MAGICAL TRIANGLES

PATRICK GREENE

Abstract. Simplicial complexes consist of a set of vertices together with designated subsets. They can be thought of as embedded in $\mathbb{R}^n$ with the induced metric and topology, where n is large enough that the points can all be geometrically independent. The sets of points spanned by these designated subsets then form triangles or their higher or lower dimensional analogues according to some restrictions. By gluing together simplices in various ways, many compact manifolds can be approximated up to homeomorphism by finite complexes. In addition, we show that any simplicial complex of dimension n can be realized in $\mathbb{R}^{2n+1}$ without compromising the basic structure of the complex, regardless of number of vertices. Because their component parts are fairly simple, approximation with simplices can make it easier to compute properties of a space, such as the Euler characteristic. Continuous maps between spaces can also be approximated up to homotopy by linear simplicial maps, which map the simplicial structure of one space into another.

Contents

1. Introduction and Basic definitions

The properties of a space can be more easily understood if we build the space up through a finite number of smaller spaces, each of which has relatively simple properties. Similarly, building up functions through a finite number of locally linear maps defined on these smaller spaces gives an easier way to deal with functions. Developing the framework for approximating spaces and functions in this way with homeomorphic simplicial complexes and homotopic simplicial maps is one of the main purposes of this paper. These tools can then be used to compute homology and prove more important theorems about, for example, embedding manifolds in $\mathbb{R}^n$.

Definition 1.1. A set of points $\{a_i\}_{i=0,..,n}$ in $\mathbb{R}^m$ is *geometrically independent* if whenever

Date: August 20, 2008.

$$\sum_{i=0}^{n} \lambda_i a_i = 0 \tag{1.2}$$

and

$$\sum_{i=0}^{n} \lambda_i = 0, \tag{1.3}$$

$\lambda_i = 0$ for all i.

By combining the two equations above, we get the equivalent statement that the set of points $\{a_i\}$ are geometrically independent if the vectors $(a_i - a_0)$, $i = 1, ..., n$ are linearly independent.

Conversely, the set $\{a_i\}_{i=0,..,n}$ is geometrically dependent if there exist $\lambda_0, \lambda_1, ..., \lambda_n$ such that the first two equations above hold, but $\lambda_i \neq 0$ for some i. Solving the second equation for λ_i and substituting it into the first equation, we get that

$$(\lambda_0 + \cdots + \lambda_n)a_i = \lambda_0 a_0 + \cdots + \lambda_n a_n, \tag{1.4}$$

where neither side of the equation includes a λ_i term.

Dividing by $\lambda_0 + \cdots + \lambda_n$, we see that a_i can then be written as a linear combination of the other points, with coefficients η_j such that

$$\sum_{j=0}^{n} \eta_j = 1. \tag{1.5}$$

This gives us an equivalent definition of geometric dependence, and we can say in general that a point b is geometrically dependent on a set of points a_i if it can be written as a linear combination of the a_i with coefficients which add up to 1. From now on we will refer to to geometrically independent points as simply "independent," and likewise for geometrically dependent points.

Definition 1.6. Given a set of $n+1$ independent points $a_0, a_1, ..., a_n$ in $\mathbb{R}^m$, the set of points dependent on them, with $\lambda_i > 0$ for all i, is called the *n-simplex* s_n, and the a_i are called its *vertices*.

We could alternatively follow the common practice of defining a simplex as the set of all dependent points with $\lambda_i \geq 0$ for all i; however, for this paper we follow Hilton's [1] convention of insisting that the λ_i be strictly greater than 0.

A set of vertices uniquely determines a simplex, since by independence of the vertices, any point that can be written as a linear combination (with the above restrictions) is in the simplex, and can only be written one way. We say that a simplex is *spanned* by its vertices. Also, note that the combination of the conditions that $\lambda_i > 0$ and that $\sum \lambda_i = 1$ implies that simplices are points, line segments, triangles, or higher dimensional generalizations such as tetrahedrons.

Definition 1.7. The boundary ∂s_n of a simplex s_n is the set of all points d such that

$$d = \sum_{i=0}^{n} \lambda_i a_i, \tag{1.8}$$

where $\lambda_i = 0$ for one or more i, and $\sum_i \lambda_i = 1$.

Definition 1.9. Given a subset of a simplex's vertices, the simplex spanned by this subset is called a *face* of the simplex. Write $s_n \prec s_m$ if s_n is a face of s_m.

Note that a simplex is a face of itself. Proper subsets of vertices form proper faces, although we will just use the term face in both situations unless there is some ambiguity.

Definition 1.10. The *closure* of s_n, $\overline{s_n}$, is the union of a simplex with its boundary.

Note that the condition on simplices that $\lambda_i > 0$ for all i implies that simplices do not include their boundaries, and are therefore open in the closed simplex defined by the same vertices.

Proposition 1.11. *Simplices are convex.*

This is an important reason why simplices are so useful, since it makes finding homotopies between functions to simplices trivial (use the straight-line homotopy), provided their images land in the closure of the same simplex.

Proof. Take two points b_0 and b_1 in a simplex spanned by $\{a_0, .., a_n\}$. Then we can write

$b_0 = \sum_i \lambda_i a_i$, where $\sum_i \lambda_i = 1$ with $\lambda_i > 0$ for all i.

