AN ELUCIDATION OF VECTOR CALCULUS THROUGH DIFFERENTIAL FORMS

JONATHAN EMBERTON

ABSTRACT. The purpose of this paper is to introduce differential forms to readers without a background in differential geometry. With the presumption that the reader is experienced in advanced calculus, we provide motivation for learning some of the most important concepts in differential geometry by re-observing the theorems of vector calculus.

CONTENTS

1. INTRODUCTION AND OVERVIEW

The attentive student has doubtlessly been puzzled by some aspect of vector calculus in the past. This is inevitable, since some of the notation, such as the $Pdx+Qdy$ format of Green's Theorem, is usually left unexplained. In addition, students may fall under the impression that the significant operations of vector calculus (div, grad, and curl) are unrelated. The goal of this paper is to elaborate on some of the results of vector calculus, especially those results (such as the aforementioned) that cannot be explained to their fullest in introductory courses.

This paper begins by introducing various concepts of differential geometry, so that the reader may gain a firm grounding before embracing the goal of this expository piece. The proofs in this paper are admittedly presented in a rather unusual fashion - I incorporate only the proofs that I think are beneficial to the reader. For example, I omit the proofs of the set of criteria that uniquely define the wedge product, since the proofs of these properties are lengthy, though not particularly profitable to the reader since many tedious calculations are required. However, I

Date: August 22, 2008.

prove the facts that $d(d\omega) = 0$ and that the differential operator is unique - this is because such proofs offer the reader a better grasp of the usages of d. I have also included the proofs of the Poincare Lemma and Stokes' Theorem, as I believe that any serious mathematician should have at least seen them. However, I believe that if I had included the proof of every theorem, this paper would have become cluttered and possibly disheartening to the beginner.

2. Tensor Analysis

We begin with a series of definitions, particularly that of the tensor, that are employed throughout this paper. The axiomatic idea of a tensor is, in fact, not difficult. The reader may initially think of the tensor as a generalization of a linear transformation to copies of a vector space.

Definition 2.1. Let V be a vector space, and let $V^n = V \times ... \times V$. Holding $v_1, ..., v_{i-1}, v_{i+1}, ..., v_n$ constant, let $f(v_1, ..., v_{i-1}, v, v_{i+1}, ..., v_n)$ be linear. In this case, f is said to be *linear in the ith variable*. If f is linear in the ith variable for $1 \leq i \leq n$, then we say that f is a *multilinear* function.

Definition 2.2. Let $\zeta : V^k \to \mathbb{R}$ be a function. We define ζ to be a (covariant) *k-tensor*[1] on V if ζ is multilinear.

A simple example of a 2-tensor is the inner product on a vector space V over $\mathbb{R}$, particularly the dot product on $\mathbb{R}^n$. By definition of the inner product $\langle , \rangle$, for $u, v, w \in V$ and $\alpha \in \mathbb{R}$, the following properties hold:

(1) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$;
(2) $\langle v, w\rangle = \langle w, v\rangle$;
(3) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$;
(4) $\langle u, u\rangle \geq 0$; moreover, $\langle u, u\rangle = 0$ iff $u = 0$.

The first and second properties guarantee the additive requirement of multilinearity, and the second and third properties guarantee the scalar multiplicative requirement. We can, of course, drop the requirement of the 4th property and still obtain a 2-tensor on V.

Definition 2.3. The set of all k-tensors on a vector space V is denoted $T^k(V)$. For $\zeta, \eta \in T^k(V)$, and $c \in \mathbb{R}$, we then define

$$(\zeta + \eta)(v_1, ..., v_k) = \zeta(v_1, ..., v_k) + \eta(v_1, ..., v_k)$$
$$(c\zeta)(v_1, ..., v_k) = c(\zeta(v_1, ..., v_k)).$$

With this definition, we can conclude that for $k \in \mathbb{N}$, $T^k(V)$ is a vector space. To see this, one must simply check that $T^k(V)$ with addition and scalar multiplication satisfy the vector space axioms. The zero element of V is the function whose value is zero on every k-tuple of vectors.

[1]There also exists a multilinear function on V called a contravariant tensor, which maps copies of the dual space V^* of V into $\mathbb{R}$. However, since we will not use contravariant tensors in this paper, covariant k-tensors will always be referred to as simply k-tensors.

$T^1(V)$ is the dual space V^*, the set of all linear transformations $T : V \to \mathbb{R}$. We make the convention that $T^0(V) = \mathbb{R}$.

The next theorem tells us that tensors are uniquely determined by their values on basis elements.

Lemma 2.4. *Let $b_1, ..., b_n$ be a basis for a vector space V. Let $\zeta, \eta : V^k \to \mathbb{R}$ be k-tensors on V satisfying $\zeta(b_{i_1}, ..., b_{i_k}) = \eta(b_{i_1}, ..., b_{i_k})$ for every k-tuple $I = (i_1, ..., i_k)$, where $1 \leq i_m \leq n$. Then $\zeta = \eta$.*

Proof. Let $v_i = \sum_{j=1}^{n} c_{ij} b_j$. Then, using the fact that ζ and η are multilinear, we expand ζ and observe that $\zeta(v_1, ..., v_k) = \sum c_{1j_1} c_{2j_2} \cdots c_{kj_k} \zeta(b_{j_1}, ..., b_{j_k})$. An expansion of η results in the same formula. Hence, $\zeta = \eta$ if the values of ζ and η are equal on all combinations of basis elements. □

Theorem 2.5. *Let $b_1, ..., b_n$ be a basis for a vector space V. Let $I = (i_1, ..., i_k)$ be a k-tuple of integers, where $1 \leq i_m \leq n$. There exists a unique k-tensor θ_I on V such that for every k-tuple $J = (j_1, ..., j_k)$ satisfying $1 \leq j_m \leq n$,*

$$\theta_I(b_{j_1}, ..., b_{j_k}) = \delta_{IJ}.$$

Proof. Let $k = 1$. From linear algebra, we know that we can determine a 1-tensor $\theta_i : V \to \mathbb{R}$ by giving it arbitrary values on its basis elements. Then define:

$$\theta_i(b_j) = \delta_{ij}$$

These work as the unique 1-tensors stated in the theorem. If $k > 1$, define

$$\theta_J(b_1, ..., b_k) = \prod_{i=1}^{k} \theta_{j_i}(b_i). \tag{2.6}$$

These tensors easily satisfy the restrictions placed upon them in the theorem. Uniqueness follows from Lemma 2.4, and the proof is complete. □

The θ_I are often called elementary k-tensors. This set of elementary tensors is, in fact, vital to the development of the rest of this paper. They play a substantial role in the following chapters, primarily because of the following theorem.

Theorem 2.7. *The k-tensors θ_I form a basis for $T^k(V)$.*

Proof. Let ζ be a k-tensor on V. Given a k-tuple $I = (i_1, ..., i_k)$, let $c_I = \zeta(b_{i_1}, ..., b_{i_k})$. Since $T^k(V)$ is a vector space, we can define a k-tensor $\eta = \sum_I d_I \theta_I$, where J spans over all k-tuples of integers from $\{1, ..., n\}$.

By definition, $\theta_I(b_{j_1}, ..., b_{j_k}) = 0$ except when $I = J$, in which case θ_J equals 1. Therefore, $\eta(b_{j_1}, ..., b_{j_k}) = c_I = \zeta(b_{j_1}, ..., b_{j_k})$. Lemma 2.4 implies that $\zeta = \eta$ and that this representation is unique. □

As a result, given a k-tensor ζ, we can always write $\zeta = \sum_I c_I \theta_I$, where θ_I are the elementary k-tensors, and c_I are scalars. Hence, $T^k(V)$ has dimension n^k.

To see what tensors actually look like, we might resort to looking at the case $V = \mathbb{R}^n$. Let $e_1, ..., e_n$ denote the standard basis for $\mathbb{R}^n$, and let $\theta_1, ..., \theta_n$ be the basis for $T^1(\mathbb{R}^n)$. Then if $\mathbf{x} = x_1e_1 + \cdots + x_ne_n$, we have $\theta_i(x) = \theta_i(x_1e_1 + \cdots + x_ne_n) = x_1\theta_i(e_1) + \cdots + x_n\theta_i(e_n) = x_i$.

Therefore, the basis vectors for 1-tensors on $\mathbb{R}^n$ are projection functions, and it follows that 1-tensors represent functions of the form $\zeta(x) = \sum_{i=1}^{n} c_ix_i$, where $c_i \in \mathbb{R}$. Similarly, the basis vectors for 2-tensors on $\mathbb{R}^n$ are of the form $\theta_J(x_1, x_2) = \theta_{j_1}(x_1)\cdot\theta_{j_2}(x_2)$ so that the 2-tensor on $\mathbb{R}^n$ looks like $\zeta(x, y) = \sum_{i,j=1}^{n} c_{ij}x_iy_j$, where c_{ij} are scalars.

We now advance to a special kind of tensor that will play a key role in the theory of differential forms. We begin with some preliminary work in algebra.

Definition 2.8. A *permutation* σ of a set A is a bijection from A to itself. The set of all permutations of $\{1, \ldots, k\}$ is denoted by S_k.

The intuitive way to think about permutations is in terms of order: the operation of a permutation is essentially to alter the *order* of the elements of the set.

Definition 2.9. A *transposition* γ is a permutation of $\{a_1, ..., a_m\}$ such that there exist $i, j \in \{1, ..., m\}$, $i \neq j$, with $\gamma(a_j) = a_i$, $\gamma(a_i) = a_j$, and $\gamma(a_k) = a_k$ for $k \neq i, j$.

A transposition, in easier terms, swaps two elements and leaves the others alone. A transposition γ is always its own inverse. If A is a set, then $\gamma(\gamma(A)) = A$ because the first action swaps two terms, and the second action swaps them back into their original positions.

Notation 2.10. Hereinafter, for $\sigma \in S_k$, the notation $f^\sigma(x_1, ..., x_n) = f(x_{\sigma(1)}, ..., x_{\sigma(n)})$ will be used. This will greatly simplify notation in the remainder of the text.

Some nice properties of f^σ are exhibited in the following two theorems.

Theorem 2.11. *For $\sigma \in S_k$, the transformation $f \to f^\sigma$ is linear..*

Proof. It should be proved that $(af+bg)^\sigma = af^\sigma+bg^\sigma$, which is straightforward. □

Theorem 2.12. *For $\sigma, \tau \in S_k$, the equation $f^{\sigma\circ\tau} = (f^\sigma)^\tau$ holds.*

Proof.

$$(f^{\sigma\circ\tau})(v_1, \ldots, v_k) = f(v_{\sigma(\tau(1))}, \ldots, v_{\sigma(\tau(k))}) = \\ f^\sigma(v_{\tau(1)}, \ldots, v_{\tau(k)}) = (f^\sigma)^\tau(v_1, \ldots, v_k).$$

□

We now define the fundamental idea that lies behind the differential form.

