$ is the desired braid.

(2) Or $A_ix_{i\mu}A_i^{-1} \cdot A_{i+1}x_{(i+1)\mu}A_{i+1}^{-1} = A_ix_{i\mu} \cdot B_i \cdot A_{i+1}^{-1}$ where $B_i = A_i^{-1}A_{i+1}x_{(i+1)\mu}$ so that $A_i^{-1} = B_ix_{(i+1)\mu}^{-1}A_{i+1}^{-1}$. Then

$$\ell(A_i) = \ell(A_{i+1}) + 1 + \ell(B_i)$$

In this case, we compose the endomorphisms $\overline{\sigma_i^{-1}}$ and $\beta$. Again, this only affects $x_i$ and $x_{i+1}$. So

$$x_i\overline{\sigma_i^{-1}}\beta = x_{i+1}\beta = A_{i+1}x_{(i+1)\mu}A_{i+1}^{-1}$$

and

$$\begin{aligned}x_{i+1}\overline{\sigma_i^{-1}}\beta &= (x_{i+1}^{-1}x_ix_{i+1})\beta \\ &= x_{i+1}^{-1}\beta x_i\beta x_{i+1}\beta \\ &= A_{i+1}x_{(i+1)\mu}A_{i+1}^{-1} \cdot A_ix_{i\mu}A_i^{-1} \cdot A_{i+1}x_{(i+1)\mu}A_{i+1}^{-1} \\ &= A_{i+1}B_i^{-1}x_{i\mu}B_iA_{i+1}^{-1}\end{aligned}$$

Consider

$$x_j\overline{\sigma_i^{-1}}\beta = \begin{cases} x_j\beta = A_jx_{j\mu}A_j^{-1} & , \text{ if } j \neq i, i+1 \\ A_{i+1}x_{(i+1)\mu}A_{i+1}^{-1} & , \text{ if } j = i \\ A_{i+1}B_i^{-1}x_{i\mu}B_iA_{i+1}^{-1} & , \text{ if } j = i+1 \end{cases}$$

Then

$$\begin{aligned}\ell(\overline{\sigma_i^{-1}}\beta) &= \sum_{j=1}^{i-1} A_j + \ell(A_{i+1}) + \ell(A_{i+1}B_i^{-1}) + \sum_{j=i+2}^{n} A_j \\ &< \sum_{j=1}^{i-1} A_j + \ell(A_{i+1}) + \ell(A_i) + \sum_{j=i+2}^{n} A_j \\ &= \ell(\beta)\end{aligned}$$

Since $\ell(\overline{\sigma_i^{-1}}\beta)$ is less than $\ell(\beta) = m$, we know that we can find a braid $(\overline{\sigma_i^{-1}}\beta)_0$ such that $\overline{(\overline{\sigma_i^{-1}}\beta)_0} = \overline{\sigma_i^{-1}}\beta$. Then define $\beta_0$ to be $\sigma_i(\overline{\sigma_i^{-1}}\beta)_0$. Then

$$\overline{\beta_0} = \overline{\sigma_i(\overline{\sigma_i^{-1}}\beta)_0} = \overline{\sigma_i} \circ \overline{(\overline{\sigma_i^{-1}}\beta)_0} = \overline{\sigma_i} \circ \overline{\sigma_i^{-1}} \circ \beta = \beta$$

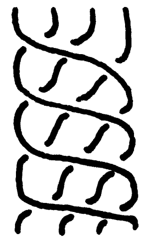

FIGURE 10

so that $\beta_0$ is the desired braid.

If no cancellation occurs, then $A_i$ is the empty word for all $i$, but then $\beta$ is the identity endomorphism, in which case $\ell(\beta)$ would be 0 not $m$. So given any endomorphism satisfying (1) and (2) we can find a braid $\beta_0$ such that $\Phi(\beta_0) = \beta$. This shows that $\Phi$ is surjective. Further, we know that $\Phi$ is invertible, and thus injective. And we finally conclude that $\Phi$ is the desired isomorphism. □

## 4. Presentation Theorem

Previously, I noted a couple of relations that hold in the braid group. I also noted that the braid group is generated by $\sigma_i$ for $1 \leq i \leq n-1$. The following theorem asserts that these generators and relations constitute the presentation of the braid group. I will not prove it here.

**Theorem 4.1.** *The group $B_n$ has defining relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$ on the generators $\sigma_1, \ldots, \sigma_{n-1}$. In other words,*

$$B_n =< \sigma_1, \ldots, \sigma_n; \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-1 >$$

Using these relations, we can more easily see whether two braids are equivalent or not. Given a braid

$$\sigma = \sigma_{i_1} \ldots \sigma_{i_k}$$

we can perform a couple of "moves" to change it into an equivalent braid. If we see $\sigma_i\sigma_j$ where $|i-j| \geq 2$ we can replace it with $\sigma_j\sigma_i$. If we see $\sigma_i\sigma_{i+1}\sigma_i$ we can replace it with $\sigma_{i+1}\sigma_i\sigma_{i+1}$ and vice versa. This procedure gives us the following exercise.

**Definition 4.2.** The center $C$ of a group $G$ is $\{x \in G \mid \forall g \in G,\ xg = gx\}$.

**Corollary 4.3.** *The center of the braid group $B_n$ is $< (\sigma_1 \ldots \sigma_{n-1})^n >$. For example, $(\sigma_1 \ldots \sigma_3)^4 \in B_5$ is seen in Figure 10.*

## 5. Build-up to Theorem of Artin and Birman

We would like to prove theorems about the fundamental groups of the complements of links. This can be made easier by associating links with braids and vice versa.