$b_1 = \sum_i \eta_i a_i$ where $\sum_i \eta_i = 1$ with $\eta_i > 0$ for all i.

Then the line from b_0 to b_1 is the set of of d such that

$$\begin{aligned} d &= t \sum_i \lambda_i a_i + (1-t) \sum_i \eta_i a_i \\ &= \sum_i (t\lambda_i + (1-t)\eta_i) a_i \end{aligned}$$

for t between 0 and 1.

Since $t\lambda_i + (1-t)\eta_i > 0$ for all such t, and

$$\sum_i t\lambda_i + (1-t)\eta_i = t + (1-t) = 1,$$

d is in s_n.

□

$\overline{s_n}$ is convex as well, by a similar argument.

It is clear that a point on the boundary of a simplex belongs to a unique proper face, since it is in the simplex spanned by the set of a_i with coefficients not equal to 0 in the expression $\sum \lambda_i a_i$, and we know all points in a simplex have a unique representation. Also, any point on a proper face is in the boundary, by our definition.

Definition 1.12. A *geometric simplicial complex* K in $\mathbb{R}^m$ is a finite set of simplices in $\mathbb{R}^m$ which satisfies two properties:

(1) Let s_p and s_q be simplices, with s_p in K. Then if $s_q \prec s_p$, s_q is in K.
(2) Distinct simplices do not intersect.

From now on, we will abbreviate "geometric simplicial complex" as simply "complex". The first condition states that a complex includes all the faces of each simplex, while the second ensures that each point in the complex lies in a unique simplex.

A closed simplex $\overline{s}_n$ is the simplest example of a complex. Further, any union of closed simplices constitutes a complex (if intersections are limited to shared faces). Conversely, since the closure of every simplex is in a complex, any complex can be written as a union of closed simplices.

Definition 1.13. A subcomplex is a subset of the simplices in a complex such that (1) and (2) above hold.

Any closed simplex or union of closed simplices constitute a subcomplex. A special case of this is the n-skeleton:

Definition 1.14. The *n-skeleton* of a complex is the set of all simplices with dimension $\leq n$.

Ignoring the simplicial structure of a complex, we can consider the underlying space consisting of points belonging to a complex K with the subspace topology from $\mathbb{R}^m$. We denote this space $|K|$, the *polyhedron* of K. If K' is a subcomplex of K, then $|K'|$ is called a *subpolyhedron* of $|K|$.

Proposition 1.15. *If K_0 and K_1 are subcomplexes of K, then $K_0 \cup K_1$ and $K_0 \cap K_1$ are also subcomplexes.*

Proof. For the first part, suppose s is a simplex in $K_0 \cup K_1$. Then it is in K_0 or K_1. So, (1) is satisfied for s, and the fact that both subcomplexes are part of the same complex implies (2) is satisfied as well. For the second part, suppose s is in $K_0 \cap K_1$. Then it is in both subcomplexes, and by definition (1) and (2) must be satisfied for s in both. Thus, $K_0 \cap K_1$ is a subcomplex. □

2. Abstract Complexes and Complexes as Spaces

If we think about a complex as a subspace of $\mathbb{R}^m$, we do not really care about its actual location relative to some set of axes, the order of its vertices or the relative sizes of simplices. All of the important topological properties of a complex come from which simplices are connected, and in what way. This information is determined completely by certain facts about its vertices, as we will show. For this reason, we define a vertex scheme and an abstract simplicial complex:

Definition 2.1. Given a finite set of vertices, a *vertex scheme* is a set of subsets of these vertices. Each subset is called a *selected set* of the vertex scheme.

Definition 2.2. An *abstract simplicial complex* consists of sets of vertices and a way of grouping them, the vertex scheme, which has the property that every vertex is in some selected set of the scheme, and that every subset of a selected set is also selected. Each of the selected sets of the vertex scheme is a *simplex* of the abstract complex.

Every subset of a selected set is called a *face* of the original set. Note that, with our abstract complex versions of the definitions for simplex and face, part (1) of definition 1.12 is satisfied, and we do not need to worry about part (2) in an abstract complex.

We can define the vertex scheme of a geometric simplicial complex by having the vertices spanning each (geometric) simplex correspond to a selected set in the scheme. This vertex scheme then has the same properties as in an abstract complex. These two things show why we use the terms "face" and "simplex" in abstract complexes.

From now on, in addition to geometric simplicial complex, we will also abbreviate "abstract simplicial complex" as just "complex." Whether we are referring to an abstract or geometric complex should be clear from the context, although the two are often interchangeable.

Example 2.3. The vertex scheme for a geometric complex consisting of a single 2-simplex would be: $\{\{a_0, a_1, a_2\}, \{a_0, a_1\}, \{a_0, a_2\}, \{a_1, a_2\}, \{a_0\}, \{a_1\}, \{a_2\}\}$.

Definition 2.4. The set of vertices in an abstract complex can be associated with a set of points $\{a_i\}$ in $\mathbb{R}^m$ through a bijective correspondence. If the points corresponding to each selected subset are independent, then the vertex scheme V of the abstract complex determines a complex in $\mathbb{R}^m$ called the geometric realization of V, $|V|$. The realization consists of the sets of points of the form $\sum_i \lambda_i a_i$ such that

(1) $\lambda_i > 0$ for all i
(2) $\sum_i \lambda_i = 1$
(3) If for some point a, $\lambda_{i_1}, ..., \lambda_{i_k}$ is its complete set of non-zero coefficients, then their corresponding $\{a_{i_1}, ..., a_{i_k}\}$ is a selected set of the scheme.

