FREE GROUPS AND GRAPHS: THE HANNA NEUMANN THEOREM

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Abstract. The relationship between free groups and graphs is one of the many examples of the interaction between seemingly foreign branches of mathematics. After defining and proving elementary relations between graphs and groups, this paper will present a result of Hanna Neumann on the intersection of free subgroups. Although Neumann's orginal upper bound on the rank of the intersection has been improved upon, her conjecture, that the upper bound is half of the one she originally proved, remains an open question.

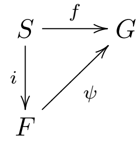
Contents

1. Groups

Definition 1.1. Recall that a *group* $\mathcal{G}$ is a set S and a binary operation $\cdot$ with the properties that $\mathcal{G}$ is closed under $\cdot$, $\cdot$ is associative, there exists an identity e such that for all $g \in (G)$, $g \cdot e = e \cdot g = g$, and there exist inverses such that for each $g \in \mathcal{G}$, there exists $g^{-1} \in \mathcal{G}$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

When working with groups, it is helpful to review the definitions of homomorphism, an operation perserving map. For groups $G_1 = (S_1, \cdot)$, $G_2 = (S_2, \star)$, the homomorphism $\phi : G_1 \to G_2$ has the property that $\phi(x \cdot y) = \phi(x) \star \phi(y)$. An isomorphism is a bijective homomorphism.

Definition 1.2. This paper will focus on a particular kind of group, those which have the least possible amount of restriction: free groups. Let $i : S \to F$. A group F is *free* on a set S if for all groups G and for all $f : S \to G$, there exists a unique homomorphism $\psi : F \to G$ so that $i \circ \psi = f$. In other words, the following diagram commutes:



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The simpliest way to understand free groups is to think about elements in the free group as reduced words on a set S.

Definitions 1.3. A *word* is a string of elements $s \in S$ and their inverses s^{-1}. The identity is the empty word with no symbols. A *reduced* word is one that contains no adjacent pairs ss^{-1}, since by definition ss^{-1} is the identity. Reduction is simply replacing ss^{-1} with the empty word, or equivalently removing these pairs. A free group F on the set $\{a, b\}$ with two elements is the set of all possible words using $\left\{a, a^{-1}, b, b^{-1}\right\}$.

When writing words, exponents are used for ease: the word $aaaaababb^{-1}abb^{-1}b^{-1}$ is written as $a^4(ab)^2b^{-1}abb^{-2}$, which then reduces to $a^5ba^2b^{-1}$. Any set with the operation concatenate and reduce words generates a free group.

Definition 1.4. A *minimal generating set* for a free group is a set of elements with the property that no generator is a word that can be made from the other generators. The minimal generating set is unique up to bijection since the sole property necessary in the construction of a free group is the number of symbols used. Minimal generating sets will be written inside $\langle\rangle$ brackets. The notation F_n will denote a free group on n minimal generators.

The sets $\langle a, b\rangle, \langle c, d\rangle, \langle xy, yx\rangle, \langle pumpkinpie, cherrypie\rangle$ all generate free groups which are isomorphic to each other since they all have two generators.

Definition 1.5. The *rank* of a free group is the cardinality of any minimal set of generators needed to create the free group.

Determining minimal generators and the rank for a free group on a set S involves determining whether any element in S is a word in the other elements of S. Evaluating the rank of F_n is trivial, so we focus on finding ranks of subgroups of a free group. However, finding the rank of a subgroup involves tedious computation.

Example 1.6. Take $S = \left\{a, a^3, b\right\}$. Since it only uses the symbols a, b, the free group on S will be a subgroup of the free group on two generators $F_2 = \langle a, b\rangle$. The minimal generating set for the free group on S is $\langle a, b\rangle$ since a^3 is a word made by a. Hence this subgroup is equal to F_2.

