PROJECTIVE GEOMETRY

KRISTIN DEAN

Abstract. This paper investigates the nature of finite geometries. It will focus on the finite geometries known as projective planes and conclude with the example of the Fano plane.

Contents

1. Basic Definitions

First, we must begin with a few basic definitions relating to geometries. A geometry can be thought of as a set of objects and a relation on those elements.

Definition 1.1. A *geometry* is denoted $\mathbf{G} = (\Omega, I)$, where Ω is a set and I a relation which is both symmetric and reflexive.

The relation on a geometry is called an *incidence* relation. For example, consider the tradional Euclidean geometry. In this geometry, the objects of the set Ω are points and lines. A point is incident to a line if it lies on that line, and two lines are incident if they have all points in common - only when they are the same line. There is often this same natural division of the elements of Ω into different kinds such as the points and lines.

Definition 1.2. Suppose $\mathbf{G} = (\Omega, I)$ is a geometry. Then a *flag* of $\mathbf{G}$ is a set of elements of Ω which are mutually incident. If there is no element outside of the flag, $\mathcal{F}$, which can be added and also be a flag, then $\mathcal{F}$ is called maximal.

Definition 1.3. A geometry $\mathbf{G} = (\Omega, I)$ has *rank* r if it can be partitioned into sets $\Omega_1, \ldots, \Omega_r$ such that every maximal flag contains exactly one element of each set. The elements of Ω_i are called elements of *type* i.

Thus, these divisions of the set Ω give a natural idea of rank. Most of the examples of geometries which are dealt with in this paper are of rank two, that is, they consist of points and lines with certain incidence structures.

Date: July, 2008.

Lemma 1.4. *Let* $\boldsymbol{G}$ *be a geometry of rank* r*. Then no two distinct elements of the same type are incident.*

Proof. Suppose not. Then there exist two distinct elements of the same type which are incident. Then these elements, by definition form a flag. Now, these elements must be elements of some maximal flag, $\mathcal{F}$. But then $\mathcal{F}$ has two elements of the same type, but this is a contradiction because $\mathbf{G}$ is a geometry of rank r. □

Thus, as we saw with the Euclidean geometry, two lines are incident if and only if they are truly the same line. Often for geometries of rank 2 the types of elements are termed *points* and *lines*. This is the case for the projective spaces which are the focus of this paper.

2. Axioms of Projective Geometry

Henceforth, let $\mathbf{G} = (P, L, I)$ be a geometry of rank two with elements of P termed points, and those of L termed lines. There are many different such geometries which satisfy the following axioms, all of which are types of projective geometries.

Axiom 1 (Line Axiom)**.** *For every two distinct points there is one distinct line incident to them.*

Axiom 2 (Veblen-Young)**.** *If there are points* A, B, C, D *such that* AB *intersects* CD*, then* AC *intersects* BD*. That is to say, any two lines of a 'plane' meet.*

Axiom 3. *Any line is incident with at least three points.*

Axiom 4. *There are at least two lines.*

In projective geometries, the above axioms imply that there are no 'parallel' lines. That is, there are no lines lying in the same plane which do not intersect. The following lemma is derived easily from these axioms.

Lemma 2.1. *Any two distinct lines are incident with at most one common point.*

Proof. Suppose g and h are two distinct lines, but share more than one common point. By Axiom 1, two distinct points cannot both be incident with two distinct points, so $g = h$. □

The above axioms are used to define the following general structures.

Definition 2.2. A *projective space* is a geometry of rank 2 which satisfies the first three axioms. If it also satisfies the fourth, it is called *nondegenerate*.

Definition 2.3. A *projective plane* is a nondegenerate projective space with Axiom 2 replaced by the stronger statement: Any two lines have at least one point in common.

It is not too difficult to show that projective planes are indeed two dimensional as expected, although the notion of dimension for a geometry is defined further into the paper. A projective plane is therefore what one might naturally consider it to be. It is a plane, according to the usual conception of such, in which all lines meet as is expected from the term projective.

3. Linear Algebra with Geometries

Many of the concepts and theorems from linear algebra can be applied to the structures of geometries which give a new approach to studying these structures. Before we can apply the tools of linear algebra however, there are a few definitions to make.

Definition 3.1. A subset $\mathcal{U}$ of the point set is called *linear* if for any two points in $\mathcal{U}$ all points on the line from one to the other are also in $\mathcal{U}$.