Braids into Links: Let $\sigma \in B_n$ be given. We can make a link by connecting $P_i$ with $Q_i$ for each $1 \leq i \leq n$. This essentially turns the paths from $P_i$ to $Q_{i\mu}$ into loops starting at $P_i$. The loop would go from

$$P_i \to Q_{i\mu} \to P_{i\mu} \to Q_{i\mu\mu} \to \cdots \to P_i$$

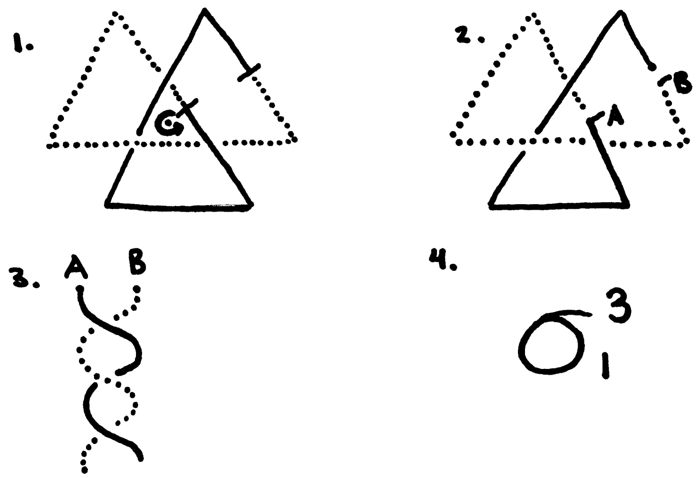


FIGURE 11. 1. The link is cut along an plane from the axis placed in the center. 2. The link is cut into three strands. 3. These strands are straightened out into a braid (4.)

The union of loops, or images of circles, forms a link. If we install an axis perpendicular to the $yz$-plane, we essentially glue strings together to form a link that encircles the axis. Given a braid $\sigma \in B_n$, we will call the link created in this way $L(\sigma)$.

Links into Braids: This process is pretty much the inverse of the other. We can again install an axis such that the link encircles the axis in a counterclockwise direction (counterclockwise if we look at the construction from a point whose $x$-coordinate is less than that of any of the points involved in the braid) and then, rather than gluing, start at the axis and move downward, cutting apart paths. This precise algorithm need not necessarily be followed. Given any link $L'$ we can find a link $L$ such that $L'$ and $L$ are string isotopic and such that there is a braid $\sigma$ so that $L(\sigma) = L$. Of course, $L'$ might be $L$.

**Examples 5.1.** Let $L$ be the left-handed trefoil. Then if $\sigma = \sigma_1^3 \in B_2$, $L = L(\sigma)$. See Figure 11.

**Examples 5.2.** Let $L$ be the figure eight knot, Then if $\sigma = (\sigma_2^{-1}\sigma_1)^2 \in B_3$, $L = L(\sigma)$. This link is turned into braid without using the algorithm given. This process is easier to see if we start with the braid and turn it into a link by gluing together the ends. Look at the two links in Figure 12. Starting at a joint, if one works around the link, reading, for example, "dots over squiggly" as appropriate, the two links are the same. For example, if we start at the dots and squiggly joint and follow the dotted line, we read "dots under squiggly," "dots over solid," "dots under squiggly," "solid over squiggly," "solid under dots," squiggly over dots," "squiggly under solid," "squiggly over dots," and then we find ourselves back where we began. This loop reads exactly the same in either diagram.

**Examples 5.3.** Let $L$ be the Borromean Rings. Then if $\sigma = (\sigma_2^{-1}\sigma_1)^3) \in B_3$, $L = L(\sigma)$. See Figure 13.

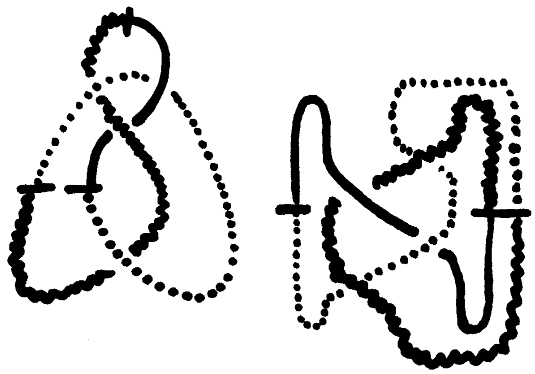

FIGURE 12

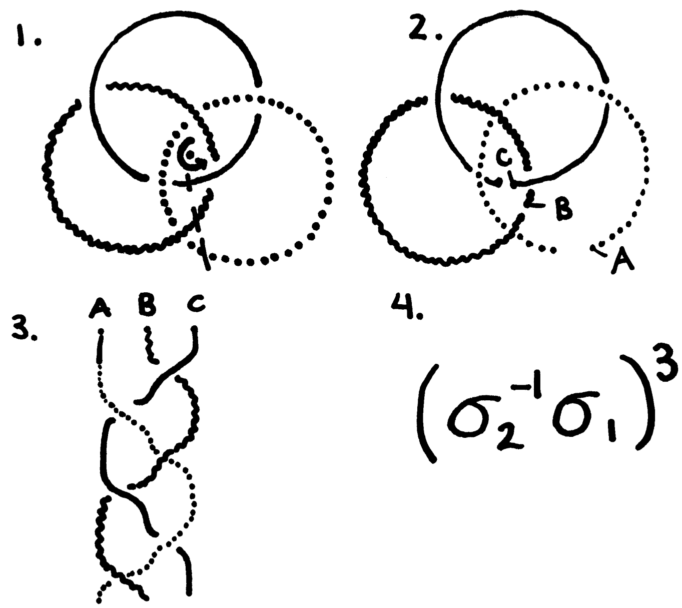


FIGURE 13. 1. The link is cut along a plane from the axis placed in the center. 2. The link is cut into three strands. 3. These strands are straightened out into a braid (4.)

**Examples 5.4.** Let $L$ be the Miller Institute link. Then if $\sigma = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^3$, $L = L(\sigma)$. With this link as well, it is easier to start by turning the braid into a link and then reading the lines around the link. The loop will be the same on either diagram. See Figure 14.

These results and others are summarized in the tables in Section 7.