Example 1.7. Although the above example is straightforward, determining the rank and minimal generators of given subset is extremely difficult combinatorially. Let $S = \left\{ab^2a^{-1}, ab, ab^3ab^{-1}\right\}$. The free subgroup of F_2 on S must have rank greater than or equal to two since there must be at least the symbols a and b. However, it cannot rank greater than three since there are three words in S. Although free groups and their subgroups are non-abelian, the inverse of the word (ab) is $b^{-1}a^{-1}$ since $abb^{-1}a^{-1}$ is the identity. The free group on S has rank 2 because $ab^3ab^{-1} = (ab^2a^{-1})(ab)(ab^2a^{-1})^{-1}(ab) = ab^2a^{-1}abab^{-2}a^{-1}ab = ab^2bab^{-2}b = ab^3ab^{-1}$.

The difficulty of the combinatorial calculation of rank gets even worse when considering intersections of free groups. The intersection L of two subgroups N, M of a free group F is defined as the set containing all the elements l such that $l \in N$ and $l \in M$.

Lemma 1.8. *$L = N \cap M$ is a (free) subgroup of N, M.*

Proof. To show L is closed under concatenation: for $l, l' \in L$, the words $ll', l'l$ are also in L since $ll', l'l \in N, M$ by the closure under concatenation for N and M. The identity of F, the empty word, must be in both N and M and hence in L. For any $l \in N, M$, there exists $l^{-1} \in N, M$, so $l^{-1} \in L$. Lastly, since associativity is inherited from F, L is a subgroup of N, M. The freeness of a subgroup of a free group will be proved later. □

Example 1.9. Let's take a very easy example: $N = \langle a^2, b \rangle$ and $M = \langle a^2, ab \rangle$ both subgroups of F_2. Then $N \cap M$ should include all words with even powers of a and no powers of b. Hence, L is the free group generated by $\langle a^2 \rangle$. Even though this is a simple example, determining how many generators are needed is not intuitive.

To help gather information about the rank of an intersection of free groups, we need Hanna Neumann's theorem. In 1955, she proved for N, M subgroups of a free group F that

$$0 \leq \operatorname{rank}(N \cap M) - 1 \leq 2(\operatorname{rank} N - 1)(\operatorname{rank} M - 1) \tag{1.10}$$

She conjectured that the 2 in her upper bound could be dropped, and this conjecture remains an open question. In order to prove her original theorem, knowledge of graphs and fundamental groups will be imperative.

2. Graphs

Definitions 2.1. A *graph* G is a set of vertices $V(G)$ together with a set $E(G)$ of edges. An oriented edge $e = (v, w)$ begins at the vertex v and ends at vertex w. The edge in the reverse direction is $e^{-1} = (w, v)$, and is called the inverse of e. All graphs in this paper will have oriented edges unless otherwise noted. The *valency* of a vertex v is the total number of edges starting and ending at v. A *loop* is an edge with the same starting and ending vertex: $l = (v, v)$. A one loop contributes twice when determining valency at a vertex. A *path* p is a finite list of edges $e_1, e_2, ..., e_n$ where the ending vertex of e_i for $i < n$ is the same as the beginning vertex for e_{i+1}. The formalized definition is that a path is a continuous map $p : [0, 1] \to G$ where $p(0)$ is the starting vertex of the path, and $p(1)$ is the ending vertex of the path. Paths are continuous movements along sequential edges in the time interval $[0, 1]$. Paths can be concatenated if the ending vertex of p_1 is the same vertex as the starting point of p_2. A *closed* path begins and ends at the same vertex. A *spur* is a subpath of p consisting of e_i followed by the inverse of e_i (a closed path that includes only one other vertex). A *tree* is a graph that has no closed paths in it. A graph is *connected* whenever given any two vertices, there exists a path between them. A *maximal tree* T is the longest non-closed path in a graph G. In a connected graph, it contains all vertices because given any tree we can lengthen it by adding the path from v_n to v_{n+1} and still never get a closed path. Examples of graphs are given in Figure 1.

