# SYMMETRIES IN $\mathbb{R}^3$

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ABSTRACT. This paper will introduce the concept of symmetries being represented as permutations and will proceed to explain the group structure of such symmetries under composition. We then describe the special orthogonal group, $SO_3$, and how its finite subgroups are various groups of symmetry. This will lead us to the conclusion that there are only five regular polyhedra.

## 1. BACKGROUND

**Definition 1.1.** A *permutation* is defined to be a bijection from a finite set to itself.

A permutation generally acts on a finite set of the form $\{1, \ldots, n\}$. The set of permutations of $\{1, \ldots, n\}$ is known as $S_n$. The permutation $\sigma$ of $\{1, 2, 3, 4, 5\}$ given by $\sigma(1) = 3, \sigma(2) = 4, \sigma(3) = 1, \sigma(4) = 2, \sigma(5) = 5$ can be written either in matrix form:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$

or in cycle structure:

$$(13)(42)(5).$$

Matrix form allows for a clearer picture of how the numbers are being moved around, whereas cycle structure shows which numbers are being switched around and which remain unchanged. If 2 had been sent to 1 and 1 sent to 3 and 3 sent back to 2, then there would have been a cycle of the form (213), (321), or (132) in the cycle structure of the permutation.

Permutations on a set can be viewed as a group under the binary operation of composition, where composition of permutations is simply composition of functions. Thus we can regard $S_n$ as a group. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

then

$$\pi \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \text{ and } \sigma \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$

Now we will relate permutation groups to symmetries of geometric objects. Looking at a square with vertices numbered 1 to 4, the set of rotations of the square by multiples of $\frac{\pi}{2}$ is a group of permutations that is a subgroup of $S_4$. The identity is a rotation by 0, and each rotation has an inverse rotation. More generally, looking at an $n$-gon with vertices labeled 1 through $n$, the rotations by multiples of $\frac{2\pi}{n}$ around the fixed center of the $n$-gon always form a symmetry group called the cyclic group, $C_n$, which is a subgroup of $S_n$. This group of rotations can be generated by rotation

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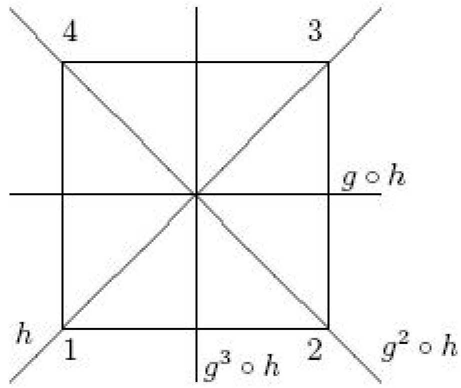


Figure 1. Symmetries of a Square [1]

by $\frac{2\pi}{n}$ since all other rotations are a composition of a certain number of copies of this rotation. Including one more element that is a reflection across a single line of symmetry of the $n$-gon expands the group into what is referred to as the dihedral group, $D_n$, which is the group of all symmetries of a regular $n$-gon. The reflection element is its own inverse, and any symmetry of a polygon can be expressed as a composition of rotations and the reflection (if needed). When considering the square (see Figure 1), if $g$, $g^2$, and $g^3$ are the three rotation elements and $h$ is the reflection across a diagonal, it is clear that these generate all symmetries of the square.

**Definition 1.2.** The group $SO_n$ is defined to be the set of $n \times n$ orthogonal matrices with determinant 1,

$$SO_n = \{A \in M_n(\mathbb{R}) \mid AA^T = A^T A = I, \det A = 1\}.$$

These matrices in two dimensions are all matrices of the form

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

and correspond to rotations of the plane. The subgroup of $SO_2$ generated by $\theta = \frac{2\pi}{n}$ geometrically describes the orientation-preserving symmetries of an $n$-gon, or the cyclic group on $n$ vertices, $C_n$.

## 2. Conceptual 3-D Group

Having thought about the rotations possible in two dimensions, the next step is to attempt to visualize the rotations possible in three dimensions. The group $SO_3$ helps in doing so because it is the group of orientation-preserving symmetries of $\mathbb{R}^3$ fixing the origin. The matrices consist of orthonormal bases that follow the right-hand rule, since $AA^T = I$ forces the inner product of the $i$-th and $j$-th columns to be $\delta_{ij}$ and $\det A = 1$ implies that the ordered column vectors are positively oriented.