**Note 5.5.** There is a little group theory that I will need in the following section. Suppose a group $G$ has presentation

$$G =< x_1, \dots, x_n; y_1 = e, \dots, y_m = e >$$

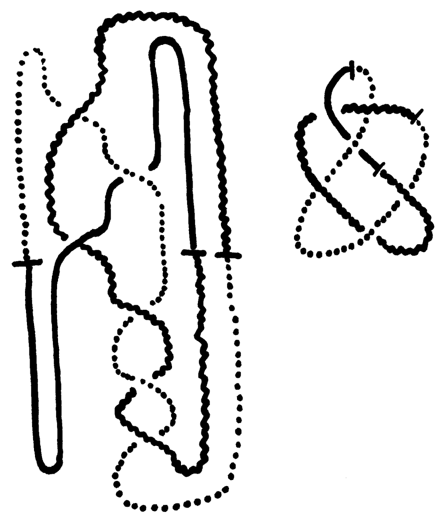

FIGURE 14

We can use Tietze transformations to simplify the presentation.

(1) If we can express an element $x$ in terms of the generators $x_1, \dots, x_n$ we can add $x$ to the generators of $G$ without affecting the group. So we can write
$$< x_1, \dots, x_n, x; y_1 = e, \dots, y_m = e > = < x_1, \dots, x_n; y_1 = e, \dots, y_m = e >$$
if $x$ can be given in the other generators. Let ADG denote this transformation in which we add a generator. For example,
$$< x; - > = < x, x^2; - >$$
and
$$< x_1, x_2; x_2x_1 = x_1x_2 > = < x_1, x_2, y; x_2x_1 = x_1x_2, y = x_1x_2x_1 >$$

(2) If an equation $y = z$ follows from the relations given, we can add $y = z$ to the relations of $G$ without affecting the group. So we can write
$$< x_1, \dots, x_n; y_1 = e, \dots, y_m = e, y = z >$$
if $y = z$ follows from the first $m$ relations. Let ADR denote this transformation in which we add a relation. For example,
$$< x_1, x_2; x_1^2 = e, x_2^3 = e > = < x_1, x_2; x_1^2 = e, x_2^3 = e, x_1^{-2}x_2^6 = e >$$

(3) If a generator is a word in the other generators, we can remove it from the generators. Further, we must replace any instance of it with the equivalent word. Let REG denote this transformation in which we remove a generator. For example,
$$< x_1, x_2, x_3; x_1 = x_2x_3, x_1^2 = e > = < x_2, x_3; (x_2x_3)^2 = e >$$

(4) If a relation follows from the others, we can remove it from the presentation. Let RER denote this transformation. For example,
$$< x_1, x_2; x_1x_2 = x_2x_1, x_1x_2x_1^{-1}x_2^{-1} = e > = < x_1, x_2; x_1x_2 = x_2x_1 >$$

**Lemma 5.6.** *Let*

$$G =< \ell, x_1, \ldots, x_n; \ell x_1 \ell^{-1} = x_1\overline{\sigma}, \ldots, \ell x_n \ell^{-1} = x_n\overline{\sigma} >$$

*where $\sigma \in B_n$ and $\overline{\sigma}$ is the automorphism as given in Theorem 3.2. Suppose the group has the property that $\ell x_i = x_i \ell$ for any $x_i$. Then $G$ is the semi-direct product of $< \ell; - >$ and $F_n$.*

*Proof.* Let $L =< \ell; - >$. For $G$ to be a semi-direct product of $L$ and $F_n$, they both need to be subgroups of $G$ and one of them needs to be normal. It is clear that $L$ is a subgroup. The identity element $e = \ell^0$. Given any $\ell^k, \ell^m \in L$, $\ell^k \ell^m = \ell^{k+m} \in L$. Finally given any $\ell^k \in L$, $\ell^{-k} \in L$ as well and $\ell^k \ell^{-k} = e$ so that all of the elements of $L$ have inverses in $L$. Then $L$ is a subgroup. We already know that $F_n$ contains the empty word or identity element, is closed, and that all of its elements have inverses in the group. So that $F_n$ is certainly a subgroup of $G$. Further, we want to show that $F_n$ is a normal subgroup of $G$. Let $g \in G$ be given. Since by assumption, $\ell x_i = x_i \ell$, every $g \in G$ can be written as $\ell^m w$ for $m \in \mathbb{Z}$ and $w \in F_n$. So $g = \ell^m w$. Take any $w_0 \in F_n$. Since we can move the $\ell$'s as we please,

$$g w_0 g^{-1} = \ell^m w w_0 w^{-1} \ell^{-m} = \ell^m \ell^{-m} w w_0 w^{-1} = w' \in F_n$$

Since $g w_0 g^{-1} \in F_n$, we conclude that $gF_n = F_n g$ and that $F_n$ is a normal subgroup of $G$. Now to look at $G/F_n$. Take $g_1, g_2 \in G$ so that $g_1 = \ell^k w_1, g_2 = \ell^m w_2$ and

$$g_1 g_2^{-1} = \ell^k w_1 w_2^{-1} \ell^{-m}$$

If $k \neq m$, $g_1 g_2^{-1} = \ell^{k-m} w_1 w_2^{-1} \notin F_n$. However, if $k = m$,

$$g_1 g_2^{-1} = \ell^{k-m} w_1 w_2^{-1} = w_1 w_2^{-1} \in F_n$$

So that two elements of $G$ are equivalent modulo $F_n$ if $\ell$ appears to the same power in both elements. We can then see that $G/F_n \cong L$. Since there is clearly an isomorphism from $L$ to $G/F_n$, $G$ is the semi-direct product of its subgroups $L$ and $F_n$. □

## 6. Theorem of Artin and Birman

We have removed braids from planes and worked with the resulting spaces. We now want to remove links from space and work with the resulting space.

**Notation 6.1.** To be clear, when talking about toruses: We can cut a torus horizontally, so that the cross section is an annulus, and a loop around the cross-section is a longitudinal loop or simply a longitude. We can cut a torus vertically, so that the cross section is a disc, and a loop around the cross-section is a meridian.

In the next big theorem, we want to find the fundamental group of the complement of a link in $\mathbb{R}^3$. We will call this the group of a link. In other words, the group of a link $L$, $G(L) = \Pi(C_{\mathbb{R}^3}(L))$. We have also seen that every link $L$ has an associated braid $\sigma$. Because of this, we can work with links of braids, $L(\sigma)$ rather than links in the abstract.