(1) and (2) say that $|V|$ includes all points dependent on the vertices in V, and all boundary points. (3) says that, if a point is on the boundary of $|V|$, then it lies on the face of some simplex.

Definition 2.5. A *vertex transformation* is a map $v : V(K) \to V(L)$, where V(K) is the vertex set of K, between vertices in an abstract or geometric complex K to vertices in an abstract or geometric complex L, which maps simplices to simplices.

Definition 2.6. A *simplicial map* $g : K \to L$, K, L geometric complexes, is the linear extension of a vertex transformation v to the interiors of simplices. Because it acts as a vertex transformation on vertices, it maps simplices in K to simplices in L.

Proposition 2.7. *Any abstract simplicial complex with vertex scheme V can be geometrically realized in $\mathbb{R}^m$ for some m.*

The idea is just to place each vertex in the abstract complex into one higher dimension than the previous vertices. This ensures geometric independence, and therefore that simplices will not intersect. We can then just take the subcomplex consisting of the faces that are selected for in V as our realization.

Proof. Let $\{b_i\}_{i=0,...,p}$ be the vertices of a single p-simplex s_p in R^{p+1}, where $p+1$ is the number of distinct vertices a_i in the abstract complex. Define a simplicial map v by $v(a_i) = b_i$. Then the subcomplex of $\overline{s_p}$ consisting of the simplices spanned by the selected subsets of V satisfies the two properties of a complex, and thus is a realization of the abstract complex. □

Definition 2.8. If there exists a vertex transformation between two abstract complexes whose inverse exists and is also a vertex transformation, the complexes are said to be *isomorphic*.

This says that isomorphic complexes are essentially the same complex with renamed vertices, since if the inverse of a vertex transformation v exists and is also a vertex transformation, v must be bijective. Now we can prove our earlier assertion that all of the important topological properties of a complex arise from the vertex scheme.

Theorem 2.9. *Isomorphic abstract complexes have homeomorphic realizations.*

Proof. Let K and L be the geometric realizations of two isomorphic abstract complexes. We know the abstract complexes have a bijective vertex transformation v between them. We can then linearly extend v to a continuous simplicial map f from K to L. f^{-1} is then just the extension of v^{-1}, and so is also continuous and therefore a homeomorphism. □

Many spaces can easily be approximated up to homeomorphism (and therefore homotopy equivalence) by a complex. For example, the sphere S^n is homeomorphic to ∂s_{n+1}, by simply putting ∂s_{n+1} inside the sphere and projecting onto it.

Theorem 2.10 (Simplicial Embedding Theorem)**.** *An abstract simplicial complex K with dimension n can be realized in $\mathbb{R}^{2n+1}$.*

Proof. This theorem is mainly a statement about the number of independent vectors needed to uniquely represent points in any two pairs of simplices. By definition, we need to find a set of points in $\mathbb{R}^{2n+1}$ that we can map the vertices in our abstract complex to, and which form a geometric complex themselves by satisfying conditions (1) and (2) in definition 1.12. So, we require that the closure of every abstract simplex is realized, and no two simplices intersect. If we can realize the closure of every simplex individually, then condition (1) will be taken care of, because every complex is the union of its closed simplices. However, this is a trivial task, since by definition the highest dimensional simplex in the abstract complex has dimension n, and thus can be realized in n dimensions, much less than $2n+1$.

The second condition is more difficult to satisfy. If two simplices intersect, then the points in their intersection can be written as linear combinations of two different sets of vertices. However, if we can find enough independent sets of points to ensure that no two n-dimensional simplices intersect, then it follows that three or more n-dimensional simplices will certainly not intersect, and also that no two (or more) p- and m-dimensional simplices, with m, $p \leq n$, will intersect either. So, it is just pairs of simplices that we need to worry about, specifically pairs of n-simplices.

Each n-simplex has $n+1$ vertices, and so requires $n+1$ geometrically independent points in $\mathbb{R}^p$ to be realized. We require that an n-simplex's intersection with any other individual simplex be empty. The worst case scenario is that the complex contains more than one n-simplex, so we want that every set of $2(n+1) = 2n+2$ points be geometrically independent. We know that the smallest $\mathbb{R}^p$ for which we can find $2n+2$ independent points in $\mathbb{R}^{2n+1}$, but can we find an arbitrary number of points in $\mathbb{R}^{2n+1}$ such that any 2n+2 of them are independent? The answer is yes, by the following method:

Recall that $2n+2$ independent points is equivalent to having $2n+1$ linearly independent vectors. We know that we can find a set of $2n+1$ linearly independent vectors in $\mathbb{R}^{2n+1}$, but not in any dimension less than this. Take the union of subspaces spanned by every set of $2n$ of these vectors. Since no finite union of subspaces with dimension less than the dimension of the whole space can equal the whole space, we can create a new vector in the space whose span does not intersect any of the previously created subspaces (except at 0). This vector is then linearly independent of any other set of $2n$ vectors. Thus we now have $2n+2$ vectors, such that any $2n+1$ of them are linearly independent. This corresponds to having $2n+3$ points, such that any $2n+2$ of them are independent. We can continue this process as long as we want, provided the number of vectors stays finite. In this way, we can create an arbitrary number of points such that any $2n+2$ of them are independent. By the above, K can then be realized in $\mathbb{R}^{2n+1}$. □

Note that the same argument shows that if K only contains one n-dimensional simplex, and its next highest simplex has dimension $m < n$, then we can realize it in $\mathbb{R}^{n+m+1}$. According to Hilton [1], this theorem is "an essential step in the proof that a compact n-dimensional metrizable space [such as a manifold] can be embedded in $\mathbb{R}^{2n+1}$."