**Definition 2.1.** A *pole* $p$ of a nontrivial symmetry $g \in SO_3$ is a point on the unit sphere $S^2 \subset \mathbb{R}^3$ left unchanged by $g$. That is, $gp = p$.

**Proposition 2.2.** *Any element $A$ of $SO_3$ fixes a line (or equivalently, two poles). Furthermore, $A$ is a rotation about the fixed line.*

*Proof.* Let $A \in SO_3$ with three eigenvalues $\in \mathbb{C}$. Note that the product of the eigenvalues is equal to $\det(A)$, which equals 1. Given eigenvector $w$ of $A$ with the corresponding eigenvalue $\lambda$,

$$w = A^T A w = A^T(Aw) = \lambda(A^T w). \tag{2.3}$$

This means that $w$ is an eigenvector of $A^T$ with an eigenvalue of $1/\lambda$, but $A^T$ has the same eigenvalues as $A$, because both matrices have the same characteristic polynomial. If $\lambda \neq 1/\lambda$, then the third eigenvalue must equal 1, since the product of all three is 1. Otherwise, $\lambda^2 = 1$. If $\lambda = -1$, then the other two eigenvalues cannot also be -1, since the product of all three must be 1. So, there is another eigenvalue $\mu$ with a corresponding eigenvector $u$. Following the same argument as in (2.3) with $u$ we have that $\mu^2 = 1$. Therefore, $A$ must have an eigenvalue equal to 1, so there exists an eigenvector $v$ such that

$$Av = v.$$

In other words, $A$ fixes the line along the vector $v$.

If the vector $\frac{v}{\|v\|}$ is defined to be one of the basis vectors, and the other two vectors are orthonormal to this vector and follow the right-hand rule, then $A$ can be written in terms of this basis by conjugating by the change of basis matrix, resulting in a new matrix, $B$. Since $A \in SO_3$ and any matrix changing one orthonormal basis to another maintaining positive orientation is also in $SO_3$, matrix $B \in SO_3$. The matrix $B$ is known to send the first basis vector $\frac{v}{\|v\|}$ (or $(1,0,0)$ in the new basis) to itself, fixing $B$'s first column to be $(1,0,0)$. Because $B \in SO_3$, the remaining entries can be determined based on the necessity for linearly independent, orthonormal column vectors and a determinant of 1, resulting in the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(x) & -\sin(x) \\ 0 & \sin(x) & \cos(x) \end{pmatrix}.$$

This matrix is a rotation about the line spanned by $(1,0,0)$. □

Now that we know that orientation-preserving symmetries in three dimensions are rotations fixing an axis, it is clear to see that if a rotation is a symmetry of a polyhedron, this axis is forced by symmetry to go through two opposing vertices, the centers of two opposing faces, or the midpoints of two opposing edges. In a regular polyhedron, any two vertices are symmetrically equivalent by way of some rotation in $SO_3$, as are any two edges or faces. This leads to two important definitions. Let $G$ be a group acting on a set $X$.

**Definition 2.4.** The *stabilizer* of $x \in X$ is defined to be the set of all rotations $g \in G$ that fix $x$,

$$\mathrm{Stab}(x) = \{g \in G \mid gx = x\}.$$

For example, if $X$ is the set of vertices of a polyhedron, then a stabilizer set can be visualized by imagining holding onto one vertex $x$ and rotating the rest.

**Definition 2.5.** The *orbit* of $x \in X$ is defined to be the set of all elements of $X$ that can be reached by way of a rotation $g \in G$,

$$\mathrm{Orb}(x) = \{gx \mid g \in G\}.$$

If we let $X$ be the set of edges of a regular two-dimensional polygon and $G$ its group of symmetries, then the orbit of any edge is the set of all of the edges, since any edge can be reached by some rotation in the group $SO_2$.

One can imagine that there is a relationship among the order of the orbit of $x$, stabilizer of $x$, and group $G$ that is somewhat intuitive.