Definitions 2.2. A *homotopy* is a path of paths or a deformation of one map to another map that is continuous with respect to $t \in [0, 1]$. More precisely, for maps $m_1 : G_1 \to G_2$ and $m_2 : G_1 \to G_2$ to be homotopic, there is a continuous map $h : [0, 1] \times G_1 \to G_2$ such that $h(0, x) = m_1(x)$ and $h(1, x) = m_2(x)$. Two spaces X, Y are homotopy equivalent spaces if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map in X and $f \circ g$ is homotopic to the identity map in Y. If a space is homotopy equivalent to a point,

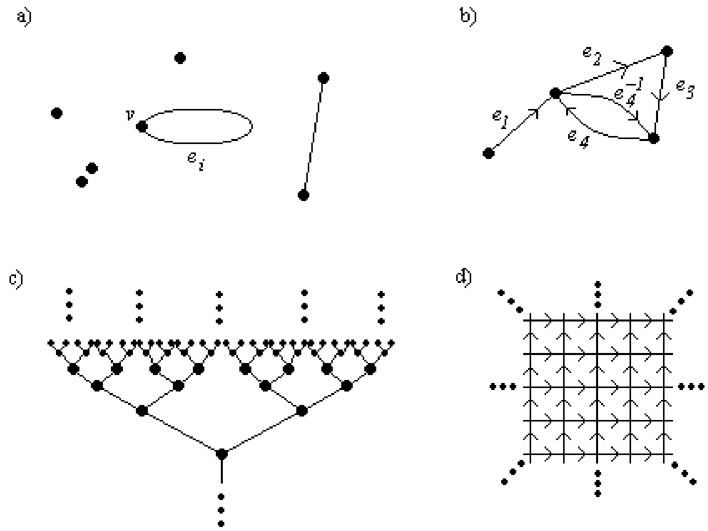

FIGURE 1. a) a disconnected, unoriented graph with a loop e_i at vertex v, which has valency 2.
b) graph with closed path $e_1e_2e_3e_4$ and spur $e_4e_4^{-1}$
c) infinite tree of valency 3 with unoriented edges
d) infinite graph of valency 4

that space is called *contractible*.
Another topological notion of equivalence is *homeomorphism*. A function $f : X \to Y$ is a homeomorphism if f is bijective, continuous, and has a continuous inverse $f^{-1} : Y \to X$.

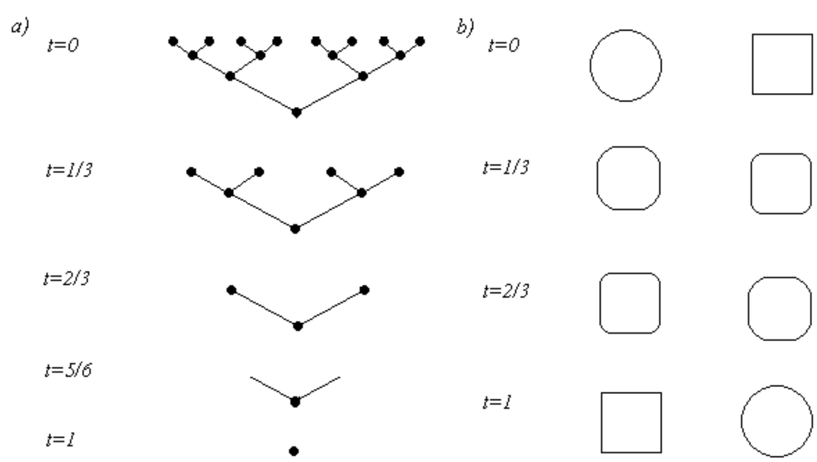

FIGURE 2. a) A tree is homotopy equivalent to a point
b) A circle is homeomorphic to a square

Now that the vocabulary of graphs has been spelled out, how do lines and dots relate to groups?

There is a group hidden within a graph called the *fundamental group* of the graph G and is denoted $\pi_1(G)$. To find this group, we take equivalence classes of closed paths. Two paths p and p' are considered equivalent if p' can be made from p via a finite number of insertions or removals of spurs in the middle or at the endpoints. Let $[p]$ be the equivalence class of closed paths p beginning (and ending) at a vertex v. Defining the operation $[p_1] \cdot [p_2] = [p_1 p_2]$ to be the equivalence class of concatenations of p_1 and p_2. This product is well-defined because adding or removing of spurs preserves the equality. The identity is the constant path at vertex v and $\left[p^{-1}\right] = [p]^{-1}$ since adding or subtracting spurs preserves the equality. Hence the fundamental group is a group. Note that paths which are homotopy equivalent are in the same equivalence class.