3. Barycentric Subdivision

In this section, we will define some concepts that will prove useful in the next two sections.

Definition 3.1. The *barycenter* of a simplex is literally its center of mass (from the Greek word *barus*, meaning "heavy"), assuming all vertices are weighted equally. It is given by the equation

$$b = \frac{1}{n+1} \sum_{i=0}^{n} a_i \tag{3.2}$$

Definition 3.3. The *barycentric subdivision* of an abstract complex K is a new complex K' defined as follows. For each simplex s with vertex set $a_0, a_1, ..., a_n$, for each selected subset $\{a_{i_0}, a_{i_1}, ..., a_{i_k}\}$ of this set, create a vertex of the form $b_{i_0,...,i_k}$ (the i_j subscript corresponds to the vertex a_{i_j} in the selected set). For example, if $\{a_{i_0}, a_{i_1}\} = \{a_2, a_3\}$ is a selected set, then we would create a new vertex labeled $b_{i_0,i_1} = b_{2,3}$. Then, forgetting about the old set of vertices, we define the selected sets of the new set of b vertices by the following rule: if a set of vertices can be arranged so that the subscripts of each are contained in the subscripts of the preceding vertex, then that set is selected. For example, the three vertices $b_1, b_{1,5,7}$, and $b_{1,7}$ form a selected set, since they can be arranged as $b_{1,5,7}, b_{1,7}, b_1$.

Note that since the original vertices each formed their own selected set, we have that $b_0 = a_0, ..., b_n = a_n$. Also, we can repeat this process as many times as we like, forming a new complex each time. The rth subdivision will be denoted $K^{(r)}$. The new vertices could be embedded in a higher dimension than that of the old simplex which they came from, but by the way we have defined the selected sets, we can embed them in the same subspace as the original simplex by associating each $b_{i_0,...,i_n}$ to the barycenter of the simplex spanned by $a_{i_o}, ..., a_{i_n}$. The embedding map defined in this way on the vertices and linearly extended to the interiors of the simplices in K' is bijective and continuous, and therefore has a continuous inverse, making it a homeomorphism between the old polyhedron and the newly subdivided polyhedron. Thus, subdivision does not change the polyhedron or underlying space of a complex.

Definition 3.4. The *mesh* of a complex K, denoted μK, is the greatest distance between any two points in the closure of the same simplex.

Note that, given any two points in the same simplex, we can move them farther apart by pushing them onto the boundary of the simplex. We can then move them even further apart by sliding them one at a time along a face of the boundary in a line, until they reach a vertex. For some vertex, moving them in any other direction would bring them closer to each other, since the simplex is convex. Since any two vertices in a simplex are connected by a 1-simplex which is part of the complex, the mesh of a complex is equal to the length of the longest 1-simplex.

Theorem 3.5. *If K is a simplicial complex with dimension $\leq n$, and K' its first barycentric subdivision, $\mu K' \leq \frac{n}{n+1}\ \mu K$.*

Proof. As mentioned above, the mesh of K' is equal to the length of the longest 1-simplex. 1-simplices in K' are lines extending from the barycenter of a simplex s in K to some point on the boundary (the barycenter of a face of s). If b is the barycenter of any simplex, then the longest line from b to any point on the boundary occurs when the endpoint is a vertex, since we can always just slide the endpoint in some direction along a straight line to increase the length of the segment, until we get to a vertex. Call this vertex a_0. For a k-dimensional simplex, the vector from b to a_0 is:

$$b - a_0 = \frac{1}{k+1} \sum_{i=0}^{k} (a_i - a_0)$$

The length of this vector is $|b - a_0|$. By the triangle inequality,

$$|b - a_0| \leq \frac{1}{k+1} \sum_{i=1}^{k} |a_i - a_0|$$

But $|a_i - a_0| \leq \mu K$, so the above equation is less than $\frac{k}{k+1}\mu K$ (note that we begin at $i = 1$, since the first term is just 0). Since dim $K \leq n$, the largest this could be is $\frac{n}{n+1}\mu K$. □

Note that since $\frac{n}{n+1} < 1$, and barycentric subdivision does not increase the dimension of a complex, we can use repeated subdivision to get the mesh of a complex as small as we like.

Definition 3.6. Recall that every point x in a complex belongs to a unique simplex, since simplices do not intersect. This simplex is called the *carrier* of x.

