AN EXPLORATION OF THE METRIZABILITY OF TOPOLOGICAL SPACES

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Abstract. A study of the conditions under which a topological space is metrizable, concluding with a proof of the Nagata Smirnov Metrization Theorem

Contents

1. Introduction

In this paper we will be exploring a basic topological notion known as metrizability, or whether or not a given topology can be understood through a distance function. We first give the reader some basic definitions. As an outline, we will be using these notions to first prove Urysohn's lemma, which we then use to prove Urysohn's metrization theorem, and we culminate by proving the Nagata Smirnov Metrization Theorem.

Definition 1.1. Let X be a topological space. The collection of subsets $\mathcal{B} \subset X$ forms a *basis* for X if for any open $U \subset X$ can be written as the union of elements of $\mathcal{B}$

Definition 1.2. Let X be a set. Let $\mathcal{B} \subset X$ be a collection of subsets of X. The topology generated by $\mathcal{B}$ is the intersection of all topologies on X containing $\mathcal{B}$.

Definition 1.3. Let X and Y be topological spaces. A map $f : X \to Y$. If f is a homeomorphism if it is a continuous bijection with a continuous inverse.

If there is a homeomorphism from X to Y, we say that X and Y are *homeomorphic*. Moreover, because continuity is a strictly topological property, a homeomorphism between two spaces guarantees that the spaces are indistinguishable topologically.

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Definition 1.4. Let $(X, \mathcal{T})$ be a topological space. Let $d : X \times X \to \mathbb{R}$ be a metric. If the topology generated by d is $\mathcal{T}$, then we say $\mathcal{T}$ is *metrizable*.

2. Product Topology

In this chapter we introduce a topology that we will use later in the paper.

Definition 2.1. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces. We define the *Product Topology* on the cartesian product $\Pi_{\alpha \in J} X_\alpha$ to be the minimal topology such that the projection maps $\pi_\beta : \Pi_{\alpha \in J} X_\alpha \to X_\beta$ given by $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$, are continuous.

Now we prove the exact form open sets take in this topology

Proposition 2.2. *Let $(X_\alpha)_{\alpha \in J}$ be topological spaces. Then in the product space $\Pi_{\alpha \in J} X_\alpha$ under the product topology all open sets are of the form $\Pi_{\alpha \in J} U_\alpha$ where U_α open in X_α, and $U_\alpha = X_\alpha$ for all but finitely many values of α*

Proof. Let $\Pi_{\alpha \in J} X_\alpha$ have the product topology. This tells us the projection maps $\pi_\beta : \Pi_{\alpha \in J} X_\alpha \to X_\beta$ are continuous for all $\beta \in J$. Take the topology generated by $S = \{f_\beta^{-1}(U_\beta) \mid \beta \in J,\ U_\beta \text{ open in } X_\beta\}$. This must be the minimum topology on $\Pi_{\alpha \in J} X_\alpha$, for if it weren't then at least one $f_\beta^{-1}(U_\beta)$ would not be open in the product space, which is a contradiction.
A basis for the topology generated by S is the set of all finite intersections of elements of S. Consider

$$f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap f_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n}) = \prod_{\alpha \in J} U_\alpha$$

Where $U_\alpha \neq X_\alpha$ for all indices in J except possibly for $\alpha_1, \alpha_2, \ldots, \alpha_n$. □

We now end the section with a proof of a theorem that we will be using in our proof of *Urysohn's Metrization Theorem.*

Lemma 2.3. *Let $f : A \to \Pi_{\alpha \in J} X_\alpha$, with the product topology. Then the function f is continuous iff each of its coordinate functions f_β is continuous*

Proof. We first prove the easier direction. Assume $f : A \to \Pi_{\alpha \in J} X_\alpha$ is continuous. We show that $\pi_\beta : \Pi_{\alpha \in J} X_\alpha \to X_\beta$ is continuous
Let $U_\beta \subset X_\beta$ such that U is open. Then $\pi_\beta^{-1}(U_\beta) = \Pi_{\alpha \in J} U_\alpha$ where $U_\alpha = X_\alpha$ except when $\alpha = \beta$. This is an open set in the product topology by prop. 2.2, and so π_β is continuous. Notice that $f_\beta = \pi_\beta \circ f$, so f_β is continuous as it is the composition of continuous functions.

