BROWNIAN MOTION

BEAU DABBS

Abstract. This paper seeks to study standard Brownian motion and some of its properties. We construct this stochastic process and demonstrate a few properties including continuity and non-differentiability.

Contents

1. Continuous Random Variables

Since this paper deals primarily with a *stochastic process*, a sequence of random variables indexed by time, we are first going to need to know a little bit of the machinery of probability in order to achieve any useful results. We assume that the reader has some familiarity with basic (discrete) probability and measure theory, particularly with respect to random variables. However since this paper deals primarily with continuous random variables we will review their properties in some detail.

Definition 1.1. We say a random variable X is *continuous* if there exists a non-negative, integrable function $f : \mathbb{R} \to \mathbb{R}$ such that for all measurable sets A

$$P\{X \in A\} = \int_A f(x)dx$$

We call this function f the *probability density function* (PDF), or just the *density*, of the random variable X.

Sometimes it is easier to instead define the probability that a random variable is greater than, or less than, a given value. For this we have a new definition.

Definition 1.2. The *cumulative distribution function* (CDF) for a random variable X, denoted $F_X(a)$, is given by

$$F_X(a) = P\{X \leq a\}$$

In the case of a continuous random variable X we also have

$$F_X(a) = P\{X \leq a\} = P\{X < a\} = \int_{-\infty}^{a} f(x)dx$$

where f(x) denotes the density as defined above.

Proposition 1.3. *Given a continuous random variable X if its cumulative distribution function is given by F and its probability density function is given by f then*

$$\frac{d}{da}F(a) = f(a)$$

Proof. By definition

$$F(a) = \int_{-\infty}^{a} f(x)dx$$

Differentiating both sides gives us the result. □

Definition 1.4. The *expected value* or *mean* of a continuous random variable X is defined to be:

$$E[X] =: \int_{-\infty}^{\infty} xf(x)dx$$

whenever the integral on the right hand side exists.

Definition 1.5. the *Variance* of a continuous random variable X is given by

$$Var(X) =: E[(X - E[X])^2]$$

again whenever the value on the right hand side exists.

These two values are of key importance when studying probability, but it is important to note that they don't necessarily exist for all random variables, one example being the Caucy distribution. However when these values do exist one can consider the expected value of a function of a random variable.

Lemma 1.6. *For any continuous random variable Y with density f_Y:*

$$E[Y] = \int_0^{\infty} P\{Y > y\}dy - \int_0^{\infty} P\{Y < -y\}dy$$

Proof. We will show this equality directly. Consider

$$\int_0^{\infty} P\{Y > y\}dy - \int_0^{\infty} P\{Y < -y\}dy = \int_0^{\infty}\int_y^{\infty} f_Y(x)dxdy - \int_0^{\infty}\int_{-\infty}^{-y} f_Y(x)dxdy$$

This equality holding simply by our definition of a probability density function. Now if we exchange the order of integration in each of these double integrals, making sure to keep the region of integration the same, we obtain

$$\begin{aligned}
&= \int_0^{\infty}\int_y^{\infty} f_Y(x)dxdy - \int_0^{\infty}\int_{-\infty}^{-y} f_Y(x)dxdy \\
&= \int_0^{\infty}\int_0^{x} f_Y(x)dydx - \int_{-\infty}^{0}\int_0^{-x} f_Y(x)dydx \\
&= \int_0^{\infty}\left(\int_0^{x} dy\right) f_Y(x)dx - \int_{-\infty}^{0}\left(\int_0^{-x} dy\right) f_Y(x)dx \\
&= \int_0^{\infty} xf_Y(x)dx + \int_{-\infty}^{0} xf_Y(x)dx \\
&= \int_{-\infty}^{\infty} xf_Y(x)dx \\
&= E[Y]
\end{aligned}$$