Definition 2.13. Let V be a set. A function $\zeta : V^n \to \mathbb{R}$ is said to be *alternating* if, given any transposition γ on $\{1, ..., n\}$, $f = -f^\gamma$.

The set of all alternating k-tensors on a vector space V is denoted by $\Lambda^k(V)$. In fact, the set $\Lambda^k(V)$ is a vector space. Because $T^1(V)$ is trivially alternating, $T^1(V) = \Lambda^1(V)$. We also use the convention that $T^0(V) = \Lambda^0(V) = \mathbb{R}$.

In terms of bases, the alternating tensor has many of the same features as the tensor. There is a bit of a quirk, however. The bases are defined over all *ascending* elements of a given k-tuple I.

Lemma 2.14. *Every permutation σ can be written as a composition of transpositions. That is, $\sigma = \gamma_1 \circ \cdots \circ \gamma_n$, where γ_k is a transposition. Although the number of transpositions used in such a decomposition is not unique, the parity is well-defined.*

Definition 2.15. Let $\sigma = \gamma_n \circ \cdots \circ \gamma_1$. Then we define sgn $\sigma = (-1)^n$.

By Lemma 2.14, sgn σ is a well-defined function. In addition, the equation $\text{sgn}(\sigma \circ \tau) = \text{sgn } \sigma \cdot \text{sgn } \tau$ holds.

Theorem 2.16. *If f is alternating, then $f^\sigma = (sgn\ \sigma)f$. Moreover, if f is alternating with $v_p = v_q$ $(p \neq q)$, then $f(v_1, ..., v_k) = 0$.*

Proof. Let $\sigma = \gamma_m \circ \cdots \circ \gamma_1$. Then, because f is alternating, $f^\sigma = f^{\gamma_m \circ \cdots \circ \gamma_1} = (-1)^m f = (\text{sgn } \sigma)f$. Now let $v_p = v_q$, with $p \neq q$. Let γ be the transposition switching p and q. Then we have $f^\gamma(v_1, ..., v_k) = f(v_1, ..., v_k)$ since $v_p = v_q$, and $f^\gamma(v_1, ..., v_k) = (\text{sgn } \gamma)f(v_1, ..., v_k) = -f(v_1, ..., v_k)$. Hence, $f(v_1, ..., v_k) = 0$. □

It is also trivially true that if $f^\sigma = (\text{sgn } \sigma)f$, f is alternating. Now we can begin our study of the bases of $\Lambda^k(V)$.

Lemma 2.17. *Let $b_1, ..., b_n$ be a basis for V. If $\rho, \xi \in \Lambda^k(V)$ and if $\rho(b_{i_1}, ..., b_{i_k}) = \xi(b_{i_1}, ..., b_{i_k})$ for every ascending k-tuple $I = (i_1, ..., i_k)$, where $1 \leq i_m \leq n$, then $\rho = \xi$.*

Proof. It is sufficient to prove that ρ and ξ are equal on every arbitrary k-tuple of basis elements. The result will then follow from Lemma 2.4.

Let $J = (j_1, ..., j_k)$. If $j_p = j_q$ for some p, q with $p \neq q$, then by Theorem 2.16, $\rho = \xi = 0$. If all indices are distinct, let σ be a permutation of $\{1, ..., k\}$ such that $I = (j_{\sigma(1)}, ..., j_{\sigma(k)})$ is ascending. Then we have that $\rho(b_{i_1}, ..., b_{i_k}) = \rho^\sigma(b_{j_1}, ..., b_{j_k}) = (\text{sgn } \sigma)\rho(b_{j_1}, ..., b_{j_k})$, since ρ is alternating. Similarly, we have $\xi^\sigma(b_{i_1}, ..., b_{i_k}) = (\text{sgn } \sigma)\xi(b_{j_1}, ..., b_{j_k})$, and so agreement on ascending basis vectors amounts to agreement on all basis vectors. □

Theorem 2.18. *Let V be a vector space, and let $b_1, ..., b_n$ form a basis for V. Let $I = (i_1, ..., i_k)$ be an ascending k-tuple, where $1 \leq i_m \leq n$. There exists a unique alternating k-tensor $\tilde{\theta}_I$ defined on V such that for every ascending k-tuple $J = (j_1, ..., j_k)$ (with $1 \leq j_m \leq n$),*

$$\tilde{\theta}_I(b_{j_1}, ..., b_{j_k}) = \delta_{IJ} \tag{2.19}$$

.

Proof. We start by defining

$$\tilde{\theta}_I = \sum_{\sigma \in S_k} (\text{sgn } \sigma)(\theta_I)^\sigma. \tag{2.20}$$

Now, let τ be a permutation. Then

$$(\tilde{\theta}_I)^\tau = \sum_\sigma (\text{sgn } \sigma)\left((\theta_I)^\sigma\right)^\tau = \sum_\sigma (\text{sgn } \sigma)(\theta_I)^{\tau\circ\sigma} = \sum_\sigma \left(\text{sgn } (\tau^{-1}\circ\tau\circ\sigma)\right)(\theta_I)^{\tau\circ\sigma}.$$

Now, using the facts that sgn $(\tau^{-1}) =$ sgn τ and sgn $(\sigma\circ\tau) =$ (sgn σ)(sgn τ), this is equal to

$$(\text{sgn } \tau)\textstyle\sum_\sigma (\text{sgn } (\tau\circ\sigma))\,(\theta_I)^{\tau\circ\sigma} = (\text{sgn } \tau)\tilde{\theta}_I.$$

This proves that $\tilde{\theta}_I$ is alternating. To see that $\tilde{\theta}_I$ has the desired values, write $\tilde{\theta}_I(b_{j_1}, ..., b_{j_k}) = \sum_\sigma (\text{sgn } \sigma)\theta_I(b_{j_{\sigma(1)}}, ..., b_{j_{\sigma(k)}})$. By definition of θ_I, the only non-zero term in this sequence is the term for which σ is the identity permutation, i.e., the permutation $\sigma^* : I \to I$ s.t. $\sigma^*(i) = i\ \forall i \in I$. Since sgn $\sigma^* = 1$, we have $\tilde{\theta}_I = \delta_{IJ}$. Uniqueness follows from Lemma 2.17. ☐

Theorem 2.21. *The k-tensors $\tilde{\theta}_I$ form a basis for $\Lambda^k(V)$.*

Proof. We shall show that, given $\rho \in \Lambda^k(V)$, ρ can be written uniquely as a linear combination of the k-tensors $\tilde{\theta}_I$. Now, letting $I = (i_1, ..., i_k)$ (with $1 \le i_m \le n$), let $c_I = f(b_{i_1}, ..., b_{i_k})$.

Define $\xi \in \Lambda^k(V)$ to be $\xi = \sum_{<J>} c_J\tilde{\theta}_J$, where we use $< J >$ to denote the set of ascending k-tuples of $\{1, \ldots, n\}$. If $I = \{i_1, ..., i_k\}$ is an ascending k-tuple, then $\xi(b_{i_1}, ..., b_{i_k}) = c_I$. But we also have that $\rho(b_{i_1}, ..., b_{i_k}) = c_I$. It follows that $\rho = \xi$, and uniqueness follows from Lemma 2.17. ☐

Therefore, the dimension of $\Lambda^k(V)$ is $\binom{n}{k}$. From here, we observe some important functions on $T^k(V)$ and $\Lambda^k(V)$. The first two functions, the alternating projection and the tensor product, are not themselves important for our purposes, but they combine to form the wedge product, which is the single most important operation in this paper.

Definition 2.22. Let ζ be a k-tensor on a vector space V. We define a linear transformation Alt: $T^k(V) \to \Lambda^k(V)$, called the alternating projection, as follows:

$$\text{Alt}(\zeta) = \frac{1}{k!}\sum_{\sigma\in S_k}(\text{sgn } \sigma)(\zeta^\sigma) \tag{2.23}$$

Theorem 2.24. *There are several desirable properties that Alt possesses:*

1. *Given $\zeta \in T^k(V)$, Alt(ζ) $\in \Lambda^k(V)$.*
2. *If $\rho \in \Lambda^k$, then Alt(ρ) $= \rho$. In fact, ρ is alternating iff Alt(ρ) $= \rho$.*
3. *If $\zeta \in T^k$, then Alt(Alt(ζ)) $=$ Alt(ζ).*

Proof.

(1) Let $\sigma_{i,j}$ be the transposition that switches i and j and leaves all other numbers fixed. If σ is a permutation, let $\sigma^* = \sigma \circ \sigma_{i,j}$. Then we have

$$\mathrm{Alt}(\zeta)^{\sigma_{i,j}} = \frac{1}{k!}\sum_{\sigma}(\text{sgn } \sigma)\zeta^{\sigma^*} = \frac{1}{k!}\sum_{\sigma^*}(-\text{sgn } \sigma^*)\zeta^{\sigma^*} = -\mathrm{Alt}(\zeta).$$

(2) Let $\rho \in \Lambda^k(V)$. By property (1), for any permutation σ, we have $\rho^\sigma = (\text{sgn } \sigma)\rho$. Therefore,

$$\mathrm{Alt}\ \rho = \frac{1}{k!}(\sum_{\sigma} \text{sgn } \sigma)\rho^\sigma = \frac{1}{k!}\sum_{\sigma}(\text{sgn } \sigma)(\text{sgn } \sigma)\rho = \rho,$$

since there are $k!$ elements of S_k, and $(\text{sgn } \sigma)^2 = 1$ for any $\sigma \in S_k$. Hence, the second property follows from the first.

(3) This is an immediate corollary of properties (1) and (2).

∎

Definition 2.25. Let $\zeta \in T^k(V)$, $\eta \in T^l(V)$ be tensors on a vector space V. We define a new operation, $\zeta \otimes \eta : T^{k+l}(V) \to \mathbb{R}$, called the *tensor product*, by

$$\zeta \otimes \eta(v_1, ..., v_{k+l}) = \zeta(v_1, ..., v_k)\eta(v_{k+1}, ..., v_{k+l}). \tag{2.26}$$

The importance of the tensor product is simply that it carries k- and l-tensors into $(k+l)$-tensors. Also, consider the following crucial fact. From Theorem 2.5, we found that the elementary k-tensors for $T^k(V)$ take the form $\theta_J(b_1, ..., b_k) = \prod_{i=1}^k \theta_{j_i}(b_i)$. By definition of the tensor product, these elementary k-tensors can be written as

$$\theta_J = \theta_{j_1} \otimes \theta_{j_2} \otimes \cdots \otimes \theta_{j_k}. \tag{2.27}$$

We have now come to the introduction of a very important operation, one that will appear substantially in the remainder of this paper.