4. Homotopy on Complexes

One of the main reasons building a space with a simplicial complex is useful is that a lot of homotopy facts are easier to prove on complexes. Homotopy equivalence between two spaces is in some sense a weaker equivalence than homeomorphism because two homeomorphic spaces are homotopy equivalent by definition, but the converse is generally not true. An example is the unit disk in $\mathbb{R}^2$, which is homotopy equivalent to a point, but not homeomorphic, since any open set in the disk must map to the closed point.

It is easy to show that for $f, g : X \to Y$, f homotopic to g (denoted $f \simeq g$), is an actual equivalence relation in the set of maps from X to Y. Reflexivity and symmetry come straight from the definition, and transitivity can by shown by composing the homotopies and running them each at twice the speed. Similarly, homotopy equivalence between two spaces is an equivalence relation on the collection of spaces, although proving this requires a lemma.

Lemma 4.1. *If $f_0, f_1 : X \to Y$, and $f_0 \simeq f_1$, then for any continuous function $g : Y \to Z$, $g \circ f_0 \simeq g \circ f_1$. Similarly, if $h : Z \to X$, then $f_0 \circ h \simeq f_1 \circ h$.*

Proof. For the first part, let H be the homotopy between f_1 and f_0. Define $H' : X \times I \to Z$ by $H' = g \circ H$. Then H' is continuous, and provides the homotopy we seek.

For the second part, again let H be the homotopy between f_1 and f_0. Define $H' : Z \times I \to Y$ by $H'(z, t) = H(g(z), t)$. H' is continuous by continuity of g, and is a homotopy. □

Proposition 4.2. *Homotopy equivalence between spaces is an equivalence relation*

Proof. Obviously a space is equivalent to itself, and $X \simeq Y$ implies $Y \simeq X$, by definition. Suppose $X \simeq Y$ and $Y \simeq Z$, with maps $f : X \to Y, g : Y \to X, h : Y \to Z, v : Z \to Y$. Then our map from X to Z is hf and our map from Z to X is gv. We know that $vh \simeq id_Y$, so $gvhf \simeq g(id_Y)f = gf \simeq id_X$. Similarly, $fg \simeq id_X$, so $hfgv \simeq h(id_X)v = hv \simeq id_Z$.

□

Definition 4.3. Let Y be a subspace of X. Then a map $r : X \to Y$ is called a retracting map if r restricted to Y is the identity on Y (written $r|Y = id_Y$). Y is called a *retraction* of X.

Definition 4.4. A function $f : X \to X$ acts *relative* to a subspace $A \subset X$ if $f|A = id_A$.

Definition 4.5. Y is a *deformation retract* of X if there exists a homotopy $H : X \times I \to X$ relative to Y such that $H(x, 0) = x$ and $H(x, 1) \in Y$ for all x. H_1 is clearly the embedding of a retraction into X, and so we can see that H is a homotopy between the identity on X and $i \circ r$ for some retracting map $r : X \to Y$ and an embedding $i : Y \to X$. H is called a *deformation retraction.*

Proposition 4.6. *If Y is a deformation retract of X, then $Y \simeq X$.*

Proof. Let $i : Y \to X$ be an embedding of Y into itself as a subspace of X. By definition of deformation retract, there exists a retraction $H : X \times I \to X$ and a retracting map r such that $i \circ r = H_1$. Thus H itself gives us a homotopy between $i \circ r$ and id_X. On the other hand, $r \circ i = id_Y$ since i just imbeds Y into itself as a subspace of X, and by definition r acts relative to Y, and therefore leaves it unchanged. □

Definition 4.7. If s is a simplex in K, then the set of all simplices which have s as a face is called the *star* of s, and is denoted $star(s)$.

Note that $star(s)$ is open in K, since simplices are open.

Theorem 4.8 (Homotopy Extension Theorem)**.** *Let f_0 be a map from $|K| \to X$, where $|K|$ is a complex and X is a topological space. Let g be a map from a subcomplex $|L|$ to X such that g is homotopic to $f_0||L|$ by a homotopy*

$$H : |L| \times I \to X$$

(where $H(x, 0) = f_0(x)$ and $H(x, 1) = g(x)$). Then we can extend g to a map $g' : |K| \to X$ such that g' is homotopic to f_0 via a homotopy $H' : |K| \times I \to X$, where $H'|(|L| \times I) = H$.

Proof. Our main goal is just to extend g in a continuous (if arbitrary) way over $|K|-|L|$, so that the homotopy between f_0 and g in $|L|$ can be continuously extended over all of $|K|$. We begin by defining $H'(x, 0) = f_0(x)$ for all $x \in |K|$, and $H'(y, t) = H(y, t)$ for all $y \in |L|$. This then gives us a map from $\{|K|-|L|\times 0\}\cup\{|L|\times I\} \to X$. Then, as f_0 is homotoped to g in $|L|$, we want the areas around $|L|$ to gradually change towards g as well, otherwise the extension of g will not be continuous. We accomplish this by using a neat projection trick to define H' on first 0-simplices, then 1-simplices, and so on until we get to the highest dimensional simplices.