□

Proposition 1.7. *If Y is a continuous random variable with density $f_Y(x)$, then for any real-valued function g,*

$$E[g(Y)] = \int_{-\infty}^{\infty} g(x) f_Y(x) dx$$

Proof. By Lemma 1.4 we know that

$$\begin{aligned}
E[g(Y)] &= \int_0^{\infty} P\{g(Y) > y\} dy - \int_0^{\infty} P\{g(Y) < -y\} dy \\
&= \int_0^{\infty} \int_{g(x)>y} f_Y(x) dx dy - \int_0^{\infty} \int_{g(x)<-y} f_Y(x) dx dy \\
&= \int_{g(x)>0} \left(\int_0^{g(x)} dy \right) f_Y(x) dx - \int_{g(x)<0} \left(\int_0^{-g(x)} dy \right) f_Y(x) dx \\
&= \int_{g(x)>0} g(x) f_Y(x) dx + \int_{g(x)<0} g(x) f_Y(x) dx \\
&= \int_{-\infty}^{\infty} g(x) f_Y(x) dx
\end{aligned}$$

□

Corollary 1.8. *Given any constants a and b*

$$E[aX + b] = aE[X] + b$$

Proposition 1.9. *For any continuous random variable X with density f, and expected value $E[X] < \infty$*

$$Var(X) = E[X^2] - (E[X])^2$$

Proof. By the definition of variance we know:

$$Var(X) = E[(X - E[X])^2]$$

and now we can apply Proposition 1.5 repeatedly to manipulate our equation.

$$\begin{aligned}
Var(X) &= \int_{-\infty}^{\infty} (x - E[X])^2 f(x) dx \\
&= \int_{-\infty}^{\infty} (x^2 - 2E[X]x + E[X]^2) f(x) dx \\
&= \int_{-\infty}^{\infty} x^2 f(x) dx - 2E[X] \cdot \int_{-\infty}^{\infty} x f(x) dx + E[X]^2 \cdot \int_{-\infty}^{\infty} f(x) dx \\
&= E[X^2] - 2E[X] \cdot E[X] + E[X]^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}$$

□

Corollary 1.10. *Given any constants a and b*

$$Var(aX + b) = a^2 Var(X)$$

This corollary is fairly immediate, and is hence left to the reader.

2. Jointly Distributed Random Variables

In addition to considering situations involving a single continuous random variable it is many times necessary to consider multiple random variables at the same time. Now we will lay out a few definitions involving the relationship between two continuous random variables.

Definition 2.1. Two random variables X and Y are said to be *jointly continuous* if there exists a function $f(x, y)$, the *joint probability density function* of X and Y, defined for all real x and y such that for all measurable sets A and B of real numbers

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x, y)dxdy$$

Definition 2.2. We call random variables X and Y *independent* if for any two measurable sets A and B of real numbers,

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}$$

If $f(x, y)$ is the joint probability density function of X and Y and $f_X(x)$ and $f_Y(y)$ are the probability density functions of X and Y respectively, then this is equivalent to saying

$$f(x, y) = f_X(x)f_Y(y), \forall x, y$$

If random variables are not independent we call them *dependent*.

Definition 2.3. We call a collection of variables $X_1, ..., X_n$ *independent* if for any measurable sets $A_1, ..., A_n$

$$P\{X_i \in A_i : 1 \leq i \leq n\} = \prod_{i=1}^{n} P\{X_i \in A_i\}$$

We call a collection of variables *pairwise independent* if for any two variables X_i, X_j and any two measurable sets A_i, A_j, $i \neq j$

$$P\{X_i \in A_i, X_j \in A_j\} = P\{X_i \in A_i\}P\{X_j \in A_j\}$$

Remark 2.4. While a collection of variables being independent clearly implies pairwise independence, the reverse is not necessarily so.

It's also possible to consider the sum of two independent random variables. Given two independent, continuous random variables X, Y and their corresponding distribution and density functions the CDF of their sum is given by:

$$\begin{aligned}
F_{X+Y}(a) &= P\{X + Y \leq a\} \\
&= \int\int_{x+y\leq a} f_X(x)f_Y(y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)dxf_Y(y)dy \\
&= \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy
\end{aligned}$$

Then if we differentiate both sides of the equation, by proposition 1.3, we can obtain the probability density function

$$\begin{aligned} f_{X+Y} &= \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy \\ &= \int_{-\infty}^{\infty} \frac{d}{da}F_X(a-y)f_Y(y)dy \qquad (2.5) \\ &= \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy \end{aligned}$$

With the framework we have now we can introduce random walk. In one dimension random walk is simply thought of as starting at the origin, then flipping a coin and moving $+1$ if it lands heads, or -1 if it lands tails. Hence this process is a simple example of discrete random motion. But to be clear we should state this idea in mathematically.