Definition 2.28. Let $\rho \in \Lambda^k(V)$ and $\xi \in \Lambda^l(V)$. We define the *wedge product* of ρ and ξ to be the alternating $(k+l)$-tensor

$$\rho \wedge \xi = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\rho \otimes \xi). \tag{2.29}$$

The coefficient in the definition is for the sole purpose of allowing associativity to work. The importance of the wedge product is similar to that of the tensor product: the wedge product turns two alternating k- and l-tensors into an alternating $(k+l)$-tensor.

The formula for the wedge product itself isn't terribly important. What *is* important is that it is the unique operation satisfying the desirable properties stated in the following theorem.

Theorem 2.30. *Let ξ, ρ, ψ be alternating tensors of arbitrary rank (unless otherwise specified). The wedge product is the unique operation with the following properties:*

1. $\xi \wedge (\rho \wedge \psi) = (\xi \wedge \rho) \wedge \psi$
2. *If* $c \in \mathbb{R}$, $(c\xi) \wedge \rho = c(\xi \wedge \rho) = \xi \wedge (c\rho)$
3. *If ξ and ρ have the same rank, then* $(\xi + \rho) \wedge \psi = \xi \wedge \psi + \rho \wedge \psi$ *and* $\psi \wedge (\xi + \rho) = \psi \wedge \xi + \psi \wedge \rho$
4. *If* $\xi \in \Lambda^k(V)$ *and* $\rho \in \Lambda^l(V)$, *then* $\rho \wedge \xi = (-1)^{kl} \xi \wedge \rho$.
5. *Let V be a vector space. If θ_i is the basis for $T^1(V) = \Lambda^1(V)$ and $\tilde{\theta}_I$ is a basis element of $\Lambda^k(V)$, where $I = (i_1, ..., i_k)$ is an ascending k-tuple satisfying $1 \leq i_m \leq n$, then $\tilde{\theta}_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$.*

By Equation 2.27 and property 5 of the preceding theorem, we have now found simple representations for the bases of $T^k(V)$ and $\Lambda^k(V)$. This is very convenient for our purposes, and should also give the reader a better handle on the spaces T^k and Λ^k once he becomes better acquainted with the tensor and wedge products. Theorem 2.30 leads us to a very important result.

Corollary 2.31. *Let $\rho \in \Lambda^k(V)$. If k is odd, then $\rho \wedge \rho = 0$.*

Proof. By property (4), $\rho \wedge \rho = (-1)^{k^2} \rho \wedge \rho = -(\rho \wedge \rho)$. Hence, $\rho \wedge \rho$ must be 0. □

This fact is vital, as one of the most important objects in differential geometry, the 1-form dx, which assigns to each point $x \in \mathbb{R}^n$ a particular alternating 1-tensor, must follow the rule $dx \wedge dx = 0$. This is essential to intuition, as well as to many necessary results of vector calculus such as Green's Theorem.

3. The Determinant

Since the determinant is not defined in vector calculus courses, we do so here. What sometimes appears to be a complex algorithm actually comes out to be a rather concise definition.

Let $e_1, ..., e_n$ be the standard basis for $\mathbb{R}^n$. By the preceding section, we know that the space $\Lambda^n(\mathbb{R}^n)$ of alternating n-tensors on $\mathbb{R}^n$ has dimension $\binom{n}{n}$ =1, as it is spanned by its sole elementary basis vector $\tilde{\theta}_{\{1,...,n\}} = \theta_1 \wedge \cdots \wedge \theta_n$.

Definition 3.1. Let $X = [x_1 \cdots x_n]$ be an $n \times n$ matrix. The *determinant* of X is defined to be

$$\det(X) = \tilde{\theta}_{\{1,...,n\}}(x_1, ..., x_n). \tag{3.2}$$

In rigorous linear algebra courses, the determinant is often defined over an $n \times n$ matrix A as the unique function F with the following properties:

(1) If B is a matrix obtained by switching two columns of A, $F(B) = -F(A)$;
(2) F is multilinear with respect to columns;
(3) If I_n denotes the $n \times n$ identity matrix, $F(I) = 1$.

By definition, $\tilde{\theta}_{\{1,\dots,n\}}$ satisfies these three properties.

As is apparent in any elementary vector calculus course, the determinant plays an important role in the calculation of volumes. Such is the case in differential forms as well, and we will need its properties often. Here, some of the determinant's most significant properties will be presented. We start by defining a parallelepiped, with which we shall do much of our work with determinants.

Definition 3.3. Let $v_1, ..., v_k$ be a set of vectors in a vector space V. We define the *parallelepiped* spanned by $v_1, ..., v_k$ to be the set $t_1 \cdot v_1 + \cdots + t_k \cdot v_k$, where t_i takes values from the set $[0, 1]$. We denote this set by $\mathcal{P}(v_1, ..., v_k)$.

We know from linear algebra that the determinant calculates the volume of a parallelepiped in 2 and 3 dimensions by considering the spanning vectors as column vectors of a matrix. However, this only works if, for example, we're given 2 vectors in $\mathbb{R}^2$ or 3 vectors in $\mathbb{R}^3$. How would we find the volume of a parallelepiped spanned by 2 vectors in $\mathbb{R}^3$? To perform this operation, we would ideally like a function V to accomplish this, while simultaneously preserving some elements of intuition that accompany our idea of volume. We first review some facts about orthogonal transformations.

Definition 3.4. An $n \times n$ matrix A is called an *orthogonal* matrix if the column vectors of A form an orthonormal set in $\mathbb{R}^n$. This is equivalent to requiring A to satisfy the relation $A^T \cdot A = I_n$, where A^T denotes the transpose of A. We let $O(n)$ denote the set of all $n \times n$ orthogonal matrices. An *orthogonal transformation* is simply multiplication by an orthogonal matrix.

Theorem 3.5. *If P is a parallelepiped composed of n vectors in $\mathbb{R}^n$, then the volume of P is invariant under orthogonal transformations.*

It is reasonable to expect that our ideal function V be invariant under orthogonal transformations of k vectors in $\mathbb{R}^n$. It is also reasonable and intuitive to require that if the parallelopiped were to lie in $\mathbb{R}^k \times 0$ of $\mathbb{R}^n$, it would have the same volume as if it were lying in $\mathbb{R}^k$. These are the only two requirements we place on our new concept of volume. In fact, as we now prove, there exists a function V satisfying these two requirements, and moreover, it is unique. First, we recall an elementary theorem from linear algebra.

Lemma 3.6. *Let A be a k-dimensional subspace of $\mathbb{R}^n$. Then there exists an orthogonal transformation $h : \mathbb{R}^n \to \mathbb{R}^n$ that takes A onto $\mathbb{R}^k \times 0$.*

Theorem 3.7. *There exists a unique function V that assigns to every k-tuple $(v_1, ..., v_k)$ of elements in $\mathbb{R}^n$ a non-negative number satisfying the following properties:*

(1) *If $g : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation, then $V(g(v_1), ..., g(v_k)) = V(v_1, ..., v_k)$*
(2) *Let $c_1, ..., c_k \in \mathbb{R}^k \times 0$, i.e., let each c_i be of the form $c_i = [d_i\ 0]$, where $d_i \in \mathbb{R}^k$. Then $V(c_1, ..., c_k) = |\det[d_1 \cdots d_k]|$.*

Proof. Let $X = (v_1, \ldots, v_k)$. For conciseness, we denote the volume of X by $V(X)$. Define $W(X) = \det(X^T \cdot X)$. We shall show that $V(X) = W(X)^{1/2}$ is the function that satisfies the restrictions of the theorem.

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation, and write $g(x) = A \cdot x$, with $A \in O(n)$. Then $W(A \cdot X) = \det\big((A \cdot X)^T \cdot (A \cdot X)\big) = \det\big(A^T \cdot X^T \cdot A \cdot X\big) = \det(A \cdot A^T) \cdot \det(X \cdot X^T) = \det(I_n) \cdot \det(X \cdot X^T) = \det(X \cdot X^T) = W(X)$. Therefore, V is invariant under orthogonal transformations.

Next, let $C = (D, 0)$. By this, we mean that C is an array of k numbers in $\mathbb{R}^n$, where zero occupies the last $(n-k)$ places of each n-tuple. Therefore, D is like an array of k numbers in $\mathbb{R}^k$. Then we have $W(C) = \det\left([D^T\ 0] \cdot \begin{bmatrix} D \\ 0 \end{bmatrix}\right) = \det(D^T \cdot D) = (\det D)^2$. Therefore, $V(C) = \det D$, and so condition 2 is satisfied. We also see that on $\mathbb{R}^k \times 0$, V is non-negative. By Lemma 3.6 and property 1 of the theorem, V is non-negative everywhere, and the proof is complete. □

It is worth noting that $V(X) = 0$ if and only if the vectors $x_1, ..., x_k$ are dependent. We now concisely state the definition of the volume of a k-tuple in $\mathbb{R}^n$.

Definition 3.8. Let $v_1, ..., v_k$ be a set of independent vectors in $\mathbb{R}^n$. The *k-dimensional volume* of the parallelepiped $\mathcal{P}(v_1, ..., v_k)$ is defined to be the positive number $V(v_1, ..., v_k) = \big(\det(X^T \cdot X)\big)^{1/2}$.

Let's examine what we have just done and how we will apply it later. Let $f : \mathbb{R}^k \to \mathbb{R}^n$ be a differentiable function. Then Df, the Jacobian of f, can be represented as an $n \times k$ matrix, and we can therefore take its volume. This will be the significant property of V that we will use when we establish a theory of integration on differential forms, a subject to which we now divert.

4. Differential Forms

Definition 4.1. Let $x \in A \subset \mathbb{R}^n$. The *tangent space* at x, written $T_x(A)$, is defined to be the set $\{(x, v) : v \in \mathbb{R}^n\}$.[2]

Definition 4.2. Let $A \subset \mathbb{R}^n$ be open. A *tensor field* on A is a continuous function that sends each point $x \in A$ to a tensor ω on the tangent space of x.

[2] This can also be written as $T_x A$, or even $T_x\mathbb{R}^n$. This text will use all 3 notations.

We now take a moment to briefly discuss what 'continuous' means in the context of the above definition.

Let $A \subset \mathbb{R}^n$ be open, and define $TA = \{(a, v) : a \in A, v \in \mathbb{R}^n\} = A \times \mathbb{R}^n$. We give TA the subspace topology, and define a function $\pi : TA \to A$ by $\pi(a, v) = a$.