Now for the other direction. Let each of the coordinate functions f_α be continuous. We will prove the continuity of f by showing the preimage of an open set in $\Pi_{\alpha \in J} X_\alpha$ to be open in A. Let $U \subset \Pi_{\alpha \in J} X_\alpha$ be an open set in the product topology. By prop. 2.2 we know $U = \Pi_{\alpha \in J} U_\alpha$, where $U_\alpha \neq X_\alpha$ for only finitely many $\alpha \in J$. Label such α's $\alpha_1, \alpha_2, \dots, \alpha_n$. We then have:

$$\begin{aligned} f^{-1}(\Pi_{\alpha \in J} U_\alpha) &= f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})) \\ &= f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \cdots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n})) \\ &= f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n}) \end{aligned}$$

Where the last line must be open as it is the finite intersection of open sets in $\Pi_{\alpha \in J} X_\alpha$ □

3. More Product Topology and $\mathbb{R}^\omega$

In this chapter we build on our knowledge of metrizability. We first show that not all topologies are metrizable.

Example 3.1. Let X be a non-empty set with the indiscrete topology with at least 2 elements. Let d be any metric over X. We will show that d can not generate the indiscrete topology. Let $x, y \in X$. Since d is a metric, let $m = \frac{1}{2} \cdot d(x, y)$. Then $B_m(x)$ is an open set in the metric topology, but $x \in B_m(x)$, so $B_m(x) \neq \emptyset$ and $y \notin B_m(x)$, so $B_m(x) \neq X$. Therefore, $B_m(x)$ is open in the metric topology, but it is not open in the indiscrete topology,and so any topological space X containing at least two points is not metrizable under the indiscrete topology.

Next we show the space $\mathbb{R}^\omega$ is metrizable, which we will be using later in our proof of theorem 5.1.

Definition 3.2. Let d be the usual metric over $\mathbb{R}$. Define the *standard bounded metric* on $\mathbb{R}$ to be $\overline{d}(a, b) = \min\{|a - b|, 1\}$

Theorem 3.3. *The space $\mathbb{R}^\omega = \Pi_{n \in \mathbb{N}} \mathbb{R}$, the countable product of the real line, is metrizable under the product topology*

Proof. Define a metric over $\mathbb{R}^\omega$ as follows: Let $x, y \in \mathbb{R}^\omega$, then

$$D(x, y) = \sup \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\}_{i \in \mathbb{N}}$$

This defines a metric over $\mathbb{R}^\omega$. We now show that D induces the product topology over $\mathbb{R}^\omega$. We prove this by showing that a set open in the product topology can be written as the union of balls in the metric topology, and also by showing that a set open in the topology generated by D can be written as a union of sets that are open in the product topology.

Let U be open in the metric topology on $\mathbb{R}^\omega$, and let $x \in U$. As U is open in the metric topology, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. For any $y \in \mathbb{R}^\omega$ we have that $\frac{\overline{d}(x_i, y_i)}{i} \leq \frac{1}{N}$ for all $i \geq N$, and so

$$D(x, y) \leq \max\{\overline{d}(x_1, y_1), \frac{\overline{d}(x_2, y_2)}{2}, \frac{\overline{d}(x_3, y_3)}{3}, \dots, \frac{1}{N}\}$$

Consider the set

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \dots$$

which is open in the product topology on $\mathbb{R}^\omega$ by prop 2.2. Let $y \in V$. By the argument in the previous paragraph, we know

$$D(x,y) \leq \max\{\overline{d}(x_1,y_1), \frac{\overline{d}(x_2,y_2)}{2}, \frac{\overline{d}(x_3,y_3)}{3}, \ldots, \frac{1}{N}\}$$
$$D(x,y) \leq \max\left\{\min\{x_1 - y_1, 1\}, \frac{\min\{x_2 - y_2, 1\}}{2}, \frac{\min\{x_3 - y_3, 1\}}{3}, \ldots, \frac{1}{N}\right\}$$
$$D(x,y) < \epsilon$$

As each element of the above set is strictly less than or equal to ϵ. Therefore, we have found a set V open in the product topology such that $x \in V \subset U$, where U is open in the metric topology.