It is often useful to consider all the points on a give line, so we denote this by letting (g) be the set of points incident with the line g. Just as in linear algebra, the notion of a subspace is remarkably useful. For geometries, it is quite natural to consider a subset of the points of the geometry as a subspace. However, to make this well defined, we must ensure that the same incidence structure makes sense. Thus we have the following definition.

Definition 3.2. A space $\mathbf{P}(\mathcal{U}) = (\mathcal{U}, L', I')$ is a (linear) *subspace* of $\mathbf{P}$, where L' is the set of lines contained in $\mathcal{U}$ and I' is the induced incidence. Also, the *span* of subset $\mathcal{X}$ is defined as:

$$\langle \mathcal{X} \rangle = \cap \{\mathcal{U} \mid \mathcal{X} \subseteq \mathcal{U}, \text{ a linear set}\}$$

Then $\mathcal{X}$ *spans* $\langle \mathcal{X} \rangle$.

From these definitions we can finally formally define the notion of a *plane*, which we already have an intuitive conception of.

Definition 3.3. A set of points is *collinear* if all points are incident with common line; otherwise, it is called *noncollinear*. A *plane* is the span of a set of three noncollinear points.

The following Theorems and Lemmas should look familiar from linear algebra. Their proofs are not significantly different from their respective counterparts, and thus they will be given without proof as a reference for the rest of the paper.

Theorem 3.4. *A set B of points of $\boldsymbol{P}$ is a basis if and only if it is a minimal spanning set.*

An important theorem regarding the basis holds here as well. Every independent set can be completed to form a basis of the whole space. The straightforward proof, which is along similar lines as that of the corresponding proof from linear algebra, is not given here.

Theorem 3.5 (Basis Extension Theorem)**.** *Let $\boldsymbol{P}$ a finitely generated projective space. Then all bases of $\boldsymbol{P}$ have the same number of elements, and any independent set can be extened to a basis.*

The basis of a geometry is a fundamental property of a specific structure. Thus there is a name related to the number of elements which are in such a basis.

Definition 3.6. Suppose $\mathbf{P}$ is a finitely generated projective space. Then the *dimension* of $\mathbf{P}$ is one less than the number of elements in a basis.

Likewise, the subspaces of a space also have dimension, and some of these subspaces are classified accordingly.

Definition 3.7. Let $\mathbf{P}$ have diminsion d. Then subspaces of dimension 2 are called *planes*, and subspaces of dimension $d-1$ are called *hyperplanes*.

Finally, we give a very important theorem from linear algebra which appears all over mathematics. The proof is not given here, but it is not too difficult and again not far from its linear algebra counterpart.

Theorem 3.8 (Dimension Formula)**.** *Suppose $\boldsymbol{U}$ and $\boldsymbol{W}$ are subspaces of $\boldsymbol{P}$. Then*

$$dim(\langle \boldsymbol{U}, \boldsymbol{W}\rangle) = dim(\boldsymbol{U}) + dim(\boldsymbol{W}) - dim(\boldsymbol{U} \cap \boldsymbol{W}).$$

4. Quotient Geometries

Another important question to consider when looking at finite and even infinite geometries is how new ones can be found from existing ones. One method for deriving new methods is akin to projecting down to a lower dimension by making lines into points and points into lines. Thus, we define the quotient geometry.

Definition 4.1. Suppose Q is a point of the geometry $\mathbf{P}$, then the *quotient geometry* of Q is the rank 2 geometry $\mathbf{P}/Q$ whose points are the lines through Q, and whose lines are the planes through Q. The incidence structure is as induced by $\mathbf{P}$.

Once we have several geometries of the same dimension, it is quite natural to ask whether they are in fact the same geometry. Therefore we need the notion of an *isomorphism* of geometries.

Definition 4.2. Suppose there are two rank 2 geometries: $\mathbf{G} = (P, B, I)$ and $\mathbf{G}' = (P', B', I')$. If there is a map ϕ

$$\phi : P \cup B \to P' \cup B'$$

where P is mapped bijectively to P' and B to B' such that the incidence structure is preserved, then this map is an *isomorphism* from $\mathbf{G}$ to $\mathbf{G}'$. An *automorphism* is an isomorphism of a rank two geometry to itself. When the geometry has elements termed 'lines', such as for projective planes, the automorphism is alternatively called a *collineation*.