Theorem 2.3. *The fundamental group of a countable connected graph is free.*

Proof. To find the generators of the fundamental group, we pick a base point v_0 and find all closed paths from v_0. For each vertex v_i in the graph, choose a path $w_i = (v_0, v_i)$ called the approach path. For any edge $e = (v_i, v_j)$, let $e' = w_i e w_j^{-1}$ be a closed path from v_0 that includes e as a subpath. Any closed path $p = e_1 e_2 e_3 \ldots e_n$ starting and ending at v_0 is equivalent to $p' = e_1' e_2' e_3' \ldots e_n'$ since $w_1 = w_n = (v_0, v_0)$ and the approach paths cancel between successive edges.
Choose a maximal tree T. Since trees are homotopy equivalent to a point, we collapse the tree T to v_0 and consider the remaining loops attached to v_0. This is called the bouquet of circles B of $[e_1], [e_2], \ldots$ attached at v_0:

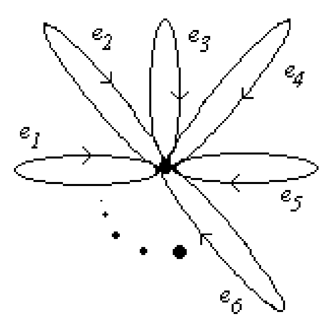

FIGURE 3. A bouquet of circles B attached at v_0.

If e_i is part of the maximal tree, then e_i' is a closed path and thus part of the equivalence class of constant paths (v_0, v_0). Hence the generators of $\pi_1(G)$ are all $[e_i']$ where $e_i \notin T$ or equivalently, $e_i \in B$. Note that the path $e_i e_j e_k \ldots$ is equal to the constant path at v_0 (the identity) only if the word $e_i e_j e_k \ldots$ is the identity in the free group on the minimal generators $\langle e_1, e_2, \ldots, e_i, \ldots \rangle$. Thus, the equivalence classes $[e_1], [e_2], \ldots$ are the free generators for $\pi_1(B)$. Since $[e_i] = [e_i']$ for $e_i \in B$ since they are equivalent closed paths, the generators for $\pi_1(G)$ are also free. An intuitive way to think about this is that paths of loops in B and their inverses form words. $\square$

Now that we have constructed a group from a graph, we will constuct a graph from a given group, mainly focusing on how to construct a covering graph from a subgroup. From this construction, we will see that subgroups of free groups are free (a combinatorial migraine otherwise).

Definition 2.4. A *covering graph* $\tilde{G}$ of G is a graph locally homeomorphic at each vertex to G. More formally, there exists a map $\phi : G \to G$ called the projection with the following properties:

(a) ϕ preserves endpoints of each edge so that for $\tilde{e} = (\tilde{v}, \tilde{w})$, $\phi(\tilde{e}) = (\phi(\tilde{v}), \phi(\tilde{w}))$.
(b) $(\phi(\tilde{e}))^{-1} = \phi(\tilde{e}^{-1})$.
(c) ϕ preserves valency: There is a one-to-one mapping from the collection of oriented edges in $\tilde{G}$ which begin at v to the collection of oriented edges in G that begin at $\phi(v)$.

Going from a path p in G to $\tilde{p}$ in $\tilde{G}$ is called lifting. Given an edge e in G and a vertex v in $\tilde{G}$, there is exactly one edge $\tilde{e}$ beginning or ending at v that covers e. Note that there is exactly one edge covering e for every vertex in $\tilde{G}$, but there are multiple edges in all of $\tilde{G}$. To find a path $\tilde{p}$ in $\tilde{G}$ which covers p, we use this lifting property recursively. Given a base point, the starting vertex v_0, there is a unique path $\tilde{p}$ formed when the vertex used for covering e_{i+1} is the ending vertex of $\tilde{e_i}$. The total number of vertices in $\tilde{G}$ which cover a given vertex in G is called the sheet number of the covering.