Now, Let $x \in U \subset \mathbb{R}^\omega$ be open an open set in the product topology. This says $U = \Pi_{n\in N} U_i$ where $U_i = \mathbb{R}$ for all but finitely many values of i, and U_j is open in $\mathbb{R}_j$ for all $j \in J$, where J is finite. For each $j \in J$ choose ϵ_j such that $x_j \in (x_j - \epsilon_j, x_j + \epsilon_j) \subset U_j$, then let $0 < \epsilon < \min\{\epsilon_j/j \mid j \in J\}$. We claim that $x \in B_\epsilon(x) \subset U$

Indeed, let $y \in B_\epsilon(x)$. This means that $\frac{\overline{d}(x_j,y_j)}{j} < \epsilon$ for all $j \in J$. This implies $\frac{\overline{d}(x_j,y_j)}{j} < \epsilon < \min\{\epsilon_j/j \mid j \in J\}$, and so we know for all $j \in J$, $\overline{d}(x_j,y_j) < \frac{\epsilon}{j} < \epsilon$, and so $d(x_j,y_j) < \epsilon_j$, $y_j \in (x_j - \epsilon_j, x_j + \epsilon_j) \subset U_j$. Moreover, for each index k such that $k \notin J$, we know that $U_k = \mathbb{R}$, and so $y_k \in U_k$ trivially. Therefore, $B_\epsilon(x) \subset U$ □

4. Separation Axioms

We are at the point now where we are almost ready to tackle *Urysohn's Lemma*. Let us remark that in order for any of the following characteristics of topological spaces to be true, we insist that all one point sets $\{x\} \subset X$ are closed. This condition is also known as the T_1 condition.

Definition 4.1. Let $(X, \mathcal{T})$ be a topological space. We say that X is *Hausdorff* , or T_2, if given any two points, $x, y \in X$, there exist $U_x, U_y \in (X, \mathcal{T})$ such that $x \in U_x$ and $y \in U_y$ with $U_x \cap U_y = \emptyset$.

Definition 4.2. If for any singleton $x \in X$ and any closed set $B \subset X$ there exists open sets U, V such that $x \in U$ and $B \subset V$, with $U \cap V = \emptyset$, then the space X is called *regular*, or T_3.

Definition 4.3. If for any two closed sets $A, B \in X$ there exist open sets U, V such that $A \subset U$ and $B \subset V$ with $U \cap V = \emptyset$, then X is called *normal*, or T_4.

Now we prove another theorem which we will use in theorem 5.1

Theorem 4.4. *Let $(X, \mathcal{T})$ be a regular topological space. Let $x \in X$. Then for any open neighborhood U of x, there exists $U_x \in \mathcal{T}$ such that $\overline{U_x} \subset U$*

Proof. Because U is open, we know $X \setminus U$ is closed. Therefore, since X is regular, there exists open sets U_x, V such that $x \in U_x$ and $X \setminus U \subset V$, with $U_x \cap V = \emptyset$. But then $U_x \subset U$, since $U_x \cap V = \emptyset$, and so all we are left to show is that $\overline{U_x} \subset U$. To show this, it will suffice to show that $\overline{U_x} \cap V = \emptyset$, since if $\overline{U_x}$ does not intersect with V, it can't intersect with $X \setminus U$, and hence must be a subset of U.

We show that $\overline{U_x} \cap V = \emptyset$: Let $y \in V$. Then V is an open neighborhood separating y from U_x, and so $y \notin \overline{U_x}$. And so our claim is proven. Here we are using the fact that a point x is in $\overline{A}$ iff for all open intervals U containing x, U is not disjoint from A.

□

Also note that the same can be done with a normal space, replacing the singleton x with a closed set $A \subset X$. The following is another proof we will be using in later chapters.

Proposition 4.5. *Let $(X, \mathcal{T})$ be a regular topological space with a countable basis $\mathcal{B}$. Then X is normal.*

Proof. Let $A, B \subset X$ be closed. Because X is regular, for each $x \in A$ there exists $U_x \in \mathcal{T}$ such that $x \in U_x$ and $U_x \cap B = \emptyset$. As $\mathcal{B}$ is a basis of X, for each $x \in A$ there exists $A_n \in \mathcal{B}$ such that $x \in A_n \subset U_x$. Notice the collection $\{A_n\}$ forms a countable open covering of A. We repeat the same construction for the closed set B.
Notice that for both $\{A_n\}$ and $\{B_n\}$, both sets will be disjoint from B and A respectively, but need not be disjoint from each-other. We fix this problem by defining the following sets:

$$A'_n = A_n \setminus \bigcup_{i=1}^{n} \overline{B_i} \text{ and } B'_n = B_n \setminus \bigcup_{i=1}^{n} \overline{A_i}$$