Definition 4.3. A continuous vector field is a function $s : A \to TA$ satisfying $\pi(s(a)) = a$. In this case, we define s to be a section of $\pi : TA \to A$.

Now, let $A \subset \mathbb{R}^k$ be open. We define $T^k(A) = \coprod_{x\in A} T^k(T_xA)$. Similar to the above situation, we have a function $\pi : T^k(A) \to A$ that takes k-tensors on the tangent space of $a \in A$ to a. We can finally rigorously define a continuous tensor field.

Definition 4.4. A k-tensor field is a continuous map $\varphi : A \to T^k(A)$ satisfying $\pi \circ \varphi(x) = x$ for all $x \in A$. We say that φ is continuous if for all continuous vector fields $s_1, \dots, s_k$ on A, the map $x \mapsto \varphi(x)\left((x; s_1(x)), \dots, (x; s_k(x))\right)$ is continuous as a function from A to $\mathbb{R}$. We can similarly define a smooth k-tensor field if we replace all instances of 'continuous' with 'smooth'.

The alternating tensor field is similarly defined, and we are finally able to define a differential form.

Definition 4.5. A *differential k-form* ω is an alternating k-tensor field. The space of differential k-forms on a vector space V is denoted by $\Omega^k(V)$.

Following in the footsteps of the preceding sections, we would like to find a kind of basis for $\Omega^k(\mathbb{R}^n)$, the set of all k-forms on $\mathbb{R}^n$. We start with the following definition.

Definition 4.6. The *elementary 1-forms* on $\mathbb{R}^n$ are defined by the equation

$$\psi_i(x)(x; e_j) = \delta_{ij}. \tag{4.7}$$

If $I = (i_1, ..., i_k)$ is ascending, we define the elementary k-forms ψ_I on $\mathbb{R}^n$ by the equation $\psi_I = \psi_{i_1} \wedge \cdots \wedge \psi_{i_k}$.

Let's see what the elementary k-forms on $\mathbb{R}^n$ look like geometrically. For a solid example, let $k = 2$ and $n = 3$. Then if $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, $\psi_{\{1,3\}}(v, w) = \det \begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix}$, by definition of the determinant. This is simply the area of the parallelogram generated by the projection of v and w onto the xz-plane. This should greatly simplify the geometric idea of the elementary differential form. The next theorem shows us that knowing the structure of the elementary k-forms gives us a good idea of the structure of arbitrary k-forms.

Proposition 4.8. *Given $\omega \in \Omega^k(\mathbb{R}^n)$, ω can be expressed as a linear combination of elementary k-forms. That is,*

$$\omega = \sum_{<I>} f_I \psi_I, \tag{4.9}$$

where $f_I : \mathbb{R}^n \to \mathbb{R}$.

Proof. This follows directly from our knowledge of alternating tensors. Take an arbitrary point $x \in \mathbb{R}^n$. Then by definition, $\omega(x)$ is an alternating k-tensor on $T_x(\mathbb{R}^n)$. We know that we can write $\omega(x) = \sum_{<J>} c_J \tilde{\theta}_J$, where $c_J \in \mathbb{R}$. Define $f_J(x) = c_j$. This defines a function $f_J : \mathbb{R}^n \to \mathbb{R}$.

The elementary k-form ψ_I simply assigns to each point $x \in \mathbb{R}^n$ the elementary alternating k-tensor $\tilde{\theta}_I$. Therefore, by construction,

$$\omega(x) = \sum_{<J>} f_J(x) \psi_I(x) \tag{4.10}$$

. □

Finally, we would like to establish that, given an open subset $A \subset \mathbb{R}^n$, $\Omega^0(A) = \{f : A \to \mathbb{R} : f \text{ is smooth}\}$. There is actually a great deal of logic behind this convention. We have $\Lambda^0(A) = \coprod_{x \in A} \mathbb{R}$, which is topologically $A \times \mathbb{R}$, the graph of functions from $A \to \mathbb{R}$.

Before moving on, we would like to extend the tensor and wedge products to fields and forms (respectively) by the following definition.

Definition 4.11. If φ and ψ are tensor fields, then $(\varphi \otimes \psi)(x) = \varphi(x) \otimes \psi(x)$. If ω and η are forms, then $(\omega \wedge \mu)(x) = \omega(x) \wedge \mu(x)$.

With this brief introduction to differential forms, we may progress to a generalization of a familiar concept.

5. The Differential Operator

Note on Notation: From here, we will be using differentiation quite a bit. As we noted before, Df will denote the Jacobian of f. $D_i f$ will signify the ith partial derivative of f.

The reader has undoubtedly seen the symbol dx from calculus and $dxdy$ from multivariable calculus. In elementary calculus, this 'notation' is employed in integrals as a representation of the small intervals being summed over. In advanced calculus, this notation is dropped, and the notation $\int f$ is used instead. The dx in the integral is actually a rigorously-defined mathematical object. When we use the notation $\int f dx$, we are actually integrating over a differential form instead of a function.

The differential operator[3] d is an operation on forms similar to the derivative of ordinary functions (i.e., 0-forms). In general, d is a linear transformation that takes k-forms to $(k+1)$-forms. We will begin by considering the differential operator on 0-forms.

[3]This is sometimes called the exterior derivative.

Recall that a 0-form on an open set $A \subset \mathbb{R}^n$ is a smooth function f on A. We want d to be a linear transformation that takes f to a 1-form. We have already seen such a function: the derivative.

Definition 5.1. Let $A \subset \mathbb{R}^n$ be open, and let $f : A \to \mathbb{R}$ be smooth. We define the 1-form df on A to be:

$$df(x)(x; v) = Df(x) \cdot v. \tag{5.2}$$

The 1-form df is called the differential of f. It is C^∞ as a function of x and v.

Theorem 5.3. *Let $\psi_1, ..., \psi_n$ be the set of elementary 1-forms in $\mathbb{R}^n$. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the ith term; that is, $\pi_i(x_1, ..., x_n) = x_i$. Then $d\pi_i = \psi_i$.*

Proof. Let $v = v_1e_1 + \cdots v_ne_n$ be a vector in $\mathbb{R}^n$. We have $d\pi_i(x)(x; v) = D\pi_i(x) \cdot v =$
$[0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_i.$

Now, consider a point $x \in \mathbb{R}^n$. Then, by definition we have $\psi_i(x)(x; v) = v_i \cdot \psi(x)(x; e_1) = v_i$. Thus, given $x \in \mathbb{R}^n$, $d\pi_i(x)(x; v) = \psi_i(x)(x; v)$. Therefore, the two are equal. □

Notation 5.4. At this point, it is appropriate to introduce a ubiquitous abuse of notation. Instead of writing the ith elementary 1-form as $d\pi_i$, we write it as dx_i. This is the dx we see when we integrate in calculus.

Therefore, from the results of the previous section, any k-form ω can be written as $\sum_{<J>} f_J dx_J$ for some scalar functions f_J. We will use this notation to represent k-forms rather often.

As we will soon see when we define the general differential operator, it would be ideal to find a nice representation for differentials of 0-forms. In fact, such representation does exist, by the following theorem.

Theorem 5.5. *Let $A \subset \mathbb{R}^n$ be open, and let $f \in \Omega^0(A)$. Then*

$$df = (D_1 f)dx_1 + \cdots + (D_n f)dx_n. \tag{5.6}$$

Proof. By definition,

$$df(x)(x; v) = Df(x) \cdot v = \sum_{i=1}^{n} D_i f(x) v_i.$$

But by the proof of Theorem 5.3, $v_i = d\pi_i(x)(x; v)$, so

$$\sum_{i=1}^{n} D_i f(x) v_i = \sum_{i=1}^{n} D_i f(x) dx_i(x)(x; v).$$

Hence, $df = (D_1 f)dx_1 + \cdots + (D_n f)dx_n$. □

Let's diverge from our notation for a moment and consider Leibnitz notation. In Leibnitz notation, this theorem takes the form

$$df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n.$$

This equation often appears in calculus books, so the reader may be more comfortable with this notation. We are finally ready to define the differential operator on arbitrary k-forms.

Definition 5.7. Let $A \subset \mathbb{R}^n$ be open, and write $\omega \in \Omega^k(A)$ as $\omega = \sum_{<I>} f_I dx_I$. We define the linear transformation $d : \Omega^k(A) \to \Omega^{k+1}(A)$ by the equation

$$d\omega = \sum_{<I>} df_I \wedge dx_I. \tag{5.8}$$

Like the wedge product, the actual formula of the differential operator is insignificant in comparison to the properties it possesses.

Theorem 5.9. *Let $A \subset \mathbb{R}^n$ be open. The differential operator $d : \Omega^k(A) \to \Omega^{k+1}(A)$ satisfies the following properties:*

(1) d is a linear transformation;
(2) If $f \in \Omega^0(A)$, $df(x)(x; v) = Df(x) \cdot v$;
(3) If $\omega \in \Omega^k(A)$ and $\mu \in \Omega^l(A)$, then $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu$.

These properties result in one of the most important characteristics of the differential operator.

Theorem 5.10. *For every differential form ω, $d(d\omega) = 0$.*

Proof. We begin by considering the case for 0-forms.

Let f be a 0-form. Then

$$d(df) = d\left(\sum_{i=1}^{n} D_i f dx_i\right) = \sum_{j=1}^{n} d(D_j f) \wedge dx_j = \sum_{i=1}^{n}\sum_{j=1}^{n} D_j D_i f dx_j \wedge dx_i.$$

Since dx_i are 1-forms, $dx_i \wedge dx_i = 0$. Hence, we can extract all terms with $i = j$. If $i > j$, then the terms with $dx_i \wedge dx_j$ and $dx_j \wedge dx_i$ will both be terms in the altered summation. The sum of these two terms will be

$$(D_j D_i f) dx_j \wedge dx_i + (D_i D_j f) dx_i \wedge dx_j = (D_j D_i - D_i D_j) f dx_j \wedge dx_i,$$

which equals 0 by commutativity of partial derivatives. Summing over all such combinations yields a summation over all non-deleted terms, and thus

$$d(df) = \sum_{i>j} (D_j D_i f - D_i D_j f) dx_j \wedge dx_i = 0.$$

Now we will expand this result to k-forms, with $k > 0$. Since d is a linear transformation, we can without loss of generality consider the case $\omega = f dx_I$. By property 3, $d(d\omega) = d(df \wedge dx_I) = d(df) \wedge dx_I - df \wedge d(dx_I)$. We have already shown that $d(df) = 0$ when f is a 0-form, and $d(dx_I) = d(1) \wedge dx_I = 0$, since $d(1) = 0$ by Theorem 5.5. It follows that $d(d\omega) = 0$. □

Theorem 5.11. *The differential operator is the only operation on k-forms satisfying the criteria of Theorems 5.9 and 5.10.*

Proof. We start with a preliminary result. We want to show (for reasons apparent later) that for any forms $\omega_1, ..., \omega_k$, $d(d\omega_1 \wedge \cdots \wedge d\omega_k) = 0$. If $k = 1$, this is a direct result from Theorem 5.10. We proceed by induction. Assume the equation works for $k - 1$ terms. Using property 3, we have

$$d(d\omega_1 \wedge \cdots \wedge d\omega_k) = d(d\omega_1) \wedge (d\omega_2 \wedge \cdots \wedge d\omega_k) \pm d\omega_1 \wedge d(d\omega_2 \wedge \cdots \wedge d\omega_k).$$

The first term is zero by Theorem 5.10, and the second also vanishes because $d(d\omega_2 \wedge \cdots \wedge d\omega_k) = 0$ by the induction hypothesis.

