# THE SHRINKING WEDGE OF CIRCLES

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Abstract. This paper describes certain properties of the fundamental group of the shrinking wedge of circles. Covering space theory is an inadequate tool since the shrinking wedge of circles is not semi-locally simply connected. We conclude with the fact that the shrinking wedge of circles is not homotopy equivalent to the wedge product of a countable number of circles.

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## 1. The Fundamental Group

The idea of the fundamental group is to associate with each space a group so that homotopic spaces have isomorphic fundamental groups. This group will be the set of equivalence classes of loops in the space with the binary operation of concatenation.

**Definition 1.1.** A *map* is a continious function.

**Definition 1.2.** A *path* in a space $X$ is a map $f : I \to X$. A *loop* is a path with$f(0) = f(1)$.

**Definition 1.3.** A *homotopy of paths* in $X$ is a family of paths $f_t : I \to X, 0 \leq t \leq 1$ such that:

(1) $f_t(0) = x$ and $f_t(1) = y$
(2) The function $F : I \times I \times I \to X$ defined by $F(t, s) = f_t(s)$ is continious.

**Definition 1.4.** Two paths $f, g$ are equivalent, denoted $f \sim g$, if they are homotopic through paths from $x$ to $y$.

**Theorem 1.5.** *Path equivalence is an equivalence relation.*

*Proof.* We have reflexivity since $f$ is equivelent to $f$ by the constant homotopy $h_t(s) = f(s)$. For symmetry suppose that $f \sim g$, then there exists a homotopy $h_t(s)$ from $f$ to $g$. Define $j_t = h_{1-t}$. Since $j$ is the composition of two maps it

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is also a map; furthermore $j_0 = h_1 = g$, $j_1 = h_0 = f$. It follows that $g \sim f$. For transitivity, suppose $\alpha \sim \beta$ via $f_t$ and $\beta \sim \gamma$ via $g_t$. Let $h(t)$ equal $f(2t)$ for $0 \leq t \leq 1/2$ and $g(2t-1)$ for $1/2 \leq t \leq 1$. This function is well defined since $f(1) = g(0) = \beta$ and continuous since it is defined on the union of the two closed sets $[0,1] \times [0,1/2]$ and $[0,1] \times [1/2,1]$ and continuous on each by the continuity of $f$ and $g$. □

**Definition 1.6.** Define $\pi_1(X,x)$ to be the set of equivalence classes of loops in the space $X$ that start and end at the point $x$.

**Definition 1.7.** For paths $f : x \to y$, $g : y \to z$, define the binary operation of *concatenation*, denoted $g \cdot f$, by traversing $f$ twice as fast and then $g$ twice as fast. Explicitly,

$$(f \cdot g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq 1/2, \\ g(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Furthermore define $c_x(t) = x$, the constant loop at $x$, and $f^{-1}(t) = f(1-t)$ the loop traversed the other way around $f$.

**Theorem 1.8.** *Concatenation is well defined on equivalence classes.*

*Proof.* If $f_0 \sim f_1$ via $f_t$ and $g_0 \sim g_1$ via $g_t$ then $f_0 \cdot g_0 \sim f_1 \cdot g_1$ via $f_t \cdot g_t$. □

**Theorem 1.9.** $(\pi_1(X,x), \cdot)$ *is a group with identity* $c_x$. *With the binary operation understood, we call* $\pi_1(X,x)$ *the fundamental group of the space* $X$ *at basepoint* $x$.

*Proof.* For closure, since the elements of $\pi_1(X,x)$ are loops beginning and ending at $x$ their concatenation is also a loop beginning and ending at $x$. For associativity, we have $h \cdot (g \cdot f) \sim (h \cdot g) \cdot f$ by a reparametrization. For the identity, $c_x \cdot f \sim f \cdot c_x \sim f$ by another reparametrization. For inverses, $f \cdot f^{-1} \sim c_x = e$ by the homotopy:

$$h(s,t) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq t/2, \\ f(t) & \text{if } t/2 \leq s \leq 1 - t/2 \\ f(2-2s) & \text{if } 1 - t/2 \leq s \leq 1 \end{cases}$$

Additionally $f^{-1} \cdot f \sim c_x$ by a similar homotopy. □

**Example 1.10.** Any convex set $X$ in $\mathbb{R}^n$ has a trivial fundamental group, since any two loops $f_0, f_1$ are homotopic via the linear homotopy $f_t(s) = (1-t)f_0(s) + tf_1(s)$

**Theorem 1.11.** *If there exists a path* $h$ *connecting* $x_0$ *to* $x_1$ *in the space* $X$ *then* $\pi_1(X,x_0)$ *is isomorphic to* $\pi_1(X,x_1)$.