And now we show that the collection of sets $\{A'_n\}_{n\in\mathbb{N}}$ and $\{B'_n\}_{n\in\mathbb{N}}$ are the open sets which separate the closed sets A and B.
It is clear that $A \subset \cup_{n\in\mathbb{N}} A'_n$ and $B \subset \cup_{n\in\mathbb{N}} B'_n$, and so we only show that sets are disjoint.
Let $x \in \cup_{n\in\mathbb{N}} A'_n$. By definition, $x \in A'_n = A_n \setminus \cup_{i=1}^{n} \overline{B_i}$ for all $n \in \mathbb{N}$. Therefore, $x \notin B_m$ for all $m \in \mathbb{N}$, and so $x \notin \cup_{n\in\mathbb{N}} \overline{B'_n}$. Hence $\{A'_n\}_{n\in\mathbb{N}}$ and $\{B'_n\}_{n\in\mathbb{N}}$ are disjoint □

5. Urysohn's Lemma

We first provide the claim.

Theorem 5.1. *(Urysohn's Lemma) Let X be a normal topological space, with A and B disjoint closed subsets of X. Then there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(y) = 1$ for all $y \in B$*

Proof. Let $Q = \{q \in \mathbb{Q} \mid q \in [0,1]\}$. As B is closed, we know $X \setminus B$ is open, and therefore $X \setminus B$ is an open neighborhood of A, as $A \cap B = \emptyset$. Let $U_1 = X \setminus B$.
Since U_1 is an open neighborhood of A, and A is closed, by Theorem 4.4 there exists some open set U_0 such that $A \subset U_0$ and $\overline{U_0} \subset U_1$. For every $p \in Q$ we wish to define an open set $A \subset U_p$ such that $\overline{U_p} \subset U_q$ for all $q > p$ in Q. We prove the collection $\{U_r\}$ with $r \in Q$, with the above condition, exists by induction.
The base case has already been proven: we know that $A \subset \overline{U_0} \subset U_1$ by our construction above.

Inductive Step: Choose a well ordering $<$ of Q. Then $Q = \{r_0 = 1, r_1, r_2, \dots, 1\}$. Suppose we have defined $U_{r_0}, U_{r_1}, \dots, U_{r_n}, U_1$ with the above property. We define $U_{r_{n+1}}$. We know r_n and 1 are the immediate predecessor and successor of r_{n+1} in $\{r_0 = 0, r_1, \dots, r_n, 1\}$ respectively. By the inductive hypothesis we have that

$\overline{U_{r_n}} \subset U_1$ where U_1 is open. Therefore, by theorem 4.4 there exists an open set $U_{r_{n+1}}$ such that $\overline{U_n} \subset \overline{U_{n+1}} \subset U_1$.

Now we have defined U_i for all $i \in Q$ by induction. We extend this to all rational numbers by letting $U_p = X$ if $p > 1$ and $U_p = \emptyset$ if $p < 0$.
Next define the function $T : X \to \mathbb{Q}$ such that $T(x) = \{p \in \mathbb{Q} \mid x \in U_p\}$. We know this function is well defined because given any $x \in X$, $x \in U_p$ for all $p > 1$. Furthermore, for any $x \in X$ we know there is no number less than 0 in $T(x)$, for $U_n = \emptyset$ for $n < 0$. Consequently, for any $x \in X$ we see that $T(x)$ is subset of the reals that is bounded below by zero, and so by the greatest lower bound property of the reals we can define:

$$f : X \to [0,1] \text{ by } f(x) = \inf T(x)$$

Now we prove that f is the function with the properties desired in the lemma.
All that is left to prove now is that $f : X \to [0,1]$ is a continuous map with $f(A) = \{1\}$ and $f(B) = \{0\}$.

Proof of Continuity of F: Let $(c,d) \subset [0,1]$ be open in the usual subspace topology on $[0,1]$. Choose some $x \in X$ such that $f(x) \in (c,d)$. Choose rational numbers j,k such that
$c < j < f(x) < k < d$. By construction, since $j < f(x) < k$, we know $x \in U_k$ and $x \notin \overline{U_j}$, so $x \in U_k \setminus \overline{U_j}$, which is an open neighborhood of x. Let $P = U_k \setminus \overline{U_j}$.
To conclude, we claim that $f(P) \subset (c,d)$.
Let $y \in P$. We have that $y \in U_k \setminus \overline{U_j}$, and hence $f(y) \in (j,k) \subset (c,d)$, and so $f(P) \subset (c,d)$. Hence, we have shown that the pre-image of an open set is open, and so our map f is continuous.
We check that f satisfies the property that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. Let $a \in A$, then by our construction we know $a \in U_0$ and so $f(a) = 0$. Let $b \in B$, then $b \notin U_p$ for any rational $p \in [0,1)$, therefore $f(b) = 1$.