**Theorem 6.2.** *Suppose $\sigma \in B_n$,. Then the group $G(L(\sigma))$ of the link $L(\sigma)$ has a presentation of the form*

$$< x_1, \ldots, x_n; x_1 = x_1\overline{\sigma}, \ldots, x_n = x_n\overline{\sigma} >$$

*where $\overline{\sigma}$ is the automorphism of $F_n$ as defined in Theorem 3.2.*

*Proof.* Let $\sigma \in B_n$ be given. We create a cylinder $S$ whose faces are in the planes $z = a$ and $z = b$ and such that $\sigma \subset S \subset \mathbb{R}^3$. We make a straight line path from the point $Q$ to the point $P$. This path will be $\ell$. As we removed $\sigma$ in the proof of Theorem 3.2, we remove $\sigma$ from $S$ and consider $C_S(\sigma)$. In $C_S(\sigma)$, $\ell x_i \ell^{-1}$ is string isotopic to $x_i\overline{\sigma}$, the same automorphism from Theorem 3.2. Both loops have $Q$ as base points and the isotopy is provided by the pushing down process. We can now make a torus $T$ out of $S$ by connecting the two faces and associating $P_i$ with $Q_i$ for every $i$. In so doing, we also turn $\sigma$ into $L(\sigma)$. There is a natural continuous and invertible map $f$ from $C_S(\sigma)$ to $C_T(L(\sigma))$. (Keep in mind that $f(P) = P$). Theorem 2.1 gives us a natural group isomorphism

$$f_\Pi : \Pi(C_S(\sigma), P) \to \Pi(C_T(L(\sigma)), P)$$

We have already seen that the fundamental group of the plane $z = a$ with $n$ holes is isomorphic to $F_n$ or $< x_1, \ldots, x_n; - >$. In the torus we can also make a longitudinal loop which is actually the path $\ell$ now that the ends have been identified. So $C_T(L(\sigma))$ is generated by $< \ell, x_1, \ldots, x_n >$. Since $\Pi(C_T(L(\sigma)), P)$ and $\Pi(C_S(\sigma), P)$ are isomorphic, relations in $\Pi(C_S(\sigma), P)$ must also hold in $\Pi(C_T(L(\sigma)), P)$. We have seen that $\ell x_i \ell^{-1} = x_i\overline{\sigma}$ in $\Pi(C_S(\sigma), P)$ so that this must hold in $\Pi(C_T(L(\sigma)), P)$. However, there might be other relations we don't know about. We will show that these are the only generators and relations. Let

$$G =< \ell, x_1, \ldots, x_n; \ell x_1 \ell^{-1} = x_1\overline{\sigma}, \ldots, \ell x_n \ell^{-1} = x_n\overline{\sigma} >$$

We want to show that $G = \Pi(C_T(L(\sigma)), P)$. There is a natural homomorphism from $G$ to $\Pi(C_T(L(\sigma)), P)$. In the fundamental group of a torus, the meridian and longitude are commutative, so that every element of $G$ can be written as $\ell^m w$ where $m \in \mathbb{Z}$ and $w \in F_n$. If $\ell^m w$ maps to the identity under the homomorphism, it is easy to see that $m$ must be 0 and the word must be the identity. So that the homomorphism is injective. Since the natural homomorphism sends generators to generators, the homomorphism must be surjective. So $G \cong \Pi(C_T(L(\sigma)), P)$, and

$$\Pi(C_T(L(\sigma)), P) =< \ell, x_1, \ldots, x_n; \ell x_1 \ell^{-1} = x_i\overline{\sigma}, \ldots, \ell x_n \ell^{-1} = x_n\overline{\sigma} >$$

Now we are going to mess around with the torus a little bit. Let $T^0$ denote the interior of the torus $T$. Then we find a slightly smaller solid torus $T'$ so that $T' \subset T^0$. We also make sure that $L(\sigma) \subset T'$ and $P \notin T'$. With these new toruses we can do some topology magic. It is clear from our setup that

$$C_{\mathbb{R}^3}(L(\sigma)) = C_{\mathbb{R}^3}(T') \cup C_{T^0}(L(\sigma));$$
$$C_{T^0}(T') = C_{\mathbb{R}^3}(T') \cap C_{T^0}(L(\sigma));$$
and that
$$P \in C_{T^0}(T')$$

These conditions satisfy Theorem 2.3, Van Kampen's Theorem so that

$$\Pi(C_{\mathbb{R}^3}(L(\sigma))) \cong \Pi(C_{\mathbb{R}^3}(T'))^{(f_1)_\Pi} \underset{\Pi(C_{T^0}(T'))}{*}{}^{(f_2)_\Pi}\, \Pi(C_{T^0}(L(\sigma)))$$

where $f_1, f_2$ are the natural maps from $C_{T^0}(T')$ to $C_{\mathbb{R}^3}(T'), C_{T^0}(L(\sigma))$ respectively that give rise to the group homomorphisms through Theorem 2.1. So now we just have to evaluate the three fundamental groups on the right.

$C_{\mathbb{R}^3}(T')$ is space with a donut removed. Two loops in $\Pi(C_{\mathbb{R}^3}(T'))$ are homotopic if they loop the donut the same number of times in the same direction. A loop around the donut is a meridian of the torus. So it is easy to see that

$$\Pi(C_{\mathbb{R}^3}(T')) \cong < m; - >$$

where $m$ stands for a meridian around the removed torus.