Definition 2.6. A *random walk* is a stochastic process S_n with

$$S_n = X_1 + ... + X_n$$

where the X_i are independent, identically distributed random variables taking on values 1 and -1 each with probability $\frac{1}{2}$.

Given this definition one can then study the properties of a random walk, such as the average distance one has moved after a given number of steps, and the frequency with which one will visit any given point. However we will not be studying this process here and will instead move on to define normal random variables, which we will use to create Brownian motion, essentially a continuous random walk.

3. Normally Distributed Random Variables

Definition 3.1. We will say a variable X has a *normal distribution* with parameters μ and σ^2 if its probability density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

We will denote this by writing $X \sim N(\mu, \sigma^2)$

In order to consider such a variable X we must first make sure that the function f actually is a probability density function. We will do this, and consider the expected value and variance of $X \sim N(\mu, \sigma^2)$ with the following theorem.

Theorem 3.2. *If a continuous random variable X has a normal distribution with parameters μ and σ^2, and probability density function denoted $f(x)$ then:*

(1)
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx = 1$$

(2) $E[X] = \mu$

(3) $Var(X) = \sigma^2$

Proof. (1)
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}}dy$$

By letting $y = (x-\mu)/\sigma$. Now it only needs to be shown that $I = \int_{-\infty}^{\infty} e^{-y^2/2} = \sqrt{2\pi}$. To do this consider I^2:

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy\right) \\ &= \int_{-\infty}^{\infty} e^{\frac{-(x^2+y^2)}{2}} dy \end{aligned}$$

This integral can then be easily evaluated by switching to polar coordinates, letting $x = r\cos\theta$, $y = r\sin\theta$, and $dxdy = rd\theta dr$.

$$\begin{aligned} I^2 &= \int_0^{\infty}\int_0^{2\pi} re^{\frac{-r^2}{2}} dr \\ &= 2\pi \int_0^{\infty} re^{\frac{-r^2}{2}} dr \\ &= 2\pi \left[-e^{\frac{-r^2}{2}} \Big|_0^{\infty} \right] \\ &= 2\pi \end{aligned}$$

(2)

$$\begin{aligned} E[X] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} xe^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \mu e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} ye^{\frac{-y^2}{2\sigma^2}} dy + \mu \int_{-\infty}^{\infty} f(x)dx \end{aligned}$$

Hence by considering $x = (x-\mu)+\mu$ we can see that the first term integrates to 0 due to symmetry when we replace $y = (x-\mu)$, and the second term is simply μ since $f(x)$ is indeed a probability density function, as we showed in (1). Hence we have shown that $E[X] = \mu$, as desired.

(3)

$$Var(X) = E[(X - E[X])^2 = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Luckily this formula works out nicely since, as we already calculated, $E[x] = \mu$. Now we can just let $y = (x-\mu)$ and begin integrating:

$$\begin{aligned} Var(X) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y^2 e^{\frac{-y^2}{2\sigma^2}} dy \\ &= \frac{1}{\sqrt{2\pi}\sigma}\left(\sigma^2 y e^{\frac{-y^2}{2\sigma^2}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\sigma^2 e^{\frac{-y^2}{2\sigma^2}} dx\right) \quad \text{Integration by Parts} \\ &= \sigma^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2}} dx \\ &= \sigma^2 \end{aligned}$$

Since again by part (1) the final integral evaluates to 1, $Var(X) = \sigma^2$ as claimed.

$\square$

It's also interesting, and useful, to consider what happens to normal random variables when they are added together. In particular we will consider what happens to the expected value and variance when two normal random variables are added together.