Theorem 4.3. *Suppose $\boldsymbol{P}$ is a projective space of dimension d, and let $Q \in \boldsymbol{P}$. Then $\boldsymbol{P}/Q$ is a projective space of dimension $d-1$.*

Proof. It is enough to show that $\mathbf{P}/Q$ is isomorphic to a hyperplane which does not pass through Q. In the first place, such a hyperplane exists. Extend Q to a basis $\{Q, P_1, \ldots, P_d\}$ of $\mathbf{P}$. Then the subspace H spanned by $P_1, \ldots, P_d$ has dimension $d-1$ and so is a hyperplane not containing Q since Q was in the basis and is thus independent of the P_i.

Next, we must show that H is isomorphic to $\mathbf{P}/Q$. Define a map ϕ from the points g and lines π of $\mathbf{P}/Q$ to those of $\mathbf{H}$ by

$$\phi : g \to g \cap \mathbf{H}. \phi : \pi \to \pi \cap \mathbf{H}.$$

Remember that the points of $Q \in \mathbf{P}$ are lines of $\mathbf{P}$ which are incident with Q and the lines are the planes of $\mathbf{P}$ incident to Q. Now, we must show that ϕ is a bijection which preserves the incidence structure:

Injective: Suppose $g, h \in \mathbf{P}/Q$, meaning they are lines going through Q. Suppose both intersect $\mathbf{H}$ at the same point X. Then they have two points, Q and X in common. Since $X \in \mathbf{H}$ and $Q \notin \mathbf{H}$, these are distinct points and thus distinct lines. Similarly, if π and σ are planes through Q intersecting $\mathbf{H}$ at the same line, then they must be the same planes since they share a line and a point which are not incident.

Surjective: Suppose $X \in \mathbf{H}$, then the line QX is a point in $\mathbf{P}/Q$ which maps to it. Likewise, if $\pi \in \mathbf{H}$ is a line, then the plane $\pi \in \mathbf{P}$ defined by Q and the line π is a line of $\mathbf{P}/Q$ which maps to it.

Incidence: Suppose g a point and π a line in $\mathbf{P}/Q$. Then

$$g \subseteq \pi \Leftrightarrow g \cap \mathbf{H} \subseteq \pi \cap \mathbf{H} \Leftrightarrow \phi(g) \subseteq \phi(\pi)$$

Thus, $\mathbf{P}/Q$ is a projective space of dimension $d-1$. □

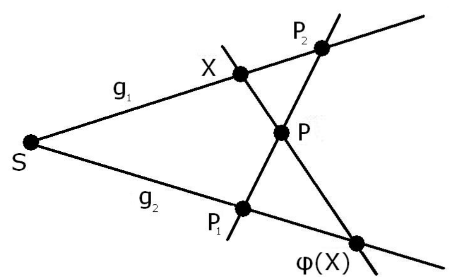
5. Finite Projective Spaces

Lemma 5.1. *Suppose g_1 and g_2 are lines of a projective space $\boldsymbol{P}$. Then there is a bijective map*

$$\phi : (g_1) \to (g_2)$$

taking the points of one line to the points on the other.

Proof. Without loss of generality, suppose $g_1 \neq g_2$. First, consider the case where the two lines intersect at some point S. Choose points P_1 on g_1 and P_2 on g_2 which are distinct from S. Then, by Axiom 3, there must be a third point, P on the line P_1P_2. Since $g_1 \neq g_2$, P is not on either line. By Axiom 2, a line through P containing $X \neq S$ on the line g_1 intersects the line g_2 at some uniquely determined point $\phi(X) \neq S$. This follows because the line XS must intersect the line PP_1 (at point P_2 in fact) and Axiom 2 gives that then XP must intersect the line SP_1 at a point distinct from S.



Consider a map defined by this procedure:

$$\phi : X \to XP \cap g_2.$$

Suppose $X_1 \neq X_2$ both mapped to the same $\phi(X)$. Then both points would be on a line incident with P and $\phi(X)$, but then by Axiom 1, they must be the same line, so $X_1 = X_2$. Thus ϕ is injective. Now suppose there is a point $\phi(X)$ on the line g_2. Then, by Axiom 2, since PP_2 intersects $S\phi(X)$, $P\phi(X)$ must intersect $SP_2 = g_1$ at some point X as desired. So ϕ maps X to $\phi(X)$ and the map is surjective. Thus ϕ is a bijection from $(g_1)\backslash\{S\}$ to $(g_2)\backslash\{S\}$. Define $\phi(S) = S$, and then ϕ is a bijection from (g_1) to (g_2) as desired.