$C_{T_0}(T')$ is a solid torus with an inner tube removed. $C_{T^0}(T') = A \times S^1$ where $A$ is an open annulus and $S^1$ is the unit circle. Then by Theorem 2.2,

$$\Pi(C_{T^0}(T')) \cong \Pi(A) \times \Pi(S^1)$$

The fundamental groups of the annulus and the circle are essentially the same- the infinite cyclic group. So we write that $\Pi(A) =< M; - >$ where $M$ stands for a meridian since a loop around the annulus is a meridian of the torus. Similarly, $\Pi(S^1) =< L; - >$ where $L$ stands for a longitude since a loop around $S^1$ is a longitudinal loop around the torus. So we find that

$$\Pi(C_{T^0}(T')) =< L; - > \times < M; - >$$

From all of our work we have found that

(1) Since $\Pi(C_T(L(\sigma)))$ and $\Pi(C_{T^0}(L(\sigma)))$ are isomorphic, we can say that

$$\Pi(C_{T^0}(L(\sigma))), P) \cong < \ell, x_1, \ldots, x_n; \ell x_1, \ell^{-1} = \overline{\sigma}, \ldots, \ell x_n \ell^{-1} = x_n \overline{\sigma} >$$

(2) $\Pi(C_{\mathbb{R}^3}(T')) \cong < m; - >$

(3) $\Pi(C_{T^0}(T')) \cong < L; - > \times < M; - >$

To evaluate $\Pi(C_{\mathbb{R}^3}(L(\sigma)))$, we need to consider the homomorphisms $(f_1)_\Pi, (f_2)_\Pi$. Remember that

$$(f_1)_\Pi : \Pi(C_{T_0}(T')) \to \Pi(C_{\mathbb{R}^3}(T'))$$
$$(f_2)_\Pi : \Pi(C_{T_0}(T')) \to \Pi(C_{T_0}(L(\sigma)))$$

The only generators of $C_{T^0}(T')$ are $L$ and $M$ so those are the only things we need to check with the homomorphisms. We can see from the topology that $(f_1)_\Pi(L) = e$ and $(f_2)_\Pi(L) = \ell$. Thus in the free product amalgamating $\Pi(C_{T^0}(T'))$, these elements are identified. Similarly, since $(f_1)_\Pi(M) = m$ and $(f_2)_\Pi(M) = x_1 \ldots x_n$, in the free product amalgamating $\Pi(C_{T^0}(T'))$ we identify $m$ and $x_1 \ldots x_n$. To put together $\Pi(C_{\mathbb{R}^3}(L(\sigma)))$, we gather the generators of $\Pi(C_{\mathbb{R}^3}(T')$ and $\Pi(C_{T^0}(L(\sigma)))$. We also gather the relations of the two subgroups, along with the relations found from the two homomorphisms. $\Pi(C_{\mathbb{R}^3}(L(\sigma)))$ is the group of the link $L(\sigma)$. So that

$$G(L(\sigma)) =< \ell, x_1, \ldots, x_n, m; \ell x_1 \ell^{-1} = x_1 \overline{\sigma}, \ldots, \ell x_n \ell^{-1} = x_n \overline{\sigma}, \ell = e, m = x_1 \ldots x_n >$$

where $\overline{\sigma}$ is the automorphism of $F_n$ as specified in Theorem 3.2.

There are a couple of details left to iron out. In the statement of the theorem,

$m$ and $\ell$ do not appear. So we can invoke Note 5.5 and apply Tietze transformations to get rid of them. Giving us

$$\begin{aligned}
&G(L(\sigma)) =< \ell, x_1, \ldots, x_n, m; x_1 = x_1\overline{\sigma}, \ldots, x_n = x_n\overline{\sigma}, m = x_1 \ldots x_n > \\
&G(L(\sigma)) =< x_1, \ldots, x_n, m; x_1 = x_1\overline{\sigma}, \ldots, x_n = x_n\overline{\sigma}, m = x_1 \ldots x_n > && \text{using REG} \\
&G(L(\sigma)) =< x_1, \ldots, x_n, x_1 \ldots x_n; x_1 = x_1\overline{\sigma}, \ldots, x_n = x_n\overline{\sigma} > && \text{using REG} \\
&G(L(\sigma)) =< x_1, \ldots, x_n; x_1 = x_1\overline{\sigma}, \ldots, x_n = x_n\overline{\sigma} > && \text{using REG}
\end{aligned}$$

This works for links made from braids. But every link is isotopic to the link of a braid from previous work. So we can find the group for any link in the form desired. ☐

**Corollary 6.3.** *Given a link $L$, for some $\sigma \in B_n$ the group of the link is*

$$< x_1, \ldots, x_n; x_1 = x_1\overline{\sigma}, \ldots, x_{i-1} = x_{i-1}\overline{\sigma}, x_{i+1} = x_{i+1}\overline{\sigma}, \ldots, x_n = x_n\overline{\sigma} > \text{ for some } 1 \leq i \leq n$$

*Proof.* Given $\sigma \in B_n$, we know that $\overline{\sigma}$ is an automorphism of $F_n$ such that

$$x_1\overline{\sigma} \ldots x_n\overline{\sigma} = (x_1 \ldots x_n)\overline{\sigma} = x_1 \ldots x_n$$

Let $1 \leq i \leq n$ be given. Then in $G(L(\sigma))$, we know that $x_j = x_j\overline{\sigma}$ for each $j \neq i$ because of the relations of the group. So that

$$\begin{aligned}
x_1 \ldots x_{i-1} x_i\overline{\sigma} x_{i+1} \ldots x_n &= x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_n \\
&\Rightarrow x_i\overline{\sigma} = x_i
\end{aligned}$$

Thus from the other $n-1$ relations, we can deduce any last one. This means that, practically, when computing the groups of links, we can disregard the most complicated relation and work with the others. ☐

**Examples 6.4.** Let $L$ be the left-handed trefoil. By previous work we know that if $\sigma = \sigma_1^3 \in B_2$, then $L = L(\sigma)$ and we know the automorphism corresponding to $\sigma$. Then by the Theorem of Artin and Birman, we know that

$$\begin{aligned}
&G(L) =< x_1, x_2; x_1 = x_1x_2x_1x_2x_1^{-1}x_2^{-1}x_1^{-1}, x_2 = x_1x_2x_1x_2^{-1}x_1^{-1} > \\
&\Rightarrow G(L) =< x_1, x_2; x_2 = x_1x_2x_1x_2^{-1}x_1^{-1} > && \text{by Corollary 5.3} \\
&\Rightarrow G(L) =< x_1, x_2; x_2x_1x_2 = x_1x_2x_1 > \\
&\Rightarrow G(L) \cong B_2 && \text{by Theorem 4.1}
\end{aligned}$$