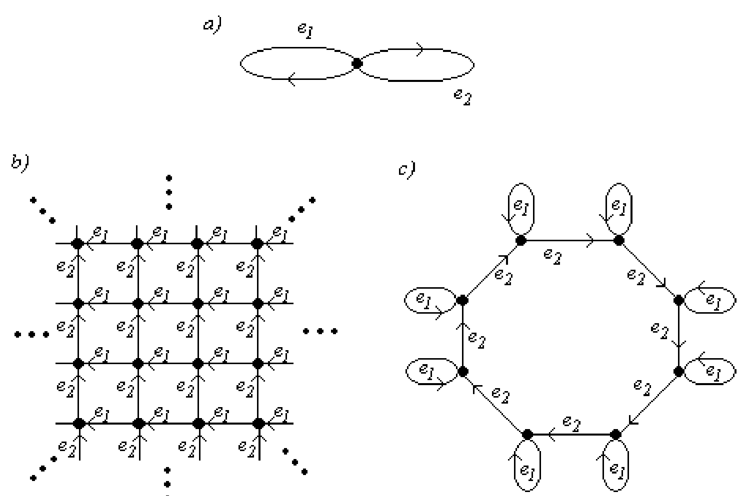

Figure 4. An example of covering maps:
a) A graph with two loops
b) Infinite sheeted covering map
c) Eight sheeted covering map

Theorem 2.5. *When we have a covering map of a graph, $\pi_1(\tilde{G})$ is isomorphic to a subgroup of $\pi_1(G)$. Classifying the set of closed paths in G which lift to nonclosed paths emanating from a chosen base point $\tilde{v_0}$ in $\tilde{G}$ by the end point $\tilde{v_f}$ of the lift path $\tilde{p}$ is in fact the right coset decomposition of $\pi_1(G)$ modulo $\pi_1(\tilde{G})$. In other words, a covering graph is identified with a subgroup of the fundamental group.*

Proof. By the properties of ϕ listed above and the lifting property, there is a one-to-one correspondence between the equivalence classes $[p]$ which form the fundamental group of G and the equivalence classes $[\tilde{p}]$ of paths, including non-closed paths, in $\tilde{G}$. In addition, ϕ takes products to products and inverses to inverses, which implies that there exists an injective homomorphism $\phi_* : \pi_1(\tilde{G}) \to \pi_1(G)$. Hence there is an isomorphism between $\pi_1(\tilde{G})$ and a subgroup of $\pi_1(G)$.
To show the right coset decomposition part of the therom, first consider when two equivalence classes of paths, $[p_1], [p_2]$ are in the same coset. Then, for some $[q] \in \pi_1(G)$, $qp_1 = p_2$. However, $\tilde{p_2} = \tilde{q}\tilde{p_2}$ where $\tilde{q}$ is a loop from $\tilde{v_0}$. Thus, $\tilde{p_1}$ and $\tilde{p_2}$ have the same starting point, namely $\tilde{v_0}$, and the same ending point $\tilde{v_f}$. Since p_1, p_2 have the same ending point in G, $\tilde{v_f}$ covers that ending point. Hence, the cosets are contained within the classification.
To go the other direction, assume that $\tilde{p_1}, \tilde{p_2}$ start at $\tilde{v_0}$ and go to the same ending vertex $\tilde{v_f}$, which covers a vertex v_f in G. Consequently, p_1 and p_2 have the same ending point (v_f) and $[p_1], [p_2]$ are in the fundamental group of G. Since the projection of the closed path $\tilde{p_2}\tilde{p_1}^{-1}$ is $[p_2]\left[p_1^{-1}\right]$, this concatenation of equivalence classes is closed and hence in the fundamental group of the covering graph: $[p_2]\left[p_1^{-1}\right] = [q] \in \pi_1(\tilde{G})$. This implies that $[p_2] = [q][p_1]$ or that $[p_1], [p_2]$ are in the same coset. □

To finalize the correspondence between subgroups and covering graphs, we look at an explicit construction of the graph of a subgroup, as well as an example.

Construction 2.6. Given a free group F of rank k, we realize it as the fundamental group of a bouquet G of k circles.
If H is a subgroup of F, then it will be isomorphic to the fundamental group of a covering $\tilde{G}$ of G. Each vertex will have one incoming and one outgoing edge for each loop in G. A connected graph is then uniquely determined since each e_i originating at a vertex corresponding to the coset $H[p]$ must end at the vertex corresponding to the coset $H[pe_i]$.
This constructed graph is such that $\pi_1(\tilde{G}) = H$. Take a path $\tilde{p}$ in $\tilde{G}$ that goes from the base point to some $\tilde{v_i}$ and covers the path $p \in G$. Then $\tilde{v_i}$ corresponds to the coset $H[p]$. The path $\tilde{p}$ is closed and thus in the fundamental group only if $H[p] = H \Rightarrow [p] \in H$. Taken with the above, this shows that $\pi_1(\tilde{G})$ is isomorphic to H and thus the constructed graph represents the subgroup.