Now, let ω be an arbitrary k-form. By Property 2, d is uniquely determined for 0-forms. Expressing ω as $\sum f_I dx_I$, we will show that $d\omega$ is uniquely determined by the f_I. Since d is a linear transformation, we can consider the case $\omega = f dx_I$ without loss of generality. We have $d\omega = d(f \wedge dx_I) = df \wedge dx_I + f \wedge d(dx_I) = df \wedge dx_I$, since $d(dx_I) = 0$ by the lemma just proven. Thus, $d\omega$ is uniquely determined by the value of d on the 0-form f. It follows that d is the unique operation satisfying the criteria of Theorems 5.9 and 5.10. □

Before we progress any further, we introduce a function on forms called a pullback, which allows us to create a form on a set $A \subset \mathbb{R}^k$ if we are given a form on a set $B \subset \mathbb{R}^n$. We introduce it here because of its relation to d and the fact that it is greatly used in Section 8.

Definition 5.12. Let $A \subset \mathbb{R}^m$, and let $f : A \to \mathbb{R}^n$ be smooth. Let $B \subset \mathbb{R}^n$ be an open set containing $f(A)$. We define the *pullback* $f^* : \Omega^k(B) \to \Omega^k(A)$ as follows: Given a k-form ω on B with $k \geq 1$, we define a k-form $f^*\omega$ on A by the equation

$$(f^*\omega)(x)(x; v_1, ..., v_m) = \omega(f(x))(Df(x) \cdot v_1, ..., Df(x) \cdot v_m). \tag{5.13}$$

If $g : B \to \mathbb{R}$ is a 0-form on B, we define a 0-form f^*g on A by the equation $(f^*g)(x) = g(f(x))$ for $x \in A$.

The pullback itself is very useful, but when combined with the differential operator, it has many serious ramifications. We will use the following proposition in the proof of Stokes' Theorem and, indeed, it is almost ubiquitous in differential geometry.

Proposition 5.14. *Let $B \subset \mathbb{R}^k$ be open, and let $f : B \to \mathbb{R}^n$ be a smooth function. If the l-form ω is defined on some open set containing $f(B)$, then on B, we have*

$$f^*(d\omega) = d(f^*\omega). \tag{5.15}$$

Let us look at an example of how this works. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(\theta) = (\sin\theta, \cos\theta)$, and let $\omega \in \Omega^1(\mathbb{R}^2)$ be defined by $\omega(x_1, x_2) = -x_2 dx_1 + x_2 dx_2$. Then

$$\begin{aligned} f^*\omega &= -(\pi_2 \circ f)d(\pi_1 \circ f) + (\pi_1 \circ f)d(\pi_2 \circ f) = -(\sin\theta)d(\cos\theta) + (\cos\theta)d(\sin\theta) \\ &= (\sin^2\theta)d\theta + (\cos^2\theta)d\theta = d\theta. \end{aligned}$$

It follows from Theorem 5.10 that $d(f^*\omega) = 0$. It is also clear that $d\omega = 0$, so that $f^*(d\omega) = 0$, and Equation 5.15 holds.

6. Grad, Curl, and Div - As Differential Operators

We start this section by reviewing the important operators of vector calculus. The reader should be somewhat comfortable with all of them, so we will not review their properties. If the reader would prefer to do so, he is referred to [4].

Definition 6.1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. The *gradient* is the vector field in $\mathbb{R}^3$ given by

$$\text{grad}(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial x}\mathbf{k}. \tag{6.2}$$

Definition 6.3. Let $\mathbf{F}$ be the vector field given by $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$. We define the *curl* of $\mathbf{F}$ to be the vector field in $\mathbb{R}^3$ given by

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}. \tag{6.4}$$

Definition 6.5. Let $\mathbf{F}$ be the vector field given by $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$. Then the *divergence* of $\mathbf{F}$ is the scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}. \tag{6.6}$$

One of the goals of this paper is to explain these three operations rigorously. There exists a close relationship between div, grad, curl, and the differential operator, which we exhibit in the following theorem.

Definition 6.7. Let V, W be vector spaces, and let $T : V \to W$ be a linear transformation. If T is also a bijection between V and W, then T is said to be a *linear isomorphism.*

Theorem 6.8. *Let $A \subset \mathbb{R}^3$ be open. There exist linear isomorphisms β_i according to the following chart:*

$$
\begin{tikzcd}
\textit{Scalar Function} \arrow[r, "\beta_0"] \arrow[d, "\nabla"'] & \Omega^0(A) \arrow[d, "d"] \\
\textit{Vector Field} \arrow[r, "\beta_1"] \arrow[d, "\nabla\times"'] & \Omega^1(A) \arrow[d, "d"] \\
\textit{Vector Field} \arrow[r, "\beta_2"] \arrow[d, "\nabla\cdot"'] & \Omega^2(A) \arrow[d, "d"] \\
\textit{Scalar Function} \arrow[r, "\beta_3"] & \Omega^3(A)
\end{tikzcd}
$$

In fact, this diagram is commutative, as these linear isomorphisms satisfy the following equations:

$$d \circ \beta_0 = \beta_1 \circ (\nabla F)$$
$$d \circ \beta_1(F) = \beta_2 \circ (\nabla \times f)$$
$$d \circ \beta_2(F) = \beta_3 \circ (\nabla \cdot F)$$

Proof. We begin the proof by explicitly defining β_i:

Let f be a scalar function on A, and let $F = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ be a vector field on A. We define:

$$\beta_0 f = f.$$
$$\beta_1 F = f_1 dx_1 + f_2 dx_2 + f_3 dx_3.$$
$$\beta_2 F = f_1 dx_2 dx_3 - f_2 dx_1 dx_3 + f_3 dx_1 dx_2.$$
$$\beta_3 f = f dx_1 \wedge \cdots \wedge dx_n.$$

By Proposition 4.6, we have that the β_i are linear isomorphisms. Therefore, it is sufficient to consider the formulas, whose proofs are merely calculations. We prove the first formula only – the second and third proofs are similar and just as simple, yet more tedious.

For the first formula, note that $d \circ \beta_0 f = df$. We also have

$$\beta_1 \circ \nabla(f) = \beta_1\left(\frac{\partial f}{\partial x_1}\mathbf{i} + \frac{\partial f}{\partial x_2}\mathbf{j} + \frac{\partial f}{\partial x_3}\mathbf{k}\right) = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = df,$$

and so the equality is proven. For the other two cases, just expand both sides of the equation. For example, one proves that both $d \circ \beta_1(F)$ and $\beta_2(\nabla \times F)$ equal

$$\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3$$

$\square$

We can, in fact, generalize this result to $\mathbb{R}^n$, although we cannot use curl in this diagram, as the two vector fields on the left column cannot be connected via a single operation. We can only draw 4 rows in this diagram, since scalar and vector fields are the only 3-dimensional analogs we have.

Theorem 6.9. *Let $A \subset \mathbb{R}^n$ be open. There exist linear isomorphisms β_i such that the following diagram commutes:*

$$\begin{tikzcd} \textit{Scalar Field} \arrow[r, "\beta_0"] \arrow[d, "\nabla"'] & \Omega^0(A) \arrow[d, "d"] \\ \textit{Vector Field} \arrow[r, "\beta_1"] & \Omega^1(A) \end{tikzcd}$$

$$\begin{tikzcd} \textit{Vector Field} \arrow[r, "\beta_{n-1}"] \arrow[d, "\nabla\cdot"'] & \Omega^{n-1}(A) \arrow[d, "d"] \\ \textit{Scalar Field} \arrow[r, "\beta_n"] & \Omega^n(A) \end{tikzcd}$$

Proof. We must prove that the additional equation $d \circ \beta_{n-1}F = \beta_n \circ (\nabla \cdot F)$ is satisfied. In addition to the equations used in the previous theorem, we define

$$\beta_{n-1}F = \sum_{i=1}^{n}(-1)^{i-1}f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

and

$$\beta_n f = f dx_1 \wedge \cdots \wedge dx_n.$$

The method of proof is similar to Theorem 6.8. □

Up to now, our work seemed to be without application, but with this new theorem connecting the differential operator to the three primary operations of vector calculus, motivation is certainly provided to the reader to master the previous content of this paper.

7. Manifolds and Orientation

The concept of orientation is a very complex topic in differential geometry, and many introductory texts devote at least a chapter to it. Instead of taking this path, we only develop the components of orientation necessary to comprehend the results of Section 10, the culmination of this paper.

Let us first observe the concept of orientation as presented in vector calculus courses. The concepts of orientation of a curve and orientation of a surface are often expressed separately, even though they are the same idea, as we shall see here. Intuitively, the orientation of a curve is the direction that it follows as it traces out its path. Does it start at a and end at b, or does it begin with b, tracing out the same path as before, and terminate at a? As we know, the integral of the first path is equal to the negative of the integral of the second path. This has nothing to do with the action of the integral itself, but with the way we choose an orientation of the curve.

Next we come to the idea of orientation of surfaces. A surface is called orientable if it is possible to choose a unit normal vector n at each point of the surface such that these vectors vary continuously from point to point. Each point of a smooth surface will have two unit normal vectors: one that points inward and one that points outward.

We can easily come up with examples of surfaces that are not orientable – the prototypical example is the Mobius strip. Start at a point P of a Mobius strip, and without loss of generality, assume that the unit normal n_P at P points inward. If we then take the unit normal of each point as we go around the strip, we will end up back at P, and the unit normal vector needed to make the variation continuous will have to point in the opposite direction of the chosen unit normal n, i.e., it will have to be outward-pointing. Hence, a Mobius strip is non-orientable.

It should be clear that every curve has 2 orientations, and the same applies to surfaces. We will now generalize this result to a mathematical abstraction called a manifold.