**Examples 6.5.** Let $L$ be the Hopf Link. Then the braid we want to work with is $\sigma_1^2 \in B_2$, and we know the automorphisms associated with this braid by previous work. So

$$\begin{aligned}
&G(L) =< x_1, x_2; x_1 = x_1x_2x_1x_2^{-1}x_1^{-1}, x_2 = x_1x_2x_1^{-1} > \\
&\Rightarrow G(L) =< x_1, x_2; x_2 = x_1x_2x_1^{-1} > && \text{by Corollary 5.3} \\
&\Rightarrow G(L) =< x_1, x_2; x_2x_1 = x_1x_2 > \\
&\Rightarrow G(L) \cong \mathbb{Z}^2
\end{aligned}$$

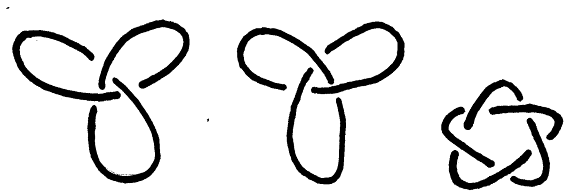

FIGURE 15. Left-Handed Trefoil on the left and Right-Handed Trefoil in the middle. Solomon's Seal on the right. All powers of $\sigma_1$ in $B_2$.

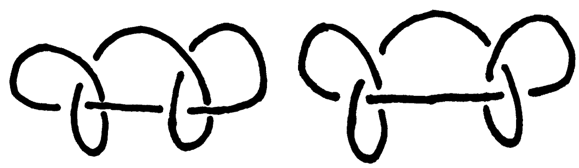

FIGURE 16. The closely related Granny Knot (left) and Square Knot (right).

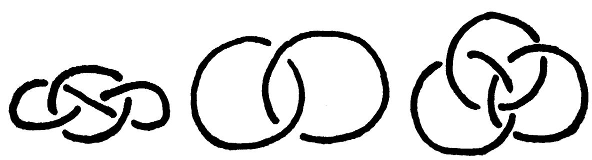

FIGURE 17. Two two-component links, the Whitehead (left) and Hopf (middle) links. A three-component link, the Borromean Rings on the right.

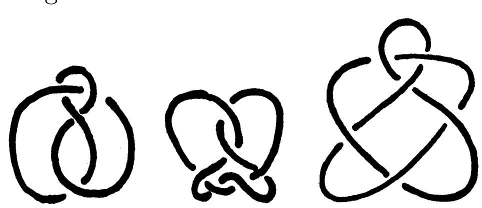

FIGURE 18. Three more complicated knots. From left to right, the Figure Eight, Stevedore's, and Miller Institute.

**Examples 6.6.** Let $L$ be the Borromean Rings. Then the braid is $(\sigma_2^{-1}\sigma_1)^3$ and the group is as follows:

$$< x_1, x_2, x_3; x_1 = x_1x_2x_1^{-1}x_3x_1x_2^{-1}x_1^{-1}x_3^{-1}x_1x_3x_1x_2x_1^{-1}x_3^{-1}x_1x_2^{-1}x_1^{-1}, x_2 = x_3^{-1}x_1^{-1}x_3x_1x_2x_1^{-1}x_3^{-1}x_1x_3 >$$
$$=< x_1, x_2, x_3; x_1 = x_1x_2x_3x_2^{-1}x_1x_2x_1^{-1}x_3^{-1}x_1x_2^{-1}x_1^{-1}, x_2 = x_3^{-1}x_1^{-1}x_3x_1x_2x_1^{-1}x_3^{-1}x_1x_3 >$$
$$=< x_1, x_2, x_3; x_1x_2x_1^{-1}x_3x_1 = x_2x_3x_2^{-1}x_1x_2, x_2 = x_3^{-1}x_1^{-1}x_3x_1x_2x_1^{-1}x_3^{-1}x_1x_3 >$$
$$=< x_1, x_2, x_3; x_1x_2x_1^{-1}x_3x_1 = x_2x_3x_2^{-1}x_1x_2, x_2x_3^{-1}x_1^{-1}x_3x_1 = x_3^{-1}x_1^{-1}x_3x_1x_2 >$$

## 7. Putting It All Together

Before proving the Theorem of Artin and Birman, we learned that links and braids are interchangeable. For pictures of several links, see Figures 15 through 18. So given a link, we can decompose it into a braid. In proving the Artin Representation Theorem, we found an automorphism of the free group $F_n$ associated with every braid. So we can start with a link, cut it apart into a braid, and then find an automorphism. Given this automorphism, the Theorem of Artin and Birman tells us how to find the fundamental group of a link in $\mathbb{R}^3$. Through a series of simple computations, we compiled the following tables.