Lemma 3.3. *Let X and Y be two independent random variables with $X \sim N(0, \sigma^2)$ and $Y \sim N(0, 1)$. Then $X + Y \sim N(0, 1 + \sigma^2)$.*

Proof. By equation (2.3) we can write

$$\begin{aligned} f_{X+Y}(a) &= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(a-y)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy \\ &= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{\frac{-(a^2-2ay+y^2)}{2\sigma^2}} e^{\frac{-y^2}{2}} dy \\ &= \frac{1}{2\pi\sigma} e^{\frac{-a^2}{2\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{1+\sigma^2}{2\sigma^2}\left(y^2 - 2y\frac{a}{1+\sigma^2}\right)} dy \end{aligned}$$

Now for simplicity let us let $c = \frac{1+\sigma^2}{\sigma^2}$. Then by completing the square we obtain

$$\begin{aligned} f_{X+Y}(a) &= \frac{1}{2\pi\sigma} e^{\frac{-a^2}{2\sigma^2}} e^{\frac{(1+\sigma^2)(a^2)}{(2\sigma^2)(1+\sigma^2)^2}} \int_{-\infty}^{\infty} e^{-\frac{c}{2}(y - \frac{a}{1+\sigma^2})^2} dy \\ &= \frac{1}{2\pi\sigma} e^{\frac{-a^2}{2(1+\sigma^2)}} \int_{-\infty}^{\infty} e^{\frac{-cx^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma c} e^{\frac{-a^2}{2(1+\sigma^2)}} \frac{\sqrt{c}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-cx^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi(1+\sigma^2)}} e^{\frac{-a^2}{2(1+\sigma^2)}} \end{aligned}$$

Which is precisely the density of a normal random variable with mean 0 and variance $1 + \sigma^2$. $\square$

Proposition 3.4. *If $X_1, X_2, ..., X_n$ are independent continuous random variables each $X_i \sim N(\mu_i, \sigma_i^2)$, $1 \leq i \leq n$, then*

$$\sum_{i=1}^{n} X_i \sim N(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2)$$

Proof. We will prove this by induction. If $n = 1$ there is nothing to prove. Now consider first the case for $n = 2$. We can write

$$X_1 + X_2 = \sigma_2(\frac{X_1 - \mu_1}{\sigma_2} + \frac{X_2 - \mu_2}{\sigma_2}) + \mu_1 + \mu_2$$

Here the random variable $\frac{X_1-\mu_1}{\sigma_2} \sim N(0, \sigma_1^2/\sigma_2^2)$, and the random variable $\frac{X_2-\mu_2}{\sigma_2} \sim N(0, 1)$ so that we can apply Lemma 3.3. Hence $\frac{X_1-\mu_1}{\sigma_2} + \frac{X_2-\mu_2}{\sigma_2}$ is normal with expected value 0 and variance $(1+(\sigma_1/\sigma_2)^2)$, and by corollaries 1.8 and 1.10 X_1+X_2 is normal with mean $0 + \mu_1 + \mu_2$ and variance $\sigma_2^2(1 + (\sigma_1/\sigma_2)^2$.

Now if we assume the result holds up to $n-1$ we can then write

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n-1} X_i + X_n$$

Then by our inductive hypothesis $\sum_{i=1}^{n-1} X_i \sim N(\sum_{i=1}^{n-1} \mu_i, \sum_{i=1}^{n-1} \sigma_i^2)$, and applying the result for $n=2$ we gain the desired result. $\square$

4. Brownian Motion

Now we finally get to the meat of this paper. Brownian motion attempts to define a continuous analogue of random walk to, in a sense, model purely random motion. In particular we would like Brownian motion to satisfy the following definition:

Definition 4.1. A *(standard) Brownian motion* is a stochastic process, W_t $0 \leq t \leq \infty$ satisfying:

(1) $W_0 = 0$
(2) Independent normal increments: If $s < t$, then $W_t - W_s \sim N(0, t-s)$ and is independent of W_r $0 \leq r \leq s$
(3) The function $t \mapsto W_t$ is continuous.

The reason we call this particular definition *standard* Brownian motion is simply due to the normalization of the random variables. We could just as easily, with the proper scaling factors, construct Brownian motion such that $W_t - W_s \sim N(0, 2(t-s))$. For simplicity this definition is set up so that $W_1 \sim N(0,1)$. But before we can consider this process, we need to be sure that Brownian motion as we defined it exists, and there are a couple of key problems in doing this. First we are going to need to find a set of random variables that will actually satisfy condition (2) on any set of numbers, let alone $\mathbb{R}$. Then given the random variables, there is still no guarantee that our map $t \mapsto W_t$ will be continuous. So to start we will attempt to define such a process on a tameable beast: the *Dyadic Rationals*.