□

With Urysohn's lemma, we now want to prove a theorem regarding the metrizability of topological space. The idea of this proof is to construct a sequence of functions using Urysohn's lemma, then use these functions as component functions to embed our topological space in the metrizable space $\mathbb{R}^\omega$

6. Urysohn's Metrization Theorem

Theorem 6.1. *(Urysohn's Metrization Theorem) Let $(X,\mathcal{T})$ be a regular topological space with a countable basis $\mathcal{B}$, then X is metrizable.*

Proof. Let $(X,\mathcal{T})$ be a regular metrizable space with countable basis $\mathcal{B}$. Idea: We will first create a countable collection of functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_m : X \to \mathbb{R}$ for all $m \in \mathbb{N}$, such that given any $x \in X$ and any open neighborhood U of x there is an index N such that $f_N(x) > 0$ and zero outside of U. Then we will use these functions to imbed X in $\mathbb{R}^\omega$.

Let $x \in X$ and let U be any open neighborhood of x. There exists $B_m \in \mathcal{B}$ such that $x \in B_m$. As X has a countable basis and is regular, by theorem 4.5 we know X is normal. Next, as B_m is open, we know by theorem 6.5 there exists some $B_n \in \mathcal{B}$ such that $\overline{B_n} \subset B_m$. Thus we now have two closed sets $\overline{B_n}$ and $X \setminus B_m$, and so we can apply Urysohn's lemma to give us a continuous function $g_{n,m} : X \to \mathbb{R}$ such that $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X \setminus B_m) = \{0\}$. Notice that this function satisfies our requirement: $g_{n,m}(y) = 0$ for $y \in X \setminus B_m$, and $g_{n,m}(x) > 0$. Notice we indexed g purposely, as it shows us that $\{g_{n,m}\}$ is indexed by $\mathbb{N} \times \mathbb{N}$, which is countable as the cross product of two countable sets is countable. With this in mind, we relable our functions $\{g_{n,m}\}_{n,m \in \mathbb{N}}$ as $\{f_n\}_{n \in \mathbb{N}}$.

We now imbed X in the metrizable space $\mathbb{R}^\omega$, (if you don't remember the metric used on $\mathbb{R}^\omega$, see theorem 3.4).
Let $F : X \to \mathbb{R}^\omega$ such that $F(x) = (f_1(x), f_2(x), f_3(x), \dots)$, where f_n are the functions constructed above. We claim that F is an imbedding of X into $\mathbb{R}^\omega$.
For F to be an imbedding we need F to be homeomorphic onto its image. First, this requires that F is a continuous bijection onto its image. We know that F is continuous by theorem 4.5, as each of its component functions f_N are continuous by construction. Now we show that F is an injection.
Let $x, y \in X$ be distinct. By the Hausdorff condition there exists open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ with $U_x \cap U_y = \emptyset$. By the construction of our maps f there exists an index $N \in \mathbb{N}$ such that $f_N(U_x) > 0$ and $F_N(X \setminus U_x) = 0$. It follows that $f_N(x) \neq f_N(y)$, and so $F(x) \neq F(y)$. Therefore, F is injective.
It is clear that F is surjective onto its image $F(X)$, and so all that is left to show is that F is an embedding is to show that for any open set $U \in X$, $F(U)$ is open in $\mathbb{R}^\omega$.
Let $U \subset X$ be open. Let $x \in U$. Choose an index N such that $f_N(x) > 0$ and $f_N(X \setminus U) = 0$. Let $F(x) = z \in F(U)$. Let $V = \pi_N^{-1}((0, \infty))$, which is simply all elements of $\mathbb{R}^\omega$ with a positive $N'th$ coordinate. Now, let $W = F(X) \cap V$. We claim that $z \in W \subset F(U)$ showing that $F(U)$ can be written as a union of open sets, hence making it open.
We first show that W is open in $F(X)$. We know that V is an open set in $\mathbb{R}^\omega$. $W = F(X) \cap V$, W is open by the definition of the subspace topology.
Next, we will first show that $f(x) = z \in W$, and then that $W \subset F(U)$. To prove our first claim, $F(x) = z$ and so $\pi_N(z) = \pi_N(F(x)) = f_N(x) > 0$, and so $\pi_N(z) > 0$ which means that $z \in \pi_N^{-1}(V)$ and of course $z \in F(x)$, and so $z \in F(X) \cap V = W$.
Now we show that $W \subset F(U)$. Let $y \in W$. This means $y \in F(X) \cap V$. This means there exists some $w \in X$ such that $F(w) = y$. But, since $y \in V$ we have that: $\pi_N(y) = \pi_N(F(w)) = f_N(w) > 0$ since $y \in V$, but $f_N(w) = 0$ for all $w \in X \setminus U$, and so $y \in F(U)$.
In conclusion, as we have shown that $F : X \to \mathbb{R}^\omega$ is a map that preserves open sets in both directions and bijective onto its image, we have shown that F is an embedding of the space X into the metrizable space $\mathbb{R}^\omega$, and X is therefore metrizable, the metric being given by the induced metric from $\mathbb{R}^\omega$. □