Corollary 2.7. *Every subgoup of a free group is free.*

[This is actually a theorem of Nielsen and Schreier, but once the graph of the subgroup is constructed, there is virtually no work.]

Proof. Given a subgroup H of a free group F, realize H as the fundamental group of a covering $\tilde{G}$. Since the covering is a graph and the fundamental group of graph is free, $\pi_1(\tilde{G}) = H$ is free. □

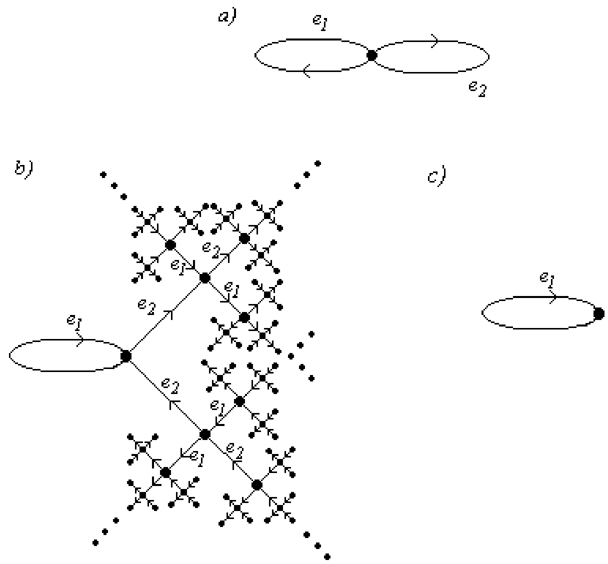

FIGURE 5. An example of covering maps:
a) A graph of $F_2 = \langle e_1, e_2 \rangle$
b) A covering graph representing the subgroup $H = \langle e_1 \rangle$
c) The spine or minimum deformation retract of b)

Definitions 2.8. A maximal branch of a graph G is the longest contractible subgraph of G that meets the rest of G only at one end of one edge. A *minimal deformation retract* R of a graph G is obtained by cutting off all maximal branches. This subgraph R is also called the *spine* of G.
The formal definition of a retraction map is as follows: Let the retraction map $r : G \to R$ be a continuous map homotopic to the identity map on R. A defomation retract is a retraction map r such that there exists $h : [0,1] \times G \to G$ which meets three conditions:
(a) for all vertices w in R, $r(w) = w$
(b) for all vertices $v \in G$, $h(0, v) = v$ and $h(1, v) = r(v)$
(c) for all vertices $v \in R$, $h(t, v) = v\ \forall t \in [0, 1]$

Theorem 2.9. *A deformation retract r induces an isomorphism $r_* : \pi_1(G) \to \pi_1(R)$.*

Proof. To show that r_* is a homomorphism, note that the continuity of r means that closed paths in G map to closed paths in R. It also maps homotopy equivalent paths to homotopy equivalent paths since r composed with any homotopy h is again a continuous map of maps, a homotopy. Since condition (a) implies that r maps products to products and inverses to inverses, r_* is a homomorphism. Condition (a) implies that r_* is onto, since closed paths in G map exactly to themselves in R. To show that r_* one-to-one, consider a closed path p in G based at w which is in R. Condition (c) implies that $r(p)$ is also based at w and condition (b) implies that $r(p)$ is homotopic to p. Hence $r(p)$ and p are in the same equivalence class in $\pi_1(R)$ and $\pi_1(G)$. Hence r_* is an isomorphism. □

Corollary 2.10. *Note that if $\pi_1(G) < \infty$, then the minimal deformation retract R is a finite graph.*

Proof. If R were infinite, it must have infinite closed paths since all infinite trees have been cut off. However, this implies that $\pi_1(R)$ is infinite and, by above, also isomorphic to a finite free group. Since this is a blatant contradiction, R is finite. □