Definition 4.2. Let $\mathbf{D_n} = \{\frac{k}{2^n} | k = 0, 1, 2, ...\}$. Then we will denote the Dyadic Rationals by:

$$\mathbb{D} = \bigcup_{0}^{\infty} \mathbf{D_n}$$

Making sure our definition satisfies condition (1) of *Brownian Motion* is easy: we will define $W_0 = 0$. Now if we import condition (2) into $\mathbb{D}$ we just need for each n, the random variables:

$$W_{k/2^n} - W_{(k-1)/2^n}, k = 1, 2, ...$$

to be independent, with mean 0 and variance 2^{-n}. But for simplification we introduce the following notation.

Let

$$J(k,n) = 2^{n/2}[W_{k/2^n} - W_{(k-1)/2^n}] \tag{4.3}$$

Then we need for each n the random variables:

$$J(1,n), J(2,n), J(3,n), ...$$

to be independent with $J(k,n) \sim N(0,1)$ $\forall k$

Now with this requirement in mind we can begin our search for a way of defining these $J(k,n)$ so as to satisfy (2). In order to do this we're first going to need a way to take two independent random variables and make new random variables. Hence we will state the following proposition without proof, for a proof see *Lawler* in the references.

Proposition 4.4. *Suppose X and Y are independent random variables, $X, Y \sim N(0,1)$, and we let:*

$$Z = \frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y$$
$$\tilde{Z} = \frac{1}{\sqrt{2}}X - \frac{1}{\sqrt{2}}Y$$

Then Z, and $\tilde{Z}$ are independent random variables, also with $Z, \tilde{Z} \sim N(0,1)$.

In order to actually define our Brownian Motion on the Dyadics we will assume that we have a countable set of independent random variables $Z_1, Z_2, ...$ such that $Z_q \sim N(0,1), \forall q$. We'll also assume that, since $\mathbb{D}$ is a countable set, they can be indexed by Z_q $q \in \mathbb{D}$. Now to define our Brownian Motion on $\mathbb{D}$ we will begin by defining

$$J(k,0) = Z_k, k = 1, 2, ...$$

Now assuming we have $J(k,n)$ for all k using only $Z_q : q \in \mathbf{D_n}$ such that each $J(k,n) \sim N(0,1)$ and independent. Then we will define

$$J(2k-1, n+1) = \frac{1}{\sqrt{2}}J(k,n) + \frac{1}{\sqrt{2}}Z_{(2k+1)/2^{n+1}}$$

$$J(2k, n+1) = \frac{1}{\sqrt{2}}J(k,n) - \frac{1}{\sqrt{2}}Z_{(2k+1)/2^{n+1}}$$

By applying Proposition 4.4 we see that for each k the $J(k, n+1)$ are independent $N(0,1)$ random variables. Then we just define our $W_{k/2^n}$ by

$$W_{k/2^n} = 2^{-n/2} \sum_{j=1}^{k} J(j,n)$$

So as to satisfy equation 4.3 above.

Now we're getting somewhere. We've defined Brownian Motion fairly successfully on $\mathbb{D}$, but the downside is, we're missing uncountably many points, and we still haven't gotten to condition (3), continuity. Luckily all we actually need to do is prove that our current definition of Brownian Motion, in fact, already defines a uniformly continuous map on any closed subset of $\mathbb{R}$ intersect $\mathbb{D}$, and we get a continuous extension to $\mathbb{R}$ for free, since $\mathbb{D}$ is a dense subset of $\mathbb{R}$. Consider the following proposition:

Proposition 4.5. *Let D be a dense subset of $\mathbb{R}$, and $f : D \to \mathbb{R}$ be a uniformly continuous function on every closed subset of $\mathbb{R} \cap D$. Then we can define a continuous extension of f to $\mathbb{R}$ s.t. f is continuous on $\mathbb{R}$.*