**Proposition 2.6** ([1], Theorem 2.1). *For any $x \in X$, we have*

$$|G| = |\mathrm{Stab}(x)||\mathrm{Orb}(x)|. \tag{2.7}$$

*Proof.* The stabilizer of an element $x$ is a subgroup of $G$, and it gives a partition of $G$ into its cosets. If the set $H$ of left cosets of $\mathrm{Stab}(x)$ is taken, there is a bijection between this set and $\mathrm{Orb}(x)$, namely if $h \in H$, $h\mathrm{Stab}(x) \mapsto hx$. We know that $\mathrm{Orb}(x)$ consists of elements $gx$ and our map carries $g\mathrm{Stab}(x)$ to $gx$, proving our map is surjective. Now if $a\mathrm{Stab}(x)$ and $b\mathrm{Stab}(x)$ have the same image, i.e. $ax = bx$, then $x = a^{-1}bx$. This implies that $a^{-1}b \in \mathrm{Stab}(x)$, so $b = as$, $s \in \mathrm{Stab}(x)$, and $b\mathrm{Stab}(x) = a\mathrm{Stab}(x)$, proving our map is injective, and altogether bijective, leading to Proposition 2.6. □

## 3. The Theorem

**Theorem 3.1** ([2], Theorem 9.1). *Every finite subgroup $G$ of $SO_3$ is one of the following:*

- *$C_k$: The cyclic group of rotations by multiples of $\frac{2\pi}{k}$ about a line*
- *$D_k$: The dihedral group of symmetries of a regular $k$-gon*
- *$T$: The tetrahedral group of twelve rotations carrying a regular tetrahedron to itself*
- *$O$: The octahedral group of order 24 of rotations of a cube or of a regular octahedron*
- *$I$: The icosahedral group of 60 rotations of a regular dodecahedron or of a regular icosahedron.*

*Proof.* Let $P$ denote the set of all poles of a finite subgroup $G$ of $SO_3$ with order $n$.

**Lemma 3.2.** *The set $P$ is carried to itself by the action of $G$ on the sphere, i.e. $G$ operates on $P$.*

*Proof.* Let $p$ be a pole of some $g \in G$ and $h$ be a different element of $G$. We want to show that $hp$ is also a pole of some element $g'$ of $G$ that is not the identity. If the element $hgh^{-1}$ is taken, $hgh^{-1}(hp) = hgp = hp$, leaving $hp$ unchanged. Note that $hgh^{-1} \neq 1$ because $g \neq 1$. □

Since $p$ is a pole, there is an element in its stabilizer $\mathrm{Stab}(p)$ that is not the identity. By Proposition 2.2, every element of $\mathrm{Stab}(p)$ fixes the line through $p$ and the origin; in fact, we proved that it is rotation about this line. Thus $\mathrm{Stab}(p)$ fixes the plane normal to this line. Geometrically, we see that an element of $\mathrm{Stab}(p)$ rotates the plane and is determined by its action on the plane. Hence we can consider $\mathrm{Stab}(p)$ as a finite subgroup of $SO_2(\mathbb{R})$. A geometric argument shows that this is a cyclic group. This means there is some smallest angle $\theta$ that is the generator of the stabilizer set. If the order of the stabilizer is $r_p$, the smallest angle of rotation would be $\frac{2\pi}{r_p}$, with $r_p > 1$. If we define $o_p$ to be the number of different poles in the orbit $\mathrm{Orb}(p)$, then by Proposition 2.6 we get that

$$r_p o_p = n. \tag{3.3}$$

Since the identity is in the stabilizer of $p$, there are $r_p - 1$ elements of the group that have $p$ as a pole. Summing all of the elements $g \in G$ that have $p$ as a pole for all $p \in P$ gives us the total number of distinct $(g, p)$ pairings. At the same time, by Proposition 2.2, every group element $g \in G$ has two poles except for the identity, making the total number of distinct $(g, p)$ pairings $2n - 2$. Using these two facts, we get the equation

$$\sum_{p \in P} (r_p - 1) = 2n - 2. \tag{3.4}$$

If two poles $p$ and $p'$ are in the same orbit, then $o_p = o_{p'}$, so $r_p = r_{p'}$ by (3.3). Because of this, we can collect the terms of the sum in (3.4) that belong to the poles of a particular orbit $\mathrm{Orb}(p)$. There are $o_p$ terms on the left equal to $r_p - 1$, so numbering the orbits as $O_1, O_2, \ldots$ gives us