*Proof.* Let $\phi_h : \pi_1(X,x_1) \to \pi_1(X,x_0)$ be defined by $\phi_h(f) = h \cdot f \cdot h^{-1}$, where $h^{-1}(t) = h(1-t)$, the string traversed in the opposite direction. We show this is an isomorphism. First it is well defined since for $f_t$ a homotopy of loops based at $x_1$ then $\phi_h(f_t)$ is a homotopy of loops based at $x_0$. $\phi_h$ is a homomorphism since $\phi_h(f \cdot g) = h \cdot f \cdot g \cdot h^{-1} = h \cdot f \cdot h^{-1} \cdot h \cdot g \cdot h^{-1} = \phi_h(f) \cdot \phi_h(g)$. Finally, it is an isomorphism since it has inverse $\phi_{h^{-1}}$. □

**Theorem 1.12.** *If a space* $X$ *retracts onto a subspace* $A$*, then the homomorphism* $i_* : \pi_1(A,x_0) \to \pi_1(X,x_0)$ *induced by the inclusion* $i : A \to X$ *is an injection. If* $A$ *is a deformation retract of* $X$*, then* $i_*$ *is an isomorphism.*

*Proof.* Suppose $r : X \to A$ is a retraction. Since $ri$ is the identity map we have that $r_* i_*$ is the identity homomorphism. Thus $i_*$ is injective. Furthermore, if $r_t : X \to X$ is a deformation retraction of $X$ onto $A$, then for loop $f$ based at $x_0 \in A$, $r_t F$ gives a homotopy of $F$ to a loop in $A$. Consequently, $i_*$ is also surjective. $\square$

**Lemma 1.13.** *If $\phi_t : X \to Y$ is a homotopy, $f \in \pi_1(Y, \phi_1(x_0))$, and $h$ is the path $\phi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the following diagram is commutative:*

$$\begin{array}{ccc} \pi_1(X, x_0) & & \\ \downarrow \phi_{1*} & \searrow^{\phi_{0*}} & \\ \pi_1(Y, \phi_1(x_0)) & \xrightarrow[h \cdot f \cdot h^{-1}]{} & \pi_1(Y, \phi_0(x_0)) \end{array}$$

We omit the proof of this lemma.

**Theorem 1.14.** *If $\phi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.*

*Proof.* Let $h_t(s) = n(ts)$, $0 \leq t \leq 1$ for $h$ given in the above lemma. If $f$ is a loop in $X$ based at $x_0$, then the product $h_t \cdot (\phi_t f) \cdot h_t^{-1}$ is a homotopy of loops at $\phi_0(x_0)$. Thus we have $\phi_{0*}(f) = h \cdot \phi_{1*}(f) \cdot h^{-1}$. $\square$

## 2. Van Kampen's Theorem

Van Kampen's Theorem allows us to determine the fundamental group of spaces that constructed in a certain manner from other spaces with known fundamental groups.

**Theorem 2.1.** *If a space $X$ is the union of path-connected open sets $A_\alpha$ each containing the basepoint $x_0 \in X$ such that each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism induced by the inclusion map from the free product of the fundamental groups of the $A_\alpha$ to the fundamental group of $X$, $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$, is surjective. Furthermore, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then* $\ker(\Phi)$ *is the normal subgroup, $N$, generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ where $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta$ in $A_\alpha$, and so we have $\pi_1(X) \approx *_\alpha \pi_1(A_\alpha)/N$*

We omit the proof of Van Kampen's Theorem.

**Example 2.2** (Wedge Sums)**.** The wedge sum of a collection of spaces $\bigvee_\alpha X_\alpha$ is the quotient space of the disjoint union of the spaces in which a basepoint $x_\alpha \in X_\alpha$ is identified to a single point $x$. Thus, if each $x_\alpha$ is a deformation retract of an open neighborhood $U_\alpha$ contained in $X_\alpha$, then $X_\alpha$ is a deformation retract of the open neighborhood $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$. Thus we have that the intersection of two or more $A_\alpha$ is the wedge product of the $U_\alpha$. These deformation retract to $x_0$ so by Van Kampen's Theorem $\pi_1(\bigvee_\alpha A_\alpha) \approx *_\alpha \pi_1(X_\alpha)$. In the specific case of the wedge sum of circles we have $\pi_1(\bigvee_\alpha S^1_\alpha) = *_\alpha \mathbb{Z}_\alpha$

## 3. Covering Space Theory

Covering Space Theory provides a tool for clarifying the structure of the fundamental group of a space.

**Definition 3.1.** A *covering space* of a space $X$ is a space $\tilde{X}$ with a map $p : \tilde{X} \to X$ such that there exists an open cover$\{U_\alpha\}$ of $X$ such that for each $\alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$ and each is mapped by $p$ homeomorphically onto $U_\alpha$.

**Definition 3.2.** A *universal cover* of a space $X$ is a covering space $(\tilde{X}, p)$ such that $\tilde{X}$ is simply connected.

**Definition 3.3.** A *lift* of a map $f : Y \to X$ in a covering space $(\tilde{X}, p)$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$.

**Theorem 3.4** (Homotopy Lifting Property)**.** *Given a covering space $p : \tilde{x} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : y \to \tilde{X}$ lifting $f_0$, then there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts $f_t$.*

We omit the proof of the Homotopy Lifting Property.