For any vertex a in $|K| - |L|$, define $H'(a, t) = f_0(a)$ for all t. For simplices in $|L|$, the value of every point at time t is already determined by H, so we will only be concerned with points outside $|L|$. For any point x at time t in the 1-simplex between two vertices, take its radial projection from the point $(b_1, 2)$, where b_1 is the barycenter of the 1-simplex, and define $H'(x, t)$ to be the value of its projection onto $\{|K| \times 0\} \cup \{K^0 \times I\}$, where K^0 is the 0-skeleton of K. Notice that for points on faces near a vertex in $|L|$, their value continuously follows that of the changing values of the vertex. Now g and our homotopy are defined on 1-simplices. Continue this technique for 2-simplices and on up to the highest dimensional simplices by taking the barycenter of each j-simplex and using the projection from $(b_j, 2)$ onto $\{|K|\times 0\}\cup\{|K^{j-1}|\times I\}$ to define $H'(x, t)$ on the j simplices via the $j-1$ simplices. This projection is well-defined on all simplices for all t, since the $j - 1$-skeleton of a complex contains the boundaries of all the j simplices (so there is something to project on to), and the projection is just the identity on the boundary. Thus $g' = H'_1$ is an extension of g to $|K|$, and in creating the extension, we have defined the desired homotopy H' between f_0 and g'. ☐

5. Approximating Functions with Homotopic Simplicial Maps

An additional advantage of approximating spaces with simplicial complexes is that continuous functions on these spaces can then be approximated arbitrarily closely with homotopic simplicial maps.

Definition 5.1. Let $g : K \to L$ and $g' : K' \to L$ be simplicial maps, where K' is a subdivision of K. Then g and g' are called *contiguous* if, for simplices $s \in K$ and $s' \in K'$, where $s' \subset s$, whenever $g(s) \prec t$ for $t \in L$, $g'(s') \prec t$.

Definition 5.2. A simplicial map $g : K \to L$, K, L complexes, is called a simplicial approximation to a continuous function $f : |K| \to |L|$ if, for any $x \in |K|$, $g(x)$ is in the closure of the carrier of $f(x)$.

Theorem 5.3 (Simplicial Approximation Theorem)**.** *Let K and L be geometric simplicial complexes, and $f : |K| \to |L|$ a continuous map. Then there exists a linear simplicial map $g : K' \to L$, where K' is some subdivision of K, such that g is a simplicial approximation to f.*

First let us get an idea about what sort of requirements a simplicial map g needs to fulfill in order to be a simplicial approximation to f.

Let x be a point in $|K|$. Then x lies in a simplex s which is contained in the open set $star(a)$, where a is a vertex of s. Recall that the definition of simplicial approximation requires only that $g(x)$ land in the closure, rather than interior, of the simplex that $f(x)$ lands in. Assuming g is a vertex transformation, g takes a to some vertex t in L. Therefore, we want to define g so that the t it maps a to is the vertex in L such that $f(x) \in |star(t)|$ for all x in $star(a)$. This implies $f(x)$ is in a simplex which has t as a face, and therefore the closure of the carrier of $f(x)$ contains t. g maps vertices to vertices, so the easiest way to see where it should map a given vertex a is to see where $f(a)$ goes, and make sure $g(a)$ gets mapped to a vertex b in L such that $|star(b)|$ contains $f(a)$. It seems reasonable that a g defined this way will be close to a simplicial approximation of f.

However, we have three problems. The first is that f may not map the open stars around every vertex in $|K|$ inside open stars in $|L|$. The open images of these stars under f may sprawl across two or more stars in $|L|$. The second problem is that we need to make sure that g defined this way is actually a simplicial map (it maps simplices to simplices), and finally, we must show that if the first two conditions are met, g is a simplicial approximation. To address the first of these problems, we begin with a lemma:

Lemma 5.4 (Lebesgue)**.** *If K is a compact metric space, then given an open covering $\{U_a\}_{a\in A}$, there exists a $\delta > 0$ such that any subspace of K with diameter $< \delta$ is entirely contained in at least one U_a.*

Proof. Let d be the metric in K. Suppose that such a δ does not exist. Then we can find an open set V_n of diameter $1/n$ for any n such that it is not contained in any U_a. Take a point a_n in each of these sets. Then the set of a_n forms a sequence in K which has a convergent subsequence. Call the point this subsequence converges to a. a is contained in some open set U_a in $\{U_a\}$, so there exists some $\epsilon > 0$ such that $B_\epsilon(a)$ is contained in U_a. But we can find an N_1 such that for all $n > N_1$, $d(a, a_n) < \epsilon/2$. Similarly, we can find an N_2 such that the V_n around each a_n has diameter $< \epsilon/2$ for all n $> N_2$. Then for all $n > \max(N_1, N_2)$, V_n is contained in $B_\epsilon(a)$, which is contained in some U_a. This contradiction implies that such a δ must exist. Any such δ is called a *Lebesgue number* of the covering. □

Now we can take care of the first problem. It is clear that $\{|star(b_i)|\}_i$ around each vertex b_i in $|L|$ form an open cover of $|L|$. Then by continuity of f, the set $f^{-1}(\{|star(b_i)|\}_i)$ forms an open cover of $|K|$. Since $|K|$ is compact and can be embedded in $\mathbb{R}^n$, it is metrizable by the induced metric from $\mathbb{R}^n$. So, we we can apply the above lemma. Let δ be a Lebesgue number of this covering. We have previously shown that we can subdivide $|K|$ so that the mesh of $|K'|$ is arbitrarily small, in particular less than $\delta/2$. Then the star around any vertex must have diameter less than δ and is therefore entirely contained in at least one open set in the cover of $|K'|$.