$$\sum_i o_i(r_i - 1) = 2n - 2. \tag{3.5}$$

As defined above, $o_i$ is the order of $O_i$ and $r_i$ is the order of $\mathrm{Stab}(p)$ for all $p \in O_i$. Since $n = r_i o_i$ by (3.3), both sides of (3.5) can be divided by $n$ to get

$$2 - \frac{2}{n} = \sum_i \left(1 - \frac{1}{r_i}\right). \tag{3.6}$$

Since $n$ is positive, the left side is less than 2, while each term of the sum in the right side is at least $\frac{1}{2}$ since there are at least two elements in any stabilizer set ($r_i \geq 2$ for all $i$). This leaves us with at most three orbits, which reduces the proof to the cases of one orbit, two orbits, or three orbits.

*Case 1: One Orbit.* If there is one orbit, then $i = 1$ and we get the equation

$$2 - \frac{2}{n} = 1 - \frac{1}{r}. \tag{3.7}$$

Since we know that $n > 1$, the left side of (3.7) must be greater than or equal to one. We also know that $r > 1$, meaning that the right side of (3.7) is strictly less than one. This means that (3.7) is impossible.

*Case 2: Two Orbits.* In this case, we have

$$\begin{aligned} 2 - \frac{2}{n} &= \left(1 - \frac{1}{r_1}\right) + \left(1 - \frac{1}{r_2}\right) \\ \frac{2}{n} &= \frac{1}{r_1} + \frac{1}{r_2}. \end{aligned} \tag{3.8}$$

The order $r_i$ of each stabilizer set must be less than or equal to $n$ since they each divide $n$. Therefore, equation (3.8) can only hold if $r_1 = r_2 = n$. By (3.3) we know that $o_1 = o_2 = 1$, meaning that there are two poles $p$ and $p'$, both of which are fixed by every element $g \in G$. This clearly makes $G$ the cyclic group $C_n$ of rotations about the line through $p$ and $p'$.

*Case 3: Three Orbits.* If there are three orbits, there are multiple possibilities for the values $r_i$. The equation yielded from (3.6) is

$$\frac{2}{n} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - 1. \tag{3.9}$$

If the $r_i$ are assumed to be in order from smallest to largest then we must have $r_1 = 2$, since otherwise the right side of (3.9) would not be positive.

If $r_1 = r_2 = 2$, then $r_3$ can be arbitrary and $n = 2r_3$. This implies that $n_3 = 2$, meaning there are two poles $p, p'$ that form the orbit $O_3$ and are either fixed or swapped by every group element $g \in G$. This makes the elements of $G$ either rotations around the line $l$ going through the two poles or a rotation by $\pi$ about a line perpendicular to $l$ (which looks like a reflection in the plane between $p$ and $p'$). This describes the dihedral group $D_r$ of a regular $r$-gon, with the vertices and centers of faces of the polygon corresponding to the other poles not equal to $p$ or $p'$ in the set $P$ of poles.

If only $r_1 = 2$, then the values of $r_2$ and $r_3$ are restricted by the fact that the right side of (3.9) must be positive. Simple algebra tells us that there are only three possible triples $(r_1, r_2, r_3)$ with $r_1 = 2$:

- $(r_1, r_2, r_3) = (2, 3, 3), n = 12$,
- $(r_1, r_2, r_3) = (2, 3, 4), n = 24$, or
- $(r_1, r_2, r_3) = (2, 3, 5), n = 60$.

The first case has $(o_1, o_2, o_3) = (6, 4, 4)$ with $G$ being the group of rotations of a tetrahedron: $G = T$. To help see this, let $p$ be one of the four poles in $O_3$ and let $q$ be one of the poles of $O_2$ closest to $p$. Given that the stabilizer of $p$ is of order three and operates on $O_2$, the number of nearest poles to $p$ obtained by the stabilizer acting on $q$ must be a multiple of three. Since there are only six poles in $O_2$ total, if the number of nearest poles to $p$ were six, then the nearest poles to all four poles in $O_3$ would be the same six poles, which is not possible. Therefore, the number of nearest poles to $p$ must be three. This leads to the conclusion that the poles in $O_3$ are the centers of faces of a polyhedron, with the poles of $O_2$ corresponding to the vertices and poles of $O_1$ being those through the centers of the edges. In this case, $o_1$ is the number of edges, while $o_2$ and $o_3$ are the number of vertices and faces of a tetrahedron.