Definition 7.1. Let A, B be topological spaces. We say that A is *locally homeomorphic* to B if for each $a \in A$, there exists a neighborhood of a that is homeomorphic to an open subset of B. Given $a \in A$, a homeomorphism $\alpha : U \to V$ satisfying this relation, where $U \subset B$ and $a \in V \subset A$, is called a *coordinate chart* around a.

Definition 7.2. An *n-manifold (without boundary) M* is a 2nd countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Note that if M is locally homeomorphic to $\mathbb{R}^n$ at $x \in M$, it cannot be locally homeomorphic to $\mathbb{R}^k$, for $k \neq n$. The reason for this is that $\mathbb{R}^k$ is not homeomorphic to $\mathbb{R}^n$. However, we might have two subsets of M, say M_1 and M_2, such that M_1 is locally homeomorphic to $\mathbb{R}^k$, whereas M_2 is locally homeomorphic to $\mathbb{R}^n$. It can be proven that if this is the case, M_1 and M_2 are not connected. Therefore, as we can easily fix this quirk if it were to occur, we ignore it and instead simply use a manifold where each point $x \in M$ is locally homeomorphic to $\mathbb{R}^n$.

Sometimes we would like to relax some of the restrictions we put on the manifold (without boundary) so that boundary points are allowed. This can make a solid definition slightly more difficult, as we would have to change $\mathbb{R}^n$ to something that itself has boundary points. This becomes possible with the upper half-space $\mathbb{H}^n = \{(x_1, ..., x_n) : x_n \geq 0\}$.

Definition 7.3. An *n-manifold (with boundary) M* is a 2nd countable Hausdorff space that is locally homeomorphic to $\mathbb{H}^n$ for some $n \in \mathbb{N}$.

It is significant that a manifold is an abstract topological space, and therefore does not a priori come embedded in an ambient space. However, we will only deal with manifolds that are subsets of $\mathbb{R}^n$ in this paper. Fortunately, by a result called Whitney's Embedding Theorem, we lose no generality with this assumption.

It is now appropriate to generalize the concept of orientation to manifolds in $\mathbb{R}^n$. We will start at an unexpected place: vector spaces.

Let's first consider the easiest example of a vector space: $\mathbb{R}$. Here, orientation's intuitive interpretation would be a choice of preferred direction of the real line. This would be determined by the basis, and so choosing the basis e_1 would yield one orientation of the line, whereas $-e_1$ would yield the other. Similarly, in $\mathbb{R}^2$, the most natural orientation would be one in which we choose the 2nd basis vector to be in the counterclockwise direction of the 1st. For example, $\{e_1, e_2\}$ would belong to this natural orientation, as well as $\{-e_1, -e_2\}$. The orientations belonging to the opposite orientation would be $\{e_1, -e_2\}$ and $\{-e_1, e_2\}$. If we look at $\mathbb{R}^3$, the natural orientation would consist of the 'right-handed' bases - that is, the bases with the property that when the fingers of your right hand curl from e_1 to e_2, your thumb points to e_3.

It should be clear from the previous examples that orientation is actually simply a choice of basis. We have a set of bases that represent the 'natural orientation', and all bases that do not belong to this group belong to the opposite orientation. However, a new problem arises here. Firstly, how do we separate the bases of vector spaces that don't have a intuitively natural orientation? Secondly, how can we find the orientation that a given basis belongs to? The following definition answers both questions.

Definition 7.4. Let $B = \{b_1, ..., b_n\}$ and $C = \{c_1, ..., c_n\}$ be two ordered bases for a vector space V of dimension n. It is a well-known result of linear algebra there there exists a unique linear transformation $T : V \to V$ carrying b_i to c_i for $1 \leq i \leq n$. We say that B and C are *consistently oriented* (or have the same orientation) if $\det(T) > 0$. If $\det(T) < 0$, then B and C are *oppositely oriented*.

Now that such questions have been handled, we define what we mean by the orientation of a manifold. The previous discussion was intended to give the reader a reasonable geometric grasp of the concept of orientation of vector spaces. Orientations of manifolds, though axiomatically similar, are more difficult to visualize.

Definition 7.5. A bijection $f : M \to N$, where $M \subset \mathbb{R}^n$ and $\mathbb{N} \subset \mathbb{R}^k$, such that both f and f^{-1} are smooth, is called a *diffeomorphism*.

Definition 7.6. Let $A, B \subset \mathbb{R}^n$, and let $g : A \to B$ be a diffeomorphism. Then g is said to be *orientation-preserving* if $\det Dg > 0$ on A and *orientation-reversing* if $\det Dg < 0$ on A.

It is important to note that, by the chain rule, $\det Dg$ can never be 0.

Definition 7.7. Let $M \subset \mathbb{R}^n$ be a k-manifold. Let $\alpha_i : U_i \to V_i$ on M for $i = 0, 1$ be coordinate charts. If $V_0 \cap V_1 \neq \emptyset$, we say that α_0 and α_1 overlap. It is said that they *overlap* positively if the function $\alpha_1^{-1} \circ \alpha_0$ is orientation-preserving. If M can be covered by a collection of coordinate charts, with each pair overlapping positively, then we say that M is *orientable*. Otherwise, M is *non-orientable*.

Definition 7.8. Let $M \subset \mathbb{R}^n$ be an orientable k-manifold. Given a collection of positively overlapping coordinate charts covering M, append to this collection all other coordinate charts on M that overlap these charts positively. The charts in this expanded collection positively overlap each other. This expanded collection is called an *orientation* for M. A manifold M with an orientation is called an oriented *manifold*.

A concept we will especially need to state Stokes' Theorem is that of the induced orientation of the boundary of a manifold. Given a manifold M, we often are interested in integrating over its boundary ∂M. The following definition tackles the problem of how to orient ∂M once we are given an orientation for M.

Definition 7.9. Let $M \subset \mathbb{R}^n$ be an orientable k-manifold, with $\partial M \neq \emptyset$. Given an orientation for M, we define the *induced orientation* of ∂M as follows: If k is even, it is the orientation obtained by restricting coordinate charts belonging to the orientation of M. Otherwise, it is the opposite of such orientation.

The first and most obvious question to ask is if such boundary orientation can even exist. Though we will not prove it, the following theorem says that this defined orientation will always exist.

Theorem 7.10. *Let $k \geq 1$. If $M \subset \mathbb{R}^n$ is an orientable k-manifold (with boundary), then ∂M is orientable.*

There is one more idea that needs to be conveyed before moving on to the next section. We have defined a manifold, but there is a special kind of manifold called a smooth manifold that is very important in differential geometry.

Definition 7.11. Let α_i be the collection of coordinate charts of a manifold M. The composition $\alpha_i \circ \alpha_j^{-1}$ of maps is called a *transition map.*

Definition 7.12. A manifold for which all transition maps are smooth is called a smooth manifold.

This definition is necessary to establish such important objects as smooth tensor fields and smooth forms.

8. Poincare's Lemma and Conservative Fields

In the latter part of vector calculus courses, it is typical to learn the concept of conservative fields, which play a titanic role in physics. In this section, we elaborate on and generalize some results of conservative fields presented in vector calculus courses.

Definition 8.1. A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is said to be *conservative* if there exists a scalar function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla \varphi$.

We now present two new definitions, the first corresponding to a vector field having curl 0, and the latter being a generalization of Definition 8.1.

Definition 8.2. A k-form ω is defined to be *closed* if $d\omega = 0$.

Definition 8.3. A k-form is said to be *exact* if there exists a $(k-1)$-form μ satisfying $d\mu = \omega$.

We can now easily prove a pertinent result.

Theorem 8.4. *Every exact form is closed.*

Proof. Let ω be exact, and write $\omega = d\mu$. Then $d\omega = d(d\mu) = 0$. ◻

Corollary 8.5. *If a vector field $\boldsymbol{F}$ is conservative, then $\nabla \times \boldsymbol{F} = 0$.*

Proof. This is clear from Theorem 6.8. $\square$

The converse of the theorem is also true with certain restrictions, as we will shortly prove. The reader should note that the following material establishes an algebraic context not strictly necessary to the main theorems.

Definition 8.6. Let $f, g : X \to Y$ be functions. If there exists a continuous function $F : X \times [0,1] \to Y$ satisfying $F(x,0) = f(x)$ and $F(x,1) = g(x)$, then f and g are said to be homotopic. If f and g are *homotopic*, then we write $f \simeq g$.

A similar idea can be applied to spaces.

Definition 8.7. Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^n$. We say that X and Y are *homotopy equivalent*, written $X \simeq Y$, if there exist maps $f : X \to Y$ and $g : Y \to X$ satisfying $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

Roughly, $f \simeq g$ if f can be continuously deformed into g. A similar statement holds for $X \simeq Y$. We now introduce a group that, given a set A, gives us an idea of the forms on A that are closed but not exact.

Definition 8.8. Let $M \subset \mathbb{R}^n$ be a smooth manifold. Let $C^k(M)$ denote the set of closed k-forms on M, and let $E^k(M)$ be the set of exact k-forms on M. The *kth deRham group* on M is defined to be the quotient vector space

$$H^k(M) = C^k(M)/E^k(M). \tag{8.9}$$

We first note that if M is connected, then $H^0(M) = \mathbb{R}$, which is similar to the property exhibited by $T^0(A)$. Since, given $\omega \in \Omega^k(M)$, $d(f^*\omega) = f^*(d\omega)$, f^* takes closed forms to closed forms and exact forms to exact forms. We now state a lemma that will lead us directly to the Poincare Lemma. However, its proof is beyond the scope of this paper.

Lemma 8.10. *If $X \simeq Y$, then $H^k(X) \cong H^k(Y)$.*

Theorem 8.11 (Poincare Lemma)**.** *If X is a smooth manifold that is homotopy equivalent to a point x_0, then all closed k-forms on X are exact, for $k \geq 1$.*

Proof. For $k \in \mathbb{N}$, $H^k(x_0) = 0$, since all $(n+1)$-forms on $\mathbb{R}^n$ are 0. Therefore, $H^k(X) = 0$ as well, and the theorem is proven. $\square$

This is a nice result, but it isn't ideal for our purposes. We would like a result that is easier to interpret, as homotopic equivalence isn't a very 'visible' criterion. There exists a more elementary restriction that we can use.

Definition 8.12. A set $A \subset \mathbb{R}^n$ is said to be *star-shaped* if there exists $x \in A$ such that for all $a \in A$, the line segment connecting x to a lies entirely within A.

It is easy to see that a star is star-shaped. If we take x in the definition to be the center of the star, then we can clearly draw a straight line to every other point in the star without going outside of it. It is not difficult to prove that every convex set is star-shaped. Therefore, it is trivial that $\mathbb{R}^n$ is star-shaped, as it is convex.