Now, in the case that the two lines do not intersect at some point, pick a point on each line and consider the line h through those two points. From the first case, there are the following bijections:

$$\phi_1 : (g_1) \to (h) \text{ and } \phi_2 : (h) \to (g_2).$$

Therefore, we can find a bijective map $\phi = \phi_2 \circ \phi_1$ from the points of g_1 to those of g_2. □

In particular, this lemma implies that all lines of a projective plane are incident with the same number of points, making projective geometries particularrly nice.

Definition 5.2. The *order* of a finite projective space is denoted by q and is one less than the number of points incident with each line (which is a fixed number by the preceding lemma).

The dimension, d, one less than the number of points in the geometry, and the order q are the two important parameters for a finite projective space. In fact, two geometries with the same order and dimension are isomorphic. Additionally, knowing these two numbers, many others calculations can be made regarding the finite projective plane.

Lemma 5.3. *Suppose* $\boldsymbol{P}$ *is a finite projective space with dimension* d *and order* q*. Then for every point* Q*,* $\boldsymbol{P}/Q$ *also has order* q*.*

Proof. By Theorem 4.3 we know that $\mathbf{P}/Q$ is isomporphic to any hyperplane, and so we have that $\mathbf{P}/Q$ is a projective space of order q. □

Theorem 5.4. *Suppose* $\boldsymbol{P}$ *is a finite projective space with dimension* d *and order* q*. Let* $\boldsymbol{U}$ *is a* t*-dimensional subspace of* $\boldsymbol{P}$*. Then:*

(a) The number of points of the subspace is:

$$q^t + q^{t-1} + \ldots + q + 1 = \frac{q^{t+1} - 1}{q - 1}.$$

(b) The number of lines of $\boldsymbol{U}$ *through a fixed point of* $\boldsymbol{U}$ *is:*

$$q^{t-1} + \ldots + q + 1.$$

(c) The total number of lines of $\boldsymbol{U}$ *is:*

$$\frac{(q^t + q^{t-1} + \ldots + q + 1)(q^{t-1} + \ldots + q + 1)}{q + 1}.$$

Proof. We start with induction on t to prove the first two claims. Suppose $t = 1$, then from Lemma 5.1 and the definition of order, the subspace clearly has $q + 1$ points, and being a line itself, has one line.

Suppose the first two claims hold for $t - 1 \geq 1$. Then, by Theorem 4.3 and Lemma 5.3 we have that the quotient geometry is a projective space of dimension $t-1$ and has order q. By induction the number of points of $\mathbf{U}/Q$ is $q^{t-1}+\ldots+q+1$, which by definition is the number of lines of $\mathbf{U}$ through Q, which gives the second claim.

Then, because there are q points on these lines through Q which are distinct from Q, and each point of the subspace $\mathbf{U}$ must lie on precisely one of these lines, we have

$$1 + (q^{t-1} + \ldots + q + 1)q = q^t + q^{t-1} + \ldots + q + 1$$

points in $\mathbf{U}$, completing the induction.

For the final part, note that $\mathbf{U}$ has $q^t + q^{t-1} + \ldots + q + 1$ points and each point is on exactly $q^{t-1}+\ldots+q+1$ lines which each have $q+1$ points. Thus the number of lines of this subspace must be

$$\frac{(q^t+q^{t-1}+\ldots+q+1)(q^{t-1}+\ldots+q+1)}{q+1}$$

as desired. □

Theorem 5.5. *Suppose **P** is a finite projective space with dimension d and order q. Then,*

(1) *The number of hyperplanes of **P** is exactly*

$$q^d+\ldots+q+1.$$

(2) *The number of hyperplanes of **P** through a fixed point is*

$$q^{d-1}+\ldots+q+1.$$

Proof. (1) Use induction on d. Suppose $d=1$. Then the claim is that any line has $q+1$ points, which follow by definition, and if $d=2$, then the claim follows from the preceding theorem.

Suppose the claim holds for dimension $d-1$. Consider a hyperplane **H** of **P**. Then all other hyperplanes intersect it in a subspace of dimension $d-2$. So any hyperplane distinct from **H** is spanned by a $d-2$ dimensional subspace **U** of **H** and a point P outside of **H**.