The Euler characteric of a graph is a powerful tool used to analyze the properties of a graph. For any graph, it is related to the rank of the fundamental group as follows:

Theorem 2.11. *Let $V(G)$ be set of vertices of G and $E(G)$ be set of edges of G. Let $\chi(G) = |V(G)| - |E(G)|$, the Euler characteric of a finite graph, and $\delta(v)$ be the valency of vertex v. If G is a finite graph, then*

$$2\big(\operatorname{rank}(\pi_1(G)) - 1\big) = 2(-\chi(G)) = \sum_{v\in V(G)} \delta(v) - 2 \tag{2.12}$$

Note that the finite condition is only necessary for the right side of the equation.

Proof. Tackling the left side of the equality, let us begin by assuming that T is a tree. We begin counting the number of vertices, beginning at an initial vertex. Every subsequent edge goes to another vertex and the number of edges increases by one, but the number of vertices also increases by one. Since every edge and vertex except the initial one cancels, $\chi(T) = 1 + (1-1) + (1-1) + \ldots = 1$. Now let the graph G have k closed paths and a maximal tree G_T. Each closed path can be thought of as a tree with one edge added to close the path. Hence

$$\chi(G) = |V(G)| - |E(G)| = |V(G_T)| - (|E(G_T)| + k) = 1 - k \tag{2.13}$$

Finally, $k = \operatorname{rank}(\pi_1(G)) = \operatorname{rank}(\pi_1(G_T))$ since the maximal tree contains no closed paths. Thus $\chi(G) = 1 - \operatorname{rank}(\pi_1(G))$ or $-\chi(G) = \operatorname{rank}(\pi_1(G)) - 1$.

The right side of the equality takes little work:

$$\sum_{v\in V(G)} \delta(v) - 2 = 2|E(G)| - 2|V(G)| = 2(-\chi(G)) \tag{2.14}$$

when G is finite. If G had infinite vertices, then the sum would be either infinity or 0, which is not necessarily equivalent to the Euler characteristic: take an infinite one-dimensional tree, which has Euler characteristic 1 while the sum is

$$\sum_{v\in V(G)} \delta(v) - 2 = \sum_{v\in V(G)} 2 - 2 = 0 \neq 1. \tag{2.15}$$

□

3. Hanna Neumann Theorem

Theorem 3.1. *To restate the thereom: Let N, M be subgroups of F, and L the non-trivial intersection of N and M. Let*

$$\rho(F) = \min\{0, \operatorname{rank} F - 1\} \tag{3.2}$$

Then,

$$\rho(L) \leq 2\rho(N)\rho(M) \tag{3.3}$$

The following proof expands the proof from W. Neumann.

Proof. Assume $\operatorname{rank} F = k$ and G is a graph with $\pi_1(G) = F$, i.e. G is a bouquet of k circles. For a subgroup H of F of rank s, let $G(H)$ be a covering of G with a fundamental group H: $\pi_1(G(H)) = H$ created as above. The vertices of $G(H)$ are identified with the elements of the set of right H-cosets. Assuming H is non-trivial, let $R(H)$ be the minimal deformation retract of $G(H)$, creating a finite graph that is homotopy equivalent to $G(H)$. By Thereom 2.8, $\pi_1(R(H))$ is isomorphic to $\pi_1(H)$, so they have the same rank. Thus, we can use the more convenient finite retracted graph to discover about rank of H. $R(H)$ is a finite graph since $\operatorname{rank} H < \infty$. Applying the theorem about Euler characteristic yields

$$2\rho(H) = \sum_{v \in V(R(H))} \delta(v) - 2 \tag{3.4}$$

$G(L)$ is a covering of both $G(N)$ and $G(M)$ because L is a subgroup of both N and M. The projection maps $G(L) \to G(N)$ and $G(L) \to G(M)$ send the spine $R(L)$ into the spines $R(N)$ and $R(M)$ respectively. Hence there are injective maps $\lambda_N : V\big(R(L)\big) \to V\big(R(N)\big)$ and $\lambda_M : V\big(R(L)\big) \to V\big(R(M)\big)$. The injectivity follows since the L-cosets are also in the set of N-cosets as well as in the set of M-cosets. The same injectiveness also holds for $\lambda = (\lambda_N, \lambda_M) : V\big(R(L)\big) \to V\big(R(N)\big) \times V\big(R(M)\big)$. For any $v \in V\big(R(L)\big)$, $\delta\big(\lambda_N(v)\big) = \delta(v)$ since covers are locally homeomorphic around each vertex. Thus, for $v \in V\big(R(L)\big)$ we have