It so happens that every open, star-shaped set is homotopy equivalent to a point. (Prove it.) Therefore, in elementary differential geometry textbooks, the following theorem is often presented as the Poincare Lemma, in place of Theorem 8.11.

Theorem 8.13 (Poincare Lemma)**.** *Let $A \subset \mathbb{R}^n$ be open and star-shaped, and let ω be a closed k-form on A. Then ω is exact.*

Corollary 8.14. *If a vector field $\boldsymbol{F}$ is defined on all of $\mathbb{R}^3$ and $\nabla \times \boldsymbol{F} = 0$, then $\boldsymbol{F}$ is conservative.*

9. Integration of Differential Forms over Manifolds

We initiate this section by defining the integral of a scalar function over a manifold. It is not defined in an obvious fashion, for precisely the reason that a simple, intuitive definition is impossible. We start by defining the integral of a scalar function f in the case where the support of f lies in a single coordinate chart. From there, the more general case is defined by means of a partition of unity. Here, we will only treat the case of a compact k-manifold, but our results can be extended to other types of manifolds in an intuitive fashion.

Definition 9.1. Let $M \subset \mathbb{R}^n$ be a compact k-manifold, and let $f : M \to \mathbb{R}$ be a continuous function. Let $C = \text{supp} f = \overline{\{x \in M : f(x) \neq 0\}}$, and assume that there exists a coordinate chart $\alpha : A \to B$ satisfying $C \subset B$. Since C is compact, $\alpha^{-1}(C)$ is also compact. Therefore, if we replace A by a smaller open set, we can assume that A is bounded without loss of generality.

The integral of f over M is defined by the equation:

$$\int_M f = \int_{\text{Int } U} (f \circ \alpha) V(D\alpha). \tag{9.2}$$

Clearly, Int $U = U$ if U is open in $\mathbb{R}^k$. We use Int U for the case that U is only open in $\mathbb{H}^k$.

Proposition 9.3. *As in the definition, assume that supp f can be covered by a single coordinate chart. Then $\int_M f$ is independent of the choice of coordinate chart, and is therefore well-defined.*

Extending this definition to the general case is more difficult and will be addressed here. In the following theorem, a family of functions $\Phi = \{\phi_1, ..., \phi_l\}$ satisfying 1-3 is called a *partition of unity.* If Φ also satisfies 4, then we say that Φ is *subordinate to the choice of coordinate charts.*

Proposition 9.4. *Let M be a compact k-manifold in $\mathbb{R}^n$. There exists a finite collection of smooth functions $\phi_1, ..., \phi_l : \mathbb{R}^n \to \mathbb{R}$ such that*

1. *For each i, $\phi_i \geq 0$.*
2. *For each $x \in M$, $\sum_i \phi_i(x) = 1$.*
3. *Given ϕ_i, there is a coordinate chart $\alpha_i : U_i \to V_i$ s.t. (supp $\phi_i \cap M) \subset V_i$.*

We now have the machinery to define the integral of a scalar function over a compact k-manifold.

Definition 9.5. Let $M \subset \mathbb{R}^n$ be a compact k-manifold. Let $f : M \to \mathbb{R}$ be a continuous function, and let $\phi_1, ..., \phi_l$ be a partition of unity on M that is subordinate to the coordinate charts of M (whose existence is guaranteed by the preceding proposition). The integral of f over M is defined by the equation

$$\int_M f = \sum_{i=1}^{l} \left[\int_M (\phi_i f) \right]. \tag{9.6}$$

This representation is independent of the choice of partition of unity. To see this, let $\phi_1, ..., \phi_l$ and $\psi_1, ..., \psi_m$ be two different partitions of unity. A quick calculation shows that

$$\sum_{j=1}^{m} \left[\int_m \psi_j f \right] = \sum_{j=1}^{m} \sum_{i=1}^{l} \left[\int_M \phi_i \psi_j f \right] = \sum_{i=1}^{l} \sum_{j=1}^{m} \left[\int_M \phi_i \psi_j f \right] = \sum_{i=1}^{l} \left[\int_M \phi_i f \right],$$

which immediately proves the statement.

Now that we can integrate scalar functions over manifolds, it is not very difficult to generalize this result to k-forms, for $k > 1$. The reason for this is due to the following definition:

Definition 9.7. Let $U \subset \mathbb{R}^k$ be open, and let ω be a k-form defined in U. We know that ω can be written uniquely as $\omega = f dx_1 \wedge \cdots \wedge dx_k$. The integral of ω over U is defined by the equation

$$\int_U \omega = \int_U f \tag{9.8}$$

From this definition, it is clear now why we use the $f dx$ notation when integrating functions. We are simply integrating the k-form $f dx$, which is equal to $\int f$ by definition! The superfluous dx is then simply a marker to remind the beginner of calculus that the integral is merely a sum over very small portions of a set.

This definition doesn't solve the problem of integrating forms over manifolds just yet. We have only discovered how to integrate forms over open sets. However, this information will be vital when we do want to integrate forms over manifolds.

We now define the integral of a k-form over a k-manifold M. As we did with scalar functions, we begin by defining the integral in the case where the support of our k-form lies in a single coordinate chart.

Definition 9.9. Let $M \subset \mathbb{R}^n$ be a k-manifold, and let ω be a k-form on M. Assume that there exists a coordinate chart $\alpha : U \to V$ such that supp $\omega \subset V$. Then we define

$$\int_M \omega = \int_{\text{Int } U} \alpha^* \omega. \tag{9.10}$$

Now, by almost exactly the same process as we used to generalize the integral of scalar functions over manifolds, we shall generalize the integral of forms over *oriented* manifolds.

Definition 9.11. Let $M \subset \mathbb{R}^n$ be an oriented, compact k-manifold. Let ω be a k-form defined in an open set of $\mathbb{R}^n$ containing M. Cover M with a collection of coordinate charts $\mathcal{A}$ associated with the orientation of M, and choose a partition of unity $\phi_1, ..., \phi_l$ on M that is subordinate to $\mathcal{A}$. The integral of ω over M is defined by the equation

$$\int_M \omega = \sum_{i=1}^{l} \left[\int_M \phi_i \omega \right]. \tag{9.12}$$

This definition is, of course, independent of the choice of partition of unity by the same logic that we used for scalar functions. We also have the following two important facts:

Proposition 9.13. *Let $M \subset \mathbb{R}^n$ be an oriented, compact k-manifold. Let ω, μ be k-forms defined in an open set of $\mathbb{R}^n$ containing M. Then, if $a, b \in \mathbb{R}$, we obtain*

$$\int_M (a\omega + b\mu) = a \int_M \omega + b \int_M \mu.$$

Proposition 9.14. *If $-M$ denotes M with the opposite orientation, then*

$$\int_{-M} \omega = - \int_M \omega.$$

We previously defined integration of forms over n-manifolds, where $n > 0$. A separate definition is necessary to handle the trivial case $n = 0$.

Definition 9.15. Let $X = \{x_1, ..., x_n\}$ be a discrete set of points. We define an orientation upon each point by the map $\mathcal{O} : X \to \{-1, 1\}$. If f is a scalar function defined on an open set of $\mathbb{R}^n$ containing X, we define the integral of f over the oriented manifold X to be

$$\int_X f = \sum_{i=1}^{n} \mathcal{O}(x_i) f(x_i). \tag{9.16}$$

Now we define the induced orientation of the boundary of a 1-manifold.

Definition 9.17. Let $M \subset \mathbb{R}^n$ be an oriented 1-manifold, with $\partial M \neq \emptyset$. Let $p \in \partial M$. We define the induced orientation of ∂M about p, denoted $\mathcal{O}(p)$, to be -1 if there exists a coordinate chart $\alpha : U \to V$ around p belonging to the orientation of M, where U is open in $\mathbb{H}^1$. Otherwise, we let $\mathcal{O}(p) = +1$.

These definitions aid in developing the following version of Stokes' Theorem. A more general result will be encountered in the next section.

Theorem 9.18 (Stokes' Theorem in 1 dimension)**.** *Let $M \subset \mathbb{R}^n$ be an oriented, compact 1-manifold, and give ∂M the induced orientation. Let f be a scalar function defined in an open set of $\mathbb{R}^n$ containing M. Then*

$$\int_M df = \int_{\partial M} f. \tag{9.19}$$

The idea of the proof is encountered in the following section, but the reader is encouraged to prove it independently.

10. Stokes' Theorem

Stokes' Theorem is one of the most important and fundamental result in mathematics. Three vital results of vector calculus are direct corollaries of it, and even the fundamental theorem of calculus can be understood as a form of it. The goal of many elementary textbooks on manifolds, including references [1] and [5], is to build up to and, eventually prove, Stokes' Theorem. Many other texts such as [2] give a superb treatment of the theorem without devoting the entire book toward such purpose. The goal of this section is to both prove the theorem and demonstrate that many results of vector calculus are direct corollaries of it.

Theorem 10.1 (Stokes' Theorem)**.** *If ω is a $(k-1)$-form on a compact manifold (with boundary) M, and ∂M denotes the boundary of M (with its induced orientation), then*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{10.2}$$

Proof. We will divide the proof into four steps.

Step 1. Let $M = (0,1] \times (0,1)^{k-1}$, and consider the $(n-1)$-form $\omega = f dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$, where the notation $\widehat{dx_j}$ means to omit this term. Then we have

$$d\omega = \left(\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{10.3}$$

By linearity and because $dx_i \wedge dx_i = 0$, this is in turn equal to

$$\frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n = (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n. \tag{10.4}$$

Using this fact, we obtain

$$\int_M d\omega = \int_0^1 \cdots \int_0^1 (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \cdots dx_n. \tag{10.5}$$

By the Fundamental Theorem of Calculus, we further have that this equals $(-1)^{j-1} \int_0^1 \cdots \widehat{\int_0^1} \cdots \int_0^1 \left(f(x_1, ..., 1, ..., x_n) - f(x_1, ..., 0, ..., x_n) \right) dx_1 \cdots \widehat{dx_j} \cdots dx_n$.

We know that both ω and f have compact support in M, so we have

$$\int_M d\omega = \begin{cases} \int_0^1 \cdots \int_0^1 f(1, x_2, ..., x_n) dx_2 \cdots x_n & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}$$

Now, $\int_{\partial M} \omega$ is meant to be understood as $\int_{\partial M} \iota^* \omega$, where $\iota : \partial M \to M$ is the inclusion map. Therefore, if $j > 1$, then $\iota^* \omega = 0$, and if $j = 1$, then $\iota^* \omega(x) = f(1, x_2, ..., x_n) dx_2 \wedge \cdots \wedge dx_n$. Therefore, $\int_{\partial M} \omega = \int_0^1 \cdots \int_0^1 f(x) dx_2 \cdots dx_n$, and so the theorem has been proven for our chosen M and ω.