The for every such subspace and point, $\langle \mathbf{U}, P\rangle$ is a hyperplane containing

$$(q^{d-1}+\ldots+q+1)-(q^{d-2}+\ldots+q+1)=q^{d-1}$$

points not in **H**. Likewise, there are q^d points of **P** outside of **H**, giving a total of q hyperplanes through every **U** which are distinct from **H**. By induction, there are $q^{d-1}+\ldots+q+1$ hyperplanes of **H**. This means that there are q hyperplanes of the space **P** corresponding to each subspace of **H**. Thus there are

$$q(q^{d-1}+\ldots+q+1)=q^d+\ldots+q+1$$

hyperplanes as desired.

(2) Suppose P is a point of **P** and **H** is a hyperplane not intersecting P. Then every hyperplane of **P** through P must intersect **H** in a hyperplane of **H**. By the previous part, there are precisely $q^{d-1}+\ldots+q+1$ such hyperplanes. □

Corollary 5.6. *Suppose **P** is a finite projective plane. Then there exists $q\geq 2$ such that any line has exactly $q+1$ points and the total number of points is q^2+q+1.*

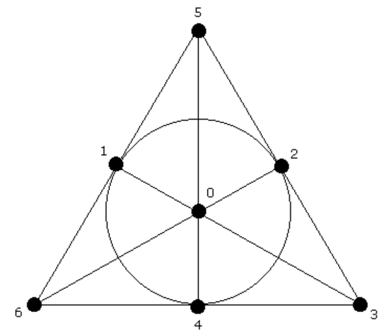
This number q is simply the order of the finite projective plane. The next section investigates a finite projective plane of order 2.

6. The Fano Plane

Simply from the equations derived in the previous section, it is easy to calculate the number of points in possible projective geometries. What is more difficult, however, is to show that such geometries exist.

Since we know that $q \geq 2$, the smallest possible projective plane, if it exists, will have $2^2 + 2 + 1 = 7$ points.

Such a plane does in fact exist, and is known as the Fano Plane. As is evident from the diagram at the side, the Fano Plane does satisfy the axioms of a projective geometry on seven points. However, not only is the Fano Plane an example of the smallest order projective geometry, but it is also the only example. This motivates the following theorem.



The Fano Plane

Theorem 6.1. *There is one unique projective plane of order 2.*

Proof. A projective plane of order 2, must have three lines through each point and three points on each line by definition. Thus there are three lines through point 1, say the lines $\{1, 0, 3\}$, $\{1, 2, 4\}$, and $\{5, 1, 6\}$. Now, the point 2 is also on two other lines. Without loss of generality, because it is only a choice of labeling, we can say these lines are $\{5, 2, 3\}$ and $\{6, 0, 2\}$. Now, there must be a line through points 4 and 5 is incident to one other point. This point cannot be on the same line as 4 or 5 already, so it must therefore be the point 0 giving the line $\{4, 0, 5\}$. Likewise, there is a line through points 6 and 4, and by the same argument, it must be point 3 giving the line $\{6, 4, 3\}$. Thus, except for relabelling, this projective plane is unique. □

There is another way to conceptualize the Fano Plane. Instead of begining with axioms to define a projective space, we can begin with a field. In a general sense we can consider a vector space over any field. A vector space will have a certain dimension. In order to make this space akin to a projective space we need to do away with parallel lines. One way that this is often accomplished is to consider the planes as lines and the lines as points. Using the field $\mathbb{R}$ gives the vector space R^n and this will give an $n - 1$ dimensional projective space.

In the case of the Fano Plane, consider the field F_2, the finite field of two elements. If we then consider the 3-dimensional vector space over this field, we get 8 different points: $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1,)$ and $(1, 1, 1)$. We derive a projective geometry from this structure by projecting lines to points and planes to lines. That is, every line through the origin becomes a point and every plane through the origin becomes a line. Since two points define a line, there are clearly seven points through the origin, giving us seven points. Likewise there are seven planes giving seven lines.

This must therefore be the Fano Plane, since it is the unique projective plane of this size. Thus, the Fano Plane can be thought of as $F_2 \times F_2 \times F_2$.

References

[1] Albrecht Beutelspacher and Ute Rosenbaum. *Projective Geometry.* Cambridge University Press. 1998.

[2] Lynn Margaret Batten. *Combinatorics of Finite Geometries.* Cambridge University Press. 1986.