Proof. For any $t \in [0, \infty)$ $\exists T > 0$ such that $t \in [0,T]$, and there is a sequence $\{t_n \in D\}$ with $\lim_{n\to\infty} t_n = t$. Hence $\{t_n\}$ is a cauchy sequence, and since f is uniformly continuous on $[0,T]$ the corresponding sequence $\{f(t_n)\}$ is also a cauchy sequence, and has a unique limit since $[0, T]$ is a compact set. So we can uniquely define the extension of f to be

$$\tilde{f}(t) = \lim_{n\to\infty} f(t_n).$$

To see that $\tilde{f}$ is a uniformly continuous function on $[0, T]$ it is enough to show that $\tilde{f}$ is continuous, since the domain is a closed and bounded interval. We know that for all $s \in [0, T]$ $\forall \epsilon > 0$ $\exists \delta > 0$ such that for all $q \in D$ if $|q - s| < \delta$, then $|f(q) - f(s)| < \epsilon/2$. Also if we instead consider any $t \in [0, T]$ such that $|t - s| < \delta$ then we know that

$$|\tilde{f}(t) - \tilde{f}(s)| \leq \sup\{|f(q) - f(s)| : q \in D\} \leq \epsilon/2 < \epsilon$$

Hence $\tilde{f}$ is continuous on $[0, T]$ and uniform continuity follows. □

Unfortunately showing that our Brownian Motion defined on $\mathbb{D}$ is uniformly continuous won't be so easy. To do this we will simply show that our Brownian Motion on the Dyadics is continuous on the interval $[0, 1]$, but an analagous proof easily follows for all positive t. But before we begin this let us consider a couple of Lemmata integral to probability.

Lemma 4.6. (Borel-Cantelli Lemma). *Suppose $E_1, E_2, \ldots$ is a collection of events such that*

$$\sum_{n=1}^{\infty} P\{E_n\} < \infty.$$

Then with probability one at most finitely many of the events occur.

Proof. Let A denote the event that infinitely many of the events, E_n occur. And let A_N denote the event that at least one of the events, $E_N, E_{N+1}, \ldots$ occurs. Then for all N

$$P\{A\} \leq P\{A_N\} = P\{\bigcup_{n=N}^{\infty} E_n\} \leq \sum_{n=N}^{\infty} P\{E_n\}$$

But since we know $\sum P\{E_n\} < \infty$, we also know that the tail of the series goes to zero as $n \to \infty$. Hence $P\{A\} = 0$. □

Lemma 4.7.

$$P\{\max\{W_{k/2^n} : k = 1, \ldots, 2^n\} > a\} \leq 2P\{W_1 > a\}$$

Proof. To show this, first fix an n and denote E_k be the event such that

$$W_{k/2^n} > a, Wj/2^n \leq a, j = 1, \ldots, k-1$$

Each of these events is mutually exclusive ($E_k \cap E_j = \emptyset, \forall k \neq j$) and their union is the right side of the statement of the lemma. Also for each k the event E_k depends on the random variables $W_{j/2^n}, 1 \leq k$, but the random variable $W_1 - W_{k/2^n}$ is independent of the event E_k. So we can write:

$$\begin{aligned} P\{E_k \cap \{W_1 > a\} &\geq P\{E_k \cap \{W_1 - W_{k/2^n} > 0\} \\ &= P\{E_k\}P\{W_1 - W_{k/2^n} > 0\} \\ &\geq \frac{1}{2}P\{E_k\} \end{aligned}$$

The last inequality holding by the reflection principle since $W_1 - W_{k/2^n}$ is a normal random variable, and hence symmetric. Now we can say

$$\begin{aligned} P\{W_1 > a\} &= P\{\bigcup_{k=1}^{2^n}(E_k \cap \{W_1 > a\})\} \\ &= \sum_{k=1}^{2^n} P\{E_k \cap \{W_1 > a\}\} \\ &\geq \frac{1}{2}\sum_{k=1}^{2^n} P\{E_k\} \\ &= \frac{1}{2}P\{\max\{W_{k/2^n} : k = 1, ..., 2^n\} > a\} \end{aligned}$$

This establishes the Lemma. □

Theorem 4.8. *If W_t is a brownian motion on the Dyadics, then with probability one $t \mapsto W_t$ is a uniformly continuous function on $[0,1] \cap \mathbb{D}$*