| **Link** | **Braid** | **Automorphism** |
|---|---|---|
| Left-handed Trefoil | $\sigma_1^3$ | $1\mu = 2, A_1 = x_1x_2x_1$<br>$2\mu = 1, A_2 = x_1x_2$ |
| Right-handed Trefoil | $\sigma_1^{-3}$ | $1\mu = 2, A_1 = x_2^{-1}x_1^{-1}$<br>$2\mu = 1, A_2 = x_2^{-1}x_1^{-1}x_2^{-1}$ |
| Solomon's Seal | $\sigma_1^5$ | $1\mu = 2, A_1 = x_1x_2x_1x_2x_1$<br>$2\mu = 1, A_2 = x_1x_2x_1x_2$ |
| Granny Knot | $\sigma_1^{-3}\sigma_2^3$ | $1\mu = 3, A_1 = x_2x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}x_2^{-1}x_1^{-1}x_2x_3x_2$<br>$2\mu = 1, A_2 = x_2x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}x_2^{-1}x_1^{-1}x_2x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}x_2^{-1}$<br>$3\mu = 2, A_3 = x_2x_3$ |
| Square Knot | $\sigma_1^{-3}\sigma_2^{-3}$ | $1\mu = 3, A_1 = x_3^{-1}x_2^{-1}x_3^{-1}x_2x_3x_1^{-1}x_3^{-1}x_2^{-1}$<br>$2\mu = 1, A_2 = x_3^{-1}x_2^{-1}x_3^{-1}xx_2x_3x_1^{-1}x_3^{-1}x_2^{-1}x_3^{-1}x_2x_3$<br>$3\mu = 2, A_3 = x_3^{-1}x_2^{-1}x_3^{-1}$ |
| Hopf Link | $\sigma_1^2$ | $1\mu = 1, A_1 = x_1x_2$<br>$2\mu = 2, A_2 = x_1$ |
| Whitehead Link | $(\sigma_1\sigma_2^{-1})^2\sigma_2$ | $1\mu = 3, A_1 = x_1x_3^{-1}x_2x_3x_1^{-1}x_3^{-1}x_2^{-1}$<br>$2\mu = 2, A_2 = x_1x_3^{-1}$<br>$3\mu = 1, A_3 = x_3^{-1}x_2^{-1}x_3^{-1}x_2x_3$ |
| Borromean Rings | $(\sigma_2^{-1}\sigma_1)^3$ | $1\mu = 1, A_1 = x_1x_2x_1^{-1}x_3x_1x_2^{-1}x_1^{-1}x_3^{-1}$<br>$2\mu = 2, A_2 = x_3^{-1}x_1^{-1}x_3x_1$<br>$3\mu = 3, A_3 = x_3^{-1}x_1^{-1}x_3x_1x_2^{-1}x_1^{-1}x_3^{-1}x_1x_3x_1x_2x_1^{-1}$ |
| Figure Eight Knot | $(\sigma_2^{-1}\sigma_1)^2$ | $1\mu = 3, A_1 = x_1x_2x_1^{-1}$<br>$2\mu = 1, A_2 = x_3^{-1}$<br>$3\mu = 2, A_3 = x_3^{-1}x_1^{-1}x_3x_1$ |
| Stevedore's Knot | $\sigma_2^{-1}\sigma_1^3\sigma_2^{-1}\sigma_1$ | $1\mu = 3, A_1 = x_1x_2x_1^{-1}x_3x_1x_2x_1^{-1}$<br>$2\mu = 1, A_2 = x_3^{-1}$<br>$3\mu = 2, A_3 = x_3^{-1}x_1^{-1}x_3x_1x_2x_1^{-1}x_3x_1$ |
| Miller Institute | $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^3$ | $1\mu = 2, A_1 = x_2x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}x_2^{-1}x_1x_2x_3x_2x_3$<br>$2\mu = 3, A_2 = x_2x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}z_2^{-1}x_1x_2x_3x_2x_3$<br>$x_2^{-1}x_3^{-1}x_2^{-1}x_3^{-1}x_2^{-1}x_1^{-1}x_2x_3x_2$<br>$3\mu = 1, A_3 = x_2x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}x_2^{-1}$ |

| **Link, L** | **Group, G(L)** |
|---|---|
| Left-Handed Trefoil | $< x_1, x_2; x_2x_1x_2 = x_1x_2x_1 > \cong B_3$ |
| Right-Handed Trefoil | $< x_1, x_2; x_1x_2x_1 = x_2x_1x_2 > \cong B_3$ |
| Solomon's Seal | $< x_1, x_2; x_1x_2x_1x_2x_1 = x_2x_1x_2x_1x_1 >$ |
| Granny Knot | $< x_1, x_2, x_3; x_1x_2^{-1}x_1^{-1} = x_2^{-1}x_1^{-1}x_2, x_3x_2x_3 = x_2x_3x_2 >$ |
| Square Knot | $< x_1, x_2, x_3; x_1x_2^{-1}x_1^{-1}x_3^{-1}x_2^{-1} = x_2^{-1}x_1^{-1}x_3^{-1}x_2^{-1}x_1, x_2x_3^{-1}x_2^{-1} = x_3^{-1}x_2^{-1}x_3 >$ |
| Hopf Link | $< x_1, x_2; x_1x_2 = x_2x_1 > \cong \mathbb{Z}^2$ |
| Whitehead Link | $< x_1, x_2, x_3; x_2x_1x_3^{-1} = x_1x_3^{-1}x_2, x_2^{-1}x_3^{-1}x_2x_3x_1 = x_3^{-1}x_2^{-1}x_3^{-1}x_2x_3 >$ |
| Borromean Rings | $< x_1, x_2, x_3; x_2x_3x_2^{-1}x_1x_2 = x_1x_2x_1^{-1}x_3x_1, x_2x_3^{-1}x_1^{-1}x_3x_1 = x_3^{-1}x_1^{-1}x_3x_1x_2 >$ |
| Figure Eight Knot | $< x_1, x_2, x_3; x_1x_2x_1^{-1} = x_2x_1^{-1}x_3, x_3x_2 = x_1x_3 >$ |
| Stevedore's Knot | $< x_1, x_2, x_3; x_2x_3^{-1} = x_3^{-1}x_1, x_2x_3x_2^{-1}x_1x_2x_1^{-1} = x_1x_2x_3x_2^{-1}x_1x_2x_1^{-1}x_3^{-1} >$ |
| Miller Institute | $< x_1, x_2, x_3; x_1 = x_2^2x_3x_2x_3^{-1}x_2^{-1}, x_2x_3x_2 = x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}x_2^{-1}x_1x_2x_3x_2x_3 >$ |

There is a collection of knots known as the prime knots, from which all other knots can be made. These include the trefoil knots, Solomon's seal knot, the figure eight knot, Stevedore's knot, and the Miller Institute knot. These important knots are found in the above tables.

## 8. Connections to Permutations

We have been discussing several kinds of knots and using them as examples of our theorems. See Figure 15 through 18 for diagrams of the most important knots. There are interesting connections between the braid group, $B_n$, and the group of permutations of n elements, $S_n$. There is a homomorphism from $B_n$ to $S_n$. As noted in the definition of braids, each braid $\sigma$ has a corresponding permutation $\mu$ that denotes where each of the $n$ strings ends up. Let $p : B_n \to S_n$ where $p(\sigma)$ is the permutation associated with $\sigma$. More formally, for any $1 \leq i \leq n-1$, $p(\sigma_i) = (i, i+1)$. So that the braid that switches two strings corresponds to the permutation that switches the same two elements. It will be useful later that $p$ is a homomorphism. Take $\sigma, \sigma' \in B_n$ so that $p(\sigma) = \mu$ and $p(\sigma') = \mu'$. In $\sigma$, the string from $P_i$ goes to $Q_{i\mu}$. Then in $\sigma'$, the string from $Q_{i\mu}$ goes to $R_{(i\mu)\mu'}$. Then in $\sigma\sigma'$, the string from $P_i$ ultimately ends up at $R_{i\mu\mu'}$ so that $\mu\mu'$ is the permutation associated with the product of the two individual braids. So we see that $p(\sigma\sigma') = p(\sigma)p(\sigma')$ where the order of composition is from left to right. Since $p$ is a homomorphism, $p(\sigma^{-1}) = p(\sigma)^{-1}$ for all $\sigma \in B_n$.