In the second case, $(o_1, o_2, o_3) = (12, 8, 6)$. Using the same argument as used in the previous case, let $p$ be one of the six poles of $O_3$ and let $q$ be one of the poles in $O_2$ closest to $p$. Given that the stabilizer of $p$ is of order four and operates on $O_2$, the number of nearest poles to $p$ obtained by the stabilizer acting on $q$ must be a multiple of four. Since there are only eight poles in $O_2$ the same logic applied previously leads to the conclusion that there are four poles closest to $p$. If the poles of $O_3$ are considered to be through the centers of the faces of a polyhedron, there are six faces with eight vertices, describing a cube. If the poles of $O_3$ are considered to be through the vertices of a polyhedron, there are six vertices with eight faces, describing an octahedron. The group of rotations of these two polyhedra is $G = O$.

In the third case, $(o_1, o_2, o_3) = (30, 20, 12)$. As before, let $p$ be one of the twelve poles of $O_3$ and let $q$ be one of the poles in $O_2$ closest to $p$. Given that the stabilizer of $p$ is of order five and operates on $O_2$, the number of nearest poles to $p$ obtained by the stabilizer acting on $q$ must be a multiple of five. If there were ten nearest poles to $p$, then the twelve poles of $O_3$ would each have ten nearest poles, requiring impossible amounts of overlap to yield only the twenty total poles of $O_2$. This

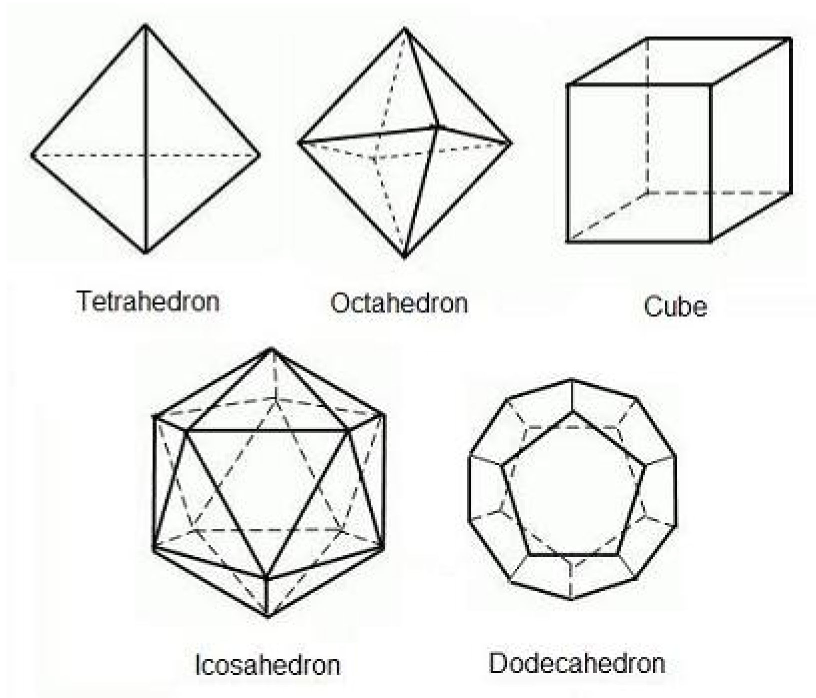


FIGURE 2. Five Polyhedra Formed by Subgroups of $SO_3$

leads to the conclusion that there are only five poles closest to $p$. If the poles of $O_3$ are considered to be through the centers of the faces of a polyhedron, there are twelve faces with twenty vertices, describing a dodecahedron. If the poles of $O_3$ are considered to be through the vertices of a polyhedron, there are twelve vertices with twenty faces, describing an icosahedron. The group of rotations of these two polyhedra is $G = I$. $\square$

## REFERENCES

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