Step 2. Now let $M = (0,1] \times (0,1)^{n-1}$, and let ω be an arbitrary $(n-1)$-form on M. Write $\omega(x) = \sum_j f_j(x) dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$. By additivity of the integral, we are reduced to the previous case.

Step 3. Let $M = (0,1)^n$, and let ω be an $(n-1)$-form on M. If we follow the proof of Step 1, we have $\int_M d\omega = 0$. Since $\partial M = \emptyset$, we have $\int_M d\omega = \int_{\partial M} \omega = 0$.

Step 4. Now let $M \subset \mathbb{R}^n$ be an arbitrary n-manifold, and let ω be an arbitrary $(n-1)$-form. Let $\mathcal{A} = (U_i, \alpha_i)$ be an oriented set of coordinate charts $\alpha_i : U_i \to M$ that cover M. We can further restrict that U_i be either $(0,1] \times (0,1)^{n-1}$ or $U_i = (0,1)^n$. (This is always possible. Check this if you are skeptical.)

Let ϕ_i be a partition of unit subordinate to $\mathcal{A}$. Then by Definition 9.11 and Theorem 5.9,

$$\int_M d\omega = \sum_i \int_M \phi_i d\omega = \sum_i \int_M d(\phi_i \omega) - \int_M \left(\sum_i d\phi_i \right) \wedge \omega = \sum_i \int_M d(\phi_i \omega),$$

since $\sum_i \phi_i = 1$ on M. In turn, by Definition 9.9,

$$\sum_i \int_M d(\phi_i \omega) = \sum_i \int_{U_i} \alpha_i^*(d(\phi_i \omega)),$$

since $\text{supp}(d(\phi_i \omega)) \subset \alpha_i(U_i)$, as $\phi_i \omega = 0$ outside $\alpha_i(U_i)$. We also have

$$\sum_i \int_{U_i} \alpha_i^*(d(\phi_i \omega)) = \sum_i \int_{U_i} d\alpha_i^*(\phi_i \omega) = \sum_i \int_{\partial U_i} \alpha_i^*(\phi_i \omega) = \int_{\partial M} \omega.$$

Hence, $\int_M d\omega = \int_{\partial M} \omega$, and we are finished. □

The resemblance of the Fundamental Theorem of Calculus to Stokes' Theorem is immediately seen. For completion, it will be stated here.

Theorem 10.6 (The Fundamental Theorem of Calculus). *Let $[a,b]$ be an interval in $\mathbb{R}$. Let f be a continuous function on $[a,b]$, and let F satisfy the equation $F'(x) = f(x)$ on the interval. Then*

$$\int_a^b f = F(b) - F(a). \tag{10.7}$$

Clearly, $[a,b]$ is a 1-manifold. Since F is differentiable, F is a 0-form, and $dF = \frac{dF}{dx}dx = fdx$ is a 1-form[4]. By Stokes' Theorem, we then have $\int_a^b fdx = \int_{\partial[a,b]} F = \int_{\{a,b\}} F = F(b) - F(a)$, where the final equality is a result of Equation 9.16. Of course, we use the concept of induced orientation here as we integrate F over $\{a,b\}$. This orientation is determined by Definition 9.17.

We briefly note that the FTC is not a true corollary to Stokes' Theorem. The reason for this is that we actually use the FTC to *prove* it. However, once Stokes' Theorem has been established, it is tempting to regard the Fundamental Theorem of Calculus as a 1-dimensional case of it.

We proceed in stating the corollaries of Stokes' Theorem that are often presented in vector calculus courses. The most direct application of Stokes' Theorem is Green's Theorem.

Corollary 10.8 (Green's Theorem)**.** *Let $M \subset \mathbb{R}^2$ be a compact 2-manifold (with boundary). Let $P, Q : M \to \mathbb{R}$ be differentiable. Then*

$$\int_{\partial M} Pdx + Qdy = \iint_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{10.9}$$

Proof. Let $\omega = Pdx + Qdy$. Then $d\omega = dP \wedge dx + dQ \wedge dy$. We further have $d\omega = dP \wedge dx + dQ \wedge dy = (\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy) \wedge dx + (\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy) \wedge dy = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy$.

The result now immediately follows from Stokes' Theorem. □

The rest of this paper will be devoted to proofs of the Gradient Theorem and the Divergence Theorem. The proof of the former is straight-forward, requiring a single lemma. The latter is more involved, and requires a bit more work.

Lemma 10.10. *Let $M \subset \mathbb{R}^n$ be an oriented, compact 1-manifold, and let $\boldsymbol{T}$ be the unit tangent vector to M corresponding to the orientation. Let $\boldsymbol{F}(x) = \sum f_i(x)e_i$ be a vector field defined in an open set containing M. By Theorem 6.9, $\boldsymbol{F}$ corresponds to the 1-form $\omega = \sum_i f_i dx_i$. Then we have*

$$\int_M \omega = \int_M \boldsymbol{F} \cdot \boldsymbol{T} \tag{10.11}$$

Theorem 10.12 (The Gradient Theorem)**.** *Let $M \in \mathbb{R}^n$ be a compact 1-manifold, and let $\boldsymbol{T}$ be a unit tangent vector field to M. Let f be a smooth function defined in an open set containing M. If $\partial M = \emptyset$, then we have*

$$\int_M (\nabla f \cdot \boldsymbol{T}) = 0 \tag{10.13}$$

[4]This provides more insight as to why we use the notation $\int fdx$. It fits perfectly into the derivation of the Fundamental Theorem of Calculus.

Proof. By Theorem 6.9, df corresponds to ∇f. Therefore, by the lemma, we have $\int_M df = \int_M (\nabla f \cdot \mathbf{T})$. But by Theorem 9.18, this implies that $\int_M (\nabla f \cdot \mathbf{T}) = \int_M df = \int_{\partial M} f = 0$, since $\partial M = \emptyset$. □

Letting $n = 3$, we have the Gradient Theorem often presented in vector calculus courses. It is finally time to develop the final theorem of this paper, the Divergence Theorem.

Definition 10.14. Let $M \subset \mathbb{R}^k$ be an oriented $(k-1)$-manifold. Let $p \in M$, and let $(p;\nu)$ be a unit vector in $T_p(\mathbb{R}^k)$ orthogonal to $T_p(M)$. Let $\alpha : U \to V$ be a coordinate chart on M around p belonging to the orientation of M, satisfying $\alpha(x) = p$. Now, select ν such that $\det(\nu, D_1\alpha(x), \ldots, D_{n-1}\alpha(x)) > 0$. Then we call the vector field $\mathbf{n}(p) = (p;\nu(p))$ the *unit normal field corresponding to the orientation of M.*

Definition 10.15. We say that a normal vector to ∂M *points inward* if the gradient of the curve traced by the boundary of M at p moves into M. Otherwise, we say that it *points outward.*

We now give a rigorous definition to the symbol dA, which will be used in the Divergence Theorem

Definition 10.16. Let $M \subset \mathbb{R}^3$ be a 2-manifold, and let $n(x)$ be the unit outward normal at $x \in M$. Then if v and w are vectors in $\mathbb{R}^3$, we define $dA(v,w) = \langle v \times w, n(x)\rangle$.

By the preceding definition, dA is a 2-form on M. It has a few desirable properties that we will use to prove the Divergence Theorem.

Theorem 10.17. *Let $M \subset \mathbb{R}^3$ be an oriented 2-manifold, and let $n(x) = n_1(x)\boldsymbol{i} + n_2(x)\boldsymbol{j} + n_3(x)\boldsymbol{k}$ be the outward unit normal. Then*

$$dA = n_1(dy \wedge dz) - n_2(dx \wedge dz) + n_3(dx \wedge dy). \tag{10.18}$$

In fact, the following equations also hold:

$$\begin{aligned} n_1 dA &= dy \wedge dx \\ n_2 dA &= -dx \wedge dz \\ n_3 dA &= dx \wedge dy \end{aligned}$$

Proof. For Equation 10.18, write $dA(v,w) = \det \begin{bmatrix} v \\ w \\ n \end{bmatrix}$. Then expand the determinant by minors along the bottom row.

For the equations, let $z \in T_x(\mathbb{R}^3)$. Since $v \times w = an(x)$ for some $a \in \mathbb{R}$,

$$\langle z, n(x)\rangle \cdot \langle v \times w, n(x)\rangle = \langle z, n(x)\rangle a = \langle z, an(x)\rangle = \langle z, v \times w\rangle.$$

Now, choose $z = (x;e_1), (x;e_2)$, and $(x;e_3)$ to obtain the equations. □

Theorem 10.19 (The Divergence Theorem)**.** *Let $M \subset \mathbb{R}^3$ be an oriented, compact 3-manifold. Let $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$ be the unit normal vector field to ∂M that points outwards from M. Let $\boldsymbol{F} = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ be a vector field defined in an open set containing M. Then*

$$\int_M \nabla \cdot \boldsymbol{F}\ dV = \int_{\partial M} \boldsymbol{F} \cdot \boldsymbol{n}\ dA. \tag{10.20}$$

Proof. Let $\omega = f_1(dy \wedge dz) + f_2(dx \wedge dz) + f_3(dx \wedge dy)$. Then $d\omega = \nabla \cdot \mathbf{F} dV$. By Theorem 10.17, $(\mathbf{F} \cdot \mathbf{n})dA = f_1 n_1 dA + f_2 n_2 dA + f_3 n_3 dA = f_1(dy \wedge dz) + f_2(dx \wedge dz) + f_3(dx \wedge dy) = \omega$. Therefore, by Stokes' Theorem,

$$\int_M \nabla \cdot \mathbf{F} dV = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} \mathbf{F} \cdot \mathbf{n} dA. \tag{10.21}$$

□

In vector calculus textbooks, the divergence theorem is often written as:

$$\iiint_M \nabla \cdot \mathbf{F} = \iint_{\partial M} \mathbf{F} \cdot \mathbf{n}. \tag{10.22}$$

References

[1] Munkres, James R. Analysis on Manifolds. Westview Press. 1991.
[2] Lee, John M. Introduction to Smooth Manifolds. Springer-Verlag. 2002.
[3] Hubbard, John H., Hubbard, Barbara Burke. Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach. Prentice Hall. 2002.
[4] Stewart, James. Multivariable Calculus. Brooks/Cole. 2003.
[5] Spivak, Michael. Calculus on Manifolds. Perseus Books Publishing, L.L.C. 1965.