On to the second problem. Fortunately, the proof that g is a simplicial map also shows us that it is an approximation to f:

Proof of Theorem 5.3. Let $x \in |K'|$. Then x is in some simplex s, with vertices $\{a_i\}_{i=1,..,n}$. So,

$$x \in \bigcap_{i=1}^{n} star(a_i)$$

This implies that

$$f(x) \in \bigcap_{i=1}^{n} f(|star(a_i)|) \subseteq \bigcap_{i=1}^{n} |star(g(a_i))|$$

since $f(|star(a_i)|) \subset |star(g(a_i))|$ for all i, by the way we've defined g and the fact that, thanks to our subdivision, f now maps stars of $|K'|$ into stars of $|L|$.

The star of a vertex is a set of simplices, so if two or more stars have a non-empty intersection, then that intersection must contain a simplex. So, if the intersection of the stars of $g(a_i)$ contains the point $f(x)$, then it must also contain the carrier of $f(x)$, which is some simplex t in $|L|$. Thus the vertices $g(a_i)$ in $|L|$ all lie on the boundary of the same simplex t, and therefore span a face of t. This proves that g is indeed a simplicial map, and since g mapped s onto a face of t, $g(x)$ is in the closure of t, the carrier of $f(x)$. So, g is a simplicial approximation to f.

□

An important result of simplicial approximation is that not only is g linear on simplices and thus easier to work with than f, but g and f are homotopic by the straight line homotopy.

Proposition 5.5. *Any two simplicial approximations are contiguous.*

Proof. If g' is another simplicial approximation of f, then it must map points in $|K|$ into the closure of the same simplices that f maps them into, which are also the same simplices g maps them into. So, g' must be a simplicial approximation of g, by defintion. Then for any simplices $s \in K$, $s' \in K'$ such that $s' \subset s$, $g(s)$ is a simplex in $|L|$, so $g'(s')$ is a simplex in the closure of that simplex. Thus, g and g' are contiguous. □

Proposition 5.6. *If $f(x) = b$, where b is some vertex in $|L|$, then $g(x) = b$ and g and f agree exactly at x.*

Proof. Note that we know this happens simply because g is a simplicial approximation to f, and the closure of a vertex is itself, so g must map x onto the same vertex. However, an explicit proof of this is helpful in illuminating how g behaves.

Since this is obvious if x is a vertex, suppose x is not a vertex. Then $x \in |star(a_i)|$ for some set of vertices $\{a_i\}$. We know that $f(|star(a_i)|) \subset |star(b_i)|$, where b_i is some vertex in in $|L|$. We also know $f(x) = b$ for $x \in |star(a_i)|$, but the only star in $|L|$ containing b is $|star(b)|$. By our subdivision of $|K|$, $f(|star(a_i)|)$ must lie entirely in some star in $|L|$, and so it must be in $|star(b)|$. This is true for all

i. Thus for all vertices in $\{a_i\}, g(a_i) = b$. Writing x out explicitly, we see that $g(x) = g(\lambda_1 a_1 + ... + \lambda_n a_n) = (\lambda_1 + ... + \lambda_n)b = b$, since $\sum_i \lambda_i = 1$.

□

Note that the last line in the proof above is true for any x inside the simplex. This shows that whenever a simplex in $|K'|$ contains even a single point that f maps to a vertex, g squashes the entire simplex into that vertex. The same is true for when f maps a point in a simplex in $|K'|$ to a face of a simplex $|L|$ (which happens whenever f stretches the image of a simplex out across two simplices without going through a vertex); g maps the entire simplex onto that face, by a similar proof: if there are n vertices in $|K|$ whose stars contain the x which gets mapped to a face, then there are $n-j$ for some $j \geq 1$ vertices in $|L|$ whose stars contain the face that has $f(x)$ in it. Again, f of the stars around the n vertices (which contain the vertex itself) each must be contained in the stars around the $n-j$ vertices due to the restriction caused by where $f(x)$ landed. So, we have that g maps n vertices in K' to the $n-j$ vertices in L whose stars contain $f(x)$, and therefore the whole simplex gets mapped onto that face by linearity of g.

Also, note that when f maps one or more simplices completely inside the same simplex in L, g expands each simplex in K' to fill the whole simplex in L, by linearity. It is clear that while g and f are homotopic, g may not be a very good approximation in the sense of mapping points close to where f takes them, since the simplices in $|L|$ may be arbitrarily big, and thus if g's approximation to f is to squash its image entirely into a vertex or edge, the distance from these points to the same points in the image of f can be arbitrarily large.

This makes it clear that if we want to make g a better approximation in this sense, we need to subdivide L.

Definition 5.7. Define the distance ρ between two functions $f, g : X \to Y$ by $\rho = \sup_{x \in X}\{d(f(x), g(x))\}$, where d is the metric in Y.