**Definition 8.1.** If a braid $\rho \in B_n$ is such that $p(\sigma) = e$, the identity element of $S_n$, $\rho$ is a pure braid. In other words, the i-th string of $\rho$ goes from $P_i$ to $Q_i$ for each $1 \leq i \leq n$. We denote the subset of pure braids $P_n \subset B_n$.

**Theorem 8.2.** $B_n/P_n \cong S_n$

*Proof.* In order to take the quotient of $B_n$ modulo $P_n$ we first need to know that $P_n$ is a normal subgroup of $B_n$.

Subgroup: Clearly, the identity or unit braid is a pure braid. If $\rho, \rho' \in P_n$,

$$p(\rho) = p(\rho') = e \in S_n$$

Then the permutation associated with $\rho\rho'$ is

$$p(\rho\rho') = p(\rho)p(\rho') = e \circ e = e$$

so that $\rho\rho' \in P_n$. It is easy to see that concatenating two pure braids results in another pure braid. Further, given $\rho \in P_n$,

$$p(\rho^{-1}) = p(\rho)^{-1} = e^{-1} = e$$

so that $\rho^{-1} \in P_n$. It is also easy to see that to unwind a pure braid, we will need a braid that sends a string from $P_i$ to $Q_i$, another pure braid. We find that $P_n$ contains the identity braid, is closed, and contains inverses of all of its elements. We conclude that $P_n$ is a subgroup of $B_n$.

Normality: Let any $\sigma \in B_n$ be given. Then if $\rho$ is any pure braid,

$$p(\sigma\rho\sigma^{-1}) = p(\sigma)p(\rho)p(\sigma^{-1}) = p(\sigma)p(\sigma)^{-1} = e$$

so that $\sigma\rho\sigma^{-1} \in P_n$. This shows us that $\sigma P_n = P_n\sigma$ so that $P_n$ is a normal subgroup.

$B_n/P_n$ : In this quotient group, all $\rho \in P_n$ are now equivalent to the identity. Any two $\sigma, \sigma' \in B_n$ are in the same equivalence class modulo $P_n$ if and only if $\sigma^{-1}\sigma' \in P_n$. If $\sigma^{-1}\sigma' \in P_n$ we know that

$$\begin{aligned} & p(\sigma^{-1}\sigma') = e \\ \Rightarrow\ & p(\sigma^{-1})p(\sigma') = e && \text{since } p \text{ is a homomorphism} \\ \Rightarrow\ & p(\sigma)^{-1}p(\sigma') = e \\ \Rightarrow\ & p(\sigma') = p(\sigma) \end{aligned}$$

This shows us that braids in $B_n$ are put into equivalence classes modulo $P_n$ according to their associated permutations. Now, $p$ can be turned into an isomorphism from $B_n/P_n$ to $S_n$ since it has already been established as a homomorphism and the permutations to which $p$ maps a braid determines its equivalence class. If $\sigma \in B_n$ is given, let

$$\sigma_\mu = \{\sigma \in B_n \mid p(\sigma) = \mu\}$$

so that $\sigma_\mu$ is an equivalence class and define a map $p' : B_n/P_n \to S_n$ so that $\sigma_\mu \mapsto \mu$. By definition $p'$ is single-valued and injective. Since $p$ is a homomorphism, $p'$ is as well. Since $p(\sigma_i) = (i, i+1)$, $\sigma_{(i,i+1)} \neq \emptyset$ for all $1 \leq i \leq n$. Any permutation $\mu \in S_n$ can be written as a finite product of permutations $(i, i+1)$ for various $i$ where $1 \leq i \leq n$. If $\mu \in S_n$ is given as

$$(i_1, i_1+1)^{k_1} \dots (i_m, i_m+1)^{k_m} = \mu$$

for $k_i \in \mathbb{Z}$ then

$$p(\sigma_{i_1}^{k_1} \dots \sigma_{i_m}^{k_m}) = \mu$$

so that $\sigma_\mu \neq \emptyset$. So given any permutation in $S_n$ we can find a braid in $B_n$ and thus an equivalence class in $B_n/P_n$ that maps to the permutation. Thus $p'$ is surjective. And we have a bijective homomorphism, i.e. an isomorphism. We conclude that $B_n/P_n \cong S_n$. □

**Theorem 8.3.** *If $\sigma \in B_n$, the link $L(\sigma)$ made from $\sigma$ has $c$ components if and only if $p(\sigma) \in S_n$ can be expressed as the product of $c$ disjoint cycles. This implies that $L(\sigma)$ is a knot if and only if $p(\sigma)$ is one $n$-cycle.*

*Proof.* We know that the permutation $p(\sigma)$ signifies where the strings of the braid $\sigma$ end up. In creating $L(\sigma)$, we connect $P_i$ with $Q_i$ for every $1 \leq i \leq n$. A component of $L(\sigma)$ is constructed if we can start at $P_i$, travel to $P_{i\mu}$, then to $Q_{i\mu}$, then to $Q_{i\mu\mu}$, then to $P_{i\mu\mu}$, etc. and end up back at $P_i$. In other words, a component is a self-contained subset of strings of $L(\sigma)$. A cycle is a set of elements of $1, 2, \ldots, n$ where we start at an element $i$ and permute again and again until we return to $i$ once more. Again, this is a self-contained subset of elements. It is clear that the two definitions correspond. From which it follows immediately that a knot, or a link with one component, must correspond to a braid with one component. $\square$

## References

[1] Siegfried Moran. The Mathematical Theory of Knots and Braids: An Introduction. Elsevier Science Publishers. 1983.
[2] Peter Cromwell. Knots and Links. Cambridge University Press. 2004.