Before we begin the proof of this theorem, let us lay down some notation to make things simpler. Let

$$M_n = \sup\{|W_t - W_s| : |t - s| \leq 2^{-n}t, s \in \mathbb{D} \cap [0,1]\}$$

So that we have W_t is uniformly continuous on $[0,1] \Leftrightarrow \lim_{n\to\infty} M_n = 0$. Clearly this statement is true, but its not so clear how to deal with this quantity. Hence we'll define an easier quantity to deal with. Let

$$K_n = \max_{k=0,\ldots,2^n-1}\{\sup\{|W_t - W_{\frac{k}{2^n}}| : \frac{k}{2^n} \leq t \leq \frac{k+1}{2^n}, t \in \mathbb{D}\}\}$$

Also we will let $Y_{k,n} = \sup\{|W_t - W_{\frac{k}{2^n}}| : \frac{k}{2^n} \leq t \leq \frac{k+1}{2^n}, t \in \mathbb{D}\}$ so that

$$K_n = \max_{k=0,\ldots,2^n-1}\{Y_{k,n}\}$$

Clearly $K_n \leq M_n$ but by the triangle inequality it can also be seen that

$$|W_t - W_s| \leq |W_t - W_{\frac{k-1}{2^n}}| + |W_{\frac{k-1}{2^n}} - W_{\frac{k}{2^n}}| + |W_{\frac{k}{2^n}} - W_s| \leq 3K_n, \forall s, t$$

Hence $M_n \leq 3K_n$ and it suffices to show that $\lim_{n\to\infty} K_n = 0$. That is we need to show that for all $a > 0$ for N large enough $P\{K_n > a\} = 0$, $\forall n \geq N$. But by the Borel-Cantelli Lemma, if we can show that

$$\sum_{n=1}^{\infty} P\{K_n > a\} < \infty$$

Then with probability one $K_n > a$ only finitely many times, and hence for N large enough $K_n \leq a$, $\forall n \geq N$.

Proof. We need to show that $\sum P\{K_n > a\} < \infty$, so let us first consider the individual $P\{K_n > a\}$. First note that K_n is the maximum of a collection of 2^n identically distributed random variables, and the probability that the maximum is greater than some value is no more than the sum of the probabilities that each individual random variable is greater than that value. Hence we can write

$$P\{K_n > a\} \leq \sum_{j=1}^{2^n} P\{Y_{j,n} > a\} = 2^n P\{Y_{1,n} > 1\}$$

Then since $Y_{1,n} \sim N(0, 2^n)$, by the scaling property of expected value and variance explored earlier in this paper(Cor 1.8, 1.10) $2^{n/2} Y_{1,0} \sim N(0, 2^n)$, so we can consider this value instead, giving us

$$\begin{aligned}
P\{K_n > a\} &\leq 2^n P\{Y_{1,0} > 2^{n/2} a\} \\
&= 2^n P\{\sup\{|W_t| : t \in \mathbb{D} \cap [0,1]\} > 2^{n/2} a\} \\
&= 2^n \lim_{m\to\infty} \{\max\{|W_t| : t \in \mathbf{D_m} \cap [0,1]\} > 2^{n/2} a\} \\
&= 2 \cdot 2^n \lim_{m\to\infty} \{\max\{W_t : t \in \mathbf{D_m} \cap [0,1]\} > 2^{n/2} a\}
\end{aligned}$$

The last inequality holding by symmetry. Now using Lemma 4.7 we can say that

$$\begin{aligned}
P\{K_n > a\} &\leq 2 \cdot 2^n \lim_{m\to\infty} \{\max\{W_t : t \in \mathbf{D_m} \cap [0,1]\} > 2^{n/2} a\} \\
&\leq 4 \cdot 2^n P\{W_1 > a 2^{n/2}\}
\end{aligned}$$