**Definition 3.5.** A space $X$ is *semi-locally simply connected* if every point $x \in X$ has a neighborhood $U$ in $X$ such that any loop in $U$ with basepoint $x$ is homotopic to $c_x$.

**Theorem 3.6.** *If a space $X$ has a universal covering space, then $X$ is semi-locally simply connected.*

*Proof.* Suppose $p : \tilde{X} \to X$ is a covering space with $\tilde{X}$ simply connected, then every point $x \in X$ has a neighborhood $U$ having a lift $\tilde{U} \subset \tilde{X}$ projecting homomorphically to $U$ by $p$. A loop in $U$ lifts to a loop in $\tilde{U}$, and this loop is homotopic to the trivial loop in $\tilde{X}$; thus, composing this homotopy with $p$, we have the loop in $U$ is homoeomorphic to the trivial loop in $X$. □

## 4. Fundamental Group of the Circle

**Theorem 4.1.** *The fundamental group of the circle $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$ via the isomorphism $\Phi : \mathbb{Z} \to \pi_1(S^1)$ sending $n \in \mathbb{Z}$ to the homotopy class of the loop $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ based at $(0, 1)$.*

*Proof.* $\mathbb{R}$ is a covering space of $S^1$ via the map $p : \mathbb{R} \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$ if we take our open cover of $S^1$ to be two open arcs whose union is $S^1$. The path $\tilde{\omega}_n(s) = ns$ lifts $\omega_n$. To show $\Phi$ is a homomorphism, $\Phi(m + n) = \omega_{m+n} \sim \omega_m \cdot \omega_n = \Phi(m) \cdot \Phi(n)$. Additionally, $\Phi(n)$ is equal to the homotopy class of the loop $p\tilde{f}$ for $\tilde{f}$ any path in $\mathbb{R}$ from 0 to $n$. For surjectivity, let $f$ be a loop at the basepoint, then by the homotopy lifting property there exists a lift $\tilde{f}$ starting at 0. This path $\tilde{f}$ ends on some integer $n$ since the preimage of the origin under $p$ is $\mathbb{Z}$, and so we have $\Phi(n) = [p\tilde{f}] = [f]$. For injectivity, $\Phi(m) = \Phi(n)$ implies that $\omega_m \sim \omega_n$, by the homotopy $f_t$. By the homotopy lifting property, $f_t$ lifts to the unique homotopy $\tilde{f}_t$ of $\tilde{\omega}_n$ and $\tilde{\omega}_m$, which implies that $m = n$ since a homotopy leaves endpoints fixed. □

## 5. The shrinking wedge of circles

The shrinking wedge of circles gives an example of a space that is not semi-locally simply connected and as a consequence has no universal covering space. This removes the possibility of using covering spaces to discover the fundamental group of this space. One might confuse this space for the wedge product of a countable number of circles, but while the fundamental group of the infinite wedge of circles is countable the fundamental group for the shrinking wedge of circles in uncountable. This uncountable group is called a Big Free Group, but it is not free.

**Definition 5.1.** The shrinking wedge of circles $C$ is a subset of the $xy$-plane that is the union of countably many circles $\bigcup_{1\leq n\leq\infty} C_n$ where $C_n$ is the circle of radius $1/n$ and center $(1/n, 0)$. The shrinking wedge of circles is compact, globally path connected, and locally path connected. Define $(0,0) = O$ to be the basepoint, and $G$ to be the fundamental group $\pi_1(C, O)$.

**Theorem 5.2.** *$C$ is not semi-locally simply connected.*

*Proof.* Every neighboorhood of $O$ contains all but a finite number of the circles $\{C_n\}$. A loop around any one of these circles is not homotopic to $C_O$. □

**Theorem 5.3.** *$G$ is uncountable.*

*Proof.* Let $r_n : C \to C_n$ be the retraction that takes all $\{C_i | i \neq n\}$ to $O$. Now take the surjection $\rho_n : \pi_1(C, O) \to \pi_1(C_n, O) \approx \mathbb{Z}$ induced by $r_n$. The product of the $\rho_n$ is the homomorphism $\rho : \pi_1(C, O) \to \prod_\infty \mathbb{Z}$. The surjectivity of $\rho$ follows from the following construction. Let $\{k_n\}$ be a sequence of integers and $f$ to be the loop that traverses $k_n$ times around $C_n$ on the interval $[1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$. This loop is continuous since every neighborhood of $O$ contains all but finitely many of the circles $C_n$. Thus, $\pi_1(C, O)$ is uncountable since $\prod_\infty \mathbb{Z}$ is uncountable. □

**Corollary 5.4.** *$C$ is not homotopy equivalent to the wedge sum of a countable number of circles.*

*Proof.* By Van Kampen's theorem the fundamental group of the wedge product of a countably infinite number of circles is equal to $*_i\mathbb{Z}$ which is a countable set. The fundamental group of $C$ is uncountable, however. □

## References

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