Theorem 5.8. *Given any continuous function $f : |K| \to |L|$ and $\epsilon > 0$, there exists a simplicial approximation $g : K^{(r)} \to L^{(s)}$ for some subdivisions r and s such that $\rho(f, g) < \epsilon$.*

Proof. Subdivide L until $\mu L < \epsilon$. Call this subdivision $L^{(s)}$. Then use the simplicial approximation theorem to find an approximation g to f.Then for any x, $g(x)$ is in the closure of the simplex $f(x)$ is in, and thus $d(f(x), g(x)) < \epsilon$. So, $\rho(f, g) < \epsilon$. □

Another way of saying this is that, if $|K|^{|L|}$ denotes the space of all continuous functions from $|K| \to |L|$ with the ρ metric above, then the set of simplicial approximations is dense in $|K|^{|L|}$.

Theorem 5.9. *If $f, g : |K| \to |L|$ are continuous and $\rho(f, g) < \delta/3$, where δ is a Lebesgue number of the open covering of $|L|$ created by the stars about its vertices, then there exists a map v such that v is a common simplicial approximation to both f and g.*

The idea is to get, for every vertex $a \in |K|$, $f(|star(a)|)$ and $g(|star(a)|)$ to land in the star around the same vertex b in $|L|$. We do this by making the open sets in the cover of L smaller so that when we bring it back to K, f and g will always map these smaller open sets into the same open star in the original larger cover of $|L|$. The simplicial approximation theorem (which defines the approximation based only on which open stars in $|L|$ f maps open stars in $|K|$ to) then gives us an approximation to both f and g.

Proof. Note that every non-vertex point in $|L|$ is contained in at least two different stars around two different vertices, each of whose boundary contains the other vertex. Also, note that these open sets extend at least δ in every direction (since by definition of Lebesgue number, if an open ball with diameter δ is in one of these stars, with the vertex of the star just inside the boundary of the ball, the entire thing must be contained inside that star, since no other star contains that vertex). So, every point in $|L|$ is greater than $\delta/3$ away from the boundary of one of the stars that contains it, since if a point is less than or equal to $\delta/3$ from the boundary of a star, it is greater than or equal to $2\delta/3$ from the vertex generating the star, so it is greater than or equal to $2\delta/3$ from the boundary of the star generated by one of the other vertices (the star whose boundary includes the first vertex).

This shows that our covering of $|L|$ by the open stars around vertices in Theorem 2.9 was a bit excessive in that we can trim a $\delta/3$ wide section off from the edges of all the open stars and still have an open cover. So, for each vertex b_i in $|L|$, define an open set

$$V_i = \{y \in star(b_i) | d(y, \partial(star(b_i))) > \delta/3\}$$

The set of V_i then form an open cover of $|L|$, so $f^{-1}(\{V_i\}_i)$ is an open cover of $|K|$. We then proceed as in 2.9: let ϵ be a Lebesgue number of this cover, then subdivide K until $\mu K < \epsilon/2$. Let a_i be a vertex in $|K'|$. Now $f(|star(a_i)|) \subset V_i \subset |star(b_i)|$, but V_i is at least $\delta/3$ away from the boundary of $|star(b_i)|$, so if $\rho(f, g) < \delta/3$, then $g(|star(a_i)|) \subset |star(b_i)|$ as well. The simplicial approximation theorem then gives us an approximation to both f and g. □

Note that we could even increase the width of the trimmed section of the open stars to anything less than $\delta/2$, by the reasoning in the first paragraph. Thus, even functions such that $\delta/3 \leq \rho(f, g) < \delta/2$ can have a common simplical approximation.

However, for the final theorem, we only require that functions with difference less than δ/n for some $n < \infty$ have a common simplicial approximation.

Theorem 5.10. *If $f_0 \simeq f_1$, then there exists a sequence of simplicial approximations $g_0, g_1, ..., g_n$, where $g_i : K^{(r_i)} \to L$, such that g_0 is an approximation of f_0, g_n is an approximation of f_1, and g_i and g_{i+1} are contiguous.*

We know that f_0 can be continuously deformed into f_1 via the homotopy H, so the idea is to split H up into a finite number of discrete steps small enough that the intermediate function at each step is within $\delta/3$ of the previous function. This way, we can use the above theorem to produce the desired sequence of approximations.

Proof. Since $I \times K$ is compact and H is continuous, $H(x, t)$ is uniformly continuous with respect to t. So, given $\epsilon > 0$, there exists a $\sigma > 0$ such that $|t - t'| < \sigma$ implies $|f_t(x) - f_{t'}(x)| < \epsilon$ for all x. In particular, there exists such a σ when $\epsilon = \delta/3$. So, we can split H into the sequence of functions $f_0, f_\sigma, f_{2\sigma}, ..., f_1$, where $\rho(f_{i\sigma}, f_{(i+1)\sigma}) < \delta/3$. By the previous theorem, any two consecutive functions

$f_{i\sigma}, f_{(i+1)\sigma}$ have a common simplicial approximation $g_{i,i+1}$. Also, since any two simplicial approximations of the same function are contiguous, consecutive g_i's in the sequence of approximations are contiguous. □

Note that non-consecutive g_i's are generally not contiguous. If they were, we could simply remove the intervening g_i's.

6. Acknowledgements

Thank you to my graduate student mentors Emily Riehl and Megan Shulman, who spent uncountably many hours answering my questions and instructing me in the wonderful and mysterious ways of the simplex.

References

[1] Hilton, P.J., Wylie, S. Homology Theory. Cambridge University Press. 1960.
[2] Maunder, C. R. F. Algebraic Topology. Courier Dover Publications. 1996