Now we can simply use the fact that this last probability is given by a normal distribution, and hence we can say

$$4{\cdot}2^n P\{W_1 > a2^{n/2}\} = \frac{2^{n+2}}{\sqrt{2\pi}} \int_{a2^{n/2}}^{\infty} e^{\frac{-x^2}{2}} dx \leq 2^{n+1} \int_{a2^{n/2}}^{\infty} e^{\frac{-x(a2^{n/2})}{2}} dx = \frac{2^{n+2}}{a2^{n/2}} e^{\frac{-a^2 2^n}{2}}$$

In particular if we take $a = 2\sqrt{n} n^{-2/n}$ we obtain that

$$\sum_{n=1}^{\infty} P\{K_n > 2\sqrt{n} 2^{-n/2}\} \leq \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} (2/e^2)^n < \infty$$

Hence with probability one $K_n \leq 2\sqrt{n} 2^{-n/2}$ holds for all n sufficiently large, and $\lim_{n\to\infty} K_n = 0$ □

Theorem 4.9. *With probability one, there is no $t \in (0,1)$ at which W_t is differentiable.*

Proof. Suppose that there is a $t \in (0,1)$ at which W_t is differentiable. Then we know that the limit

$$\lim_{\epsilon\to 0} \frac{|W_s - W_s'|}{\epsilon}, s, s' \in [t - \epsilon, t + \epsilon]$$

exists, and is hence bounded above by some constant $B < \infty$. In particular we can say that $\exists B < \infty$ such that $\forall \epsilon > 0$

$$|W_s - W_s'| \leq B\epsilon$$

For simplicity define the following values

$$M(k,n) = \max\{|W_{k/n} - W_{k-1/n}|, |W_{k+1/n} - W_{k/n}|, |W_{k+2/n} - W_{k+1n}|\}$$
$$M_n = \min\{M(1,n), ..., M(n,n)\}$$

Then again, supposing that W_t is differentiable at some point $t \in (0,1)$, $\exists C < \infty$ and an $N < \infty$ such that for all $n \geq N$, $M_n \leq C/n$. To see this just let $\epsilon = 3/n$ in the previous consideration, then for at least one of the $M(k,n)$ the three intervals fall completely in the interval $[t-3/n, t+3/n]$ and hence $M_n \leq 3B/n$ but since $3B < \infty$ we can just let $C = 3B$.

Now for all k, n we know that $M(k,n)$ is the maximum of three identically distributed, indepedent, normal random variables, each with the same distribution as $W_{1/n} \sim N(0, 1/n)$. Hence we can say the following for all $C < \infty$ and for all k, n,

$$\begin{aligned} P\{M(k,n) \leq C/n\} &\leq [P\{W_{1/n} \leq C/n\}]^3 \\ &= \left[\frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-C/n}^{C/n} e^{\frac{-nx^2}{2}} dx\right]^3 \\ &\leq \left[\frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-C/n}^{C/n} e^{\frac{-n(0)^2}{2}} dx\right]^3 \\ &= \left[\frac{\sqrt{n}}{\sqrt{2\pi}} \frac{2C}{n}\right]^3 \\ &= \left[\sqrt{\frac{2}{\pi}} \frac{C}{\sqrt{n}}\right]^3 \end{aligned}$$

Since this inequality holds for all k, n it also holds for the minimum giving us

$$\lim_{n\to\infty} P\{M_n > C/n\} = 1 - \lim_{n\to\infty} P\{M_n \leq C/n\} \geq 1 - \left[\sqrt{\frac{2}{\pi}} \frac{C}{\sqrt{n}}\right]^3 = 1$$

And since $P\{M_n \geq C/n\}$ can never be greater than 1 we must have that

$$\lim_{n\to\infty} P\{M_n > C/n\} = 1$$

Hence with probability one $M_n > C/n$. By our previous argument this implies also that with probability one W_t is not differentiable at any point $t \in (0,1)$. $\square$

We have now constructed Brownian motion, and presented a few of its characteristics. Now with the foundations we can use this process to model things beleived to be governed by continuous random motion, most notably heat flow. For more information on the uses of Brownian motion see *Lawler* in the references.

References

[1] Sheldon Ross, A First course in Probability. Prentice Hall. 1998.
[2] Gregory Lawler, 2008 REU Lecture Notes. Unpublished
[3] G.R. Grimmet and D.R. Stirzaker, Probability and Random Processes. Oxford Science Publications. 1992