$$0 \le \delta(v) - 2 \le \min\Big(\delta\big(\lambda_N(v)\big), \delta\big(\lambda_M(v)\big)\Big) \le \Big(\delta\big(\lambda_N(v)\big)\Big)\Big(\delta\big(\lambda_M(v)\big)\Big). \tag{3.5}$$

Thus, combining the above with the theorem concerning the Euler characteristic,

$$\begin{aligned} 2\rho(L) &= \sum_{v \in V\left(R(L)\right)} \delta(v) - 2 \\ &\le \sum_{v \in V\left(R(L)\right)} \Big(\delta\big(\lambda_N(v)\big) - 2\Big)\Big(\delta\big(\lambda_M(v)\big) - 2\Big) \\ &\le \sum_{(u,w) \in V\left(R(N)\right) \times V\left(R(M)\right)} \big(\delta(u) - 2\big)\big(\delta(w) - 2\big) \\ &= \Big(\sum_{u \in V\left(R(N)\right)} \big(\delta(u) - 2\big)\Big)\Big(\sum_{w \in V\left(R(M)\right)} \big(\delta(w) - 2\big)\Big) \\ &= 2\rho(N)2\rho(M) \end{aligned} \tag{3.6}$$

Simplifying gives

$$\rho(L) \le 2\rho(N)\rho(M). \tag{3.7}$$

□

The Hanna Neumann theorem and her conjecture have been intensely studied and her thereom has been stregthened or reproved several times. One easy improvement is to show her conjecture holds true when one subset has finite index:

Theorem 3.8. *If either N or M has finite index in F, then $\rho(L) \le \rho(N)\rho(M)$.*

Proof. Let the subgroup N have finite index n in F, assume $\operatorname{rank} F > 1$. Since the number of right cosets (the index) is the same as the sheet number of the cover. Hence $R(N)$ is a n-sheeted covering of $R(F)$, each of the n vertices has $\operatorname{rank} R$

outgoing edges. The maximal tree for $R(N)$ has n vertices, and $n-1$ edges. Thus there are $n(\operatorname{rank} F) - (n-1)$ edges not in the maximal tree or equivalently, there are $n(\operatorname{rank} F) - n + 1$ generators of $\pi_1\big(R(N)\big)$. A little rearraging leaves us with the equation $\operatorname{rank} N = n(\operatorname{rank} F) - n + 1 \Rightarrow \rho(N) = n\rho(F)$.
Since $N \cap M$ is a subgroup of M, N, the intersection $N \cap M$ has a finite index $i \leq n$ in N, so

$$\rho(N \cap M) = i\rho(M) \leq n\rho(M) = \rho(N)\rho(M)/\rho(F) \leq \rho(N)\rho(M). \tag{3.9}$$

□

In the 1980's, a student of John Stallings, S.M. Gersten, came up with an alternative proof of Hanna Neumann's original theorem using Stallings pullback graphs and immersions. The proof still relies upon Euler characteristics and the idea of using a retraction where all trees have been cut off.

One of the improvements upon H. Neumann's upper bound,

$$2\rho(N)\rho(M) - max\big(\rho(N), \rho(M)\big) \tag{3.10}$$

was the result of Burns' work in 1971. Another upper bound proved using the methods of Warren Dicks is

$$\rho(N)\rho(M) - \rho(N) - \rho(M) + 1. \tag{3.11}$$

Much of the more recent work relies on the idea of a strengthen Hanna Neumann theorem which considers not only asking about intersections, but also about intersections in the form $x^{-1}Nx \cap y^{-1}My$. Even though Hanna Neumann's conjecture remains an open question, its power to simplify the question of how many generators are needed for a given subgroup will continue to spark more research.

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