Example 6.2. The topology generated by the dictionary ordering on $\mathbb{R}^2$ is metrizable

Proof. By the previous theorem, all we must do to show $\mathbb{R}^2$ is metrizable in the dictionary ordering is to show that this space is regular with a countable basis.
Because the set $\{(a, b), (c, d) \mid a \leq c, b < d \; a, b, c, d \in \mathbb{R}\}$ is a basis for the dictionary ordering on $\mathbb{R}^2$, and the set of intervals with rational end-points are a basis for the

usual topology on $\mathbb{R}$, it follows that the set $\{(a,b),(c,d) \mid a \leq c, b < d\ a,b,c,d \in \mathbb{Q}\}$ is a countable basis for the dictionary ordering.
Now we show the dictionary ordering is regular. Let $a \in \mathbb{R}^2$ and $B \subseteq \mathbb{R}^2$ such that B is closed in the dictionary ordering and $a \notin B$. Let $\epsilon = \inf\{d(a,b) \mid b \in B\}$. We know $\epsilon > 0$, for if it weren't, then a would be an accumulation point of B, which is a contradiction. It follows that the open sets $((a, a-\epsilon/2),(a, a+\epsilon/2))$ and $\bigcup_{b\in B}((b, b-\epsilon/2),(b, b+\epsilon/2))$ are disjoint open sets containing a and B respectively. Therefore, the dictionary ordering over $\mathbb{R}^2$ is metrizable, as it is regular and has a countable basis

□

Remark 6.3. Recall that in this proof we proved that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ with the property that for each $x \in X$, and each neighborhood U of x, there is some $n \in \mathbb{N}$ such that $f_n(x) > 0$ and $f_n(y) = 0$ for all $y \in X \setminus U$, gives us an imbedding $F : X \to \mathbb{R}^\omega$. Notice we have the very similar result if we have a sequence of functions $\{f_j\}_{j\in J}$ with the same properties as above; given any $x \in X$ and any neighborhood U of x there exists $j \in J$ such that $f_j(x) > 0$ and $f_j(y) = 0$ for all $y \in X \setminus U$, then we have an imbedding from $X \to \mathbb{R}^J$ given by $F(x) = (f_j(x))_{j\in J}$. This is known as the *Imbedding Theorem*, and it a generalization of Urysohn's Metrization Theorem

7. Local Finiteness and G_δ sets

We have now proven our first substantial result, i.e that any regular space with a countable basis is metrizable. However, there are weaker conditions under which a given topological space is metrizable. In this chapter we will introduce the relevant notions and prove theorems we will need in the next chapter to prove theorem 8.1. We now present the definitions.

Definition 7.1. Let X be a topological space. We say a collection of subsets $\mathcal{A}$ of X is locally finite if for all $x \in X$ there is a neighborhood of U_x of x such that U_x intersects only finitely many $A \in \mathcal{A}$.

Definition 7.2. Let X be a topological space. If $A \subset X$ can be written as the countable intersection of open sets, then we say A is a G_δ set.

Definition 7.3. Let X be a topological space with $C \subset 2^X$. If $C = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite, then we say C is *countably locally finite.*

Definition 7.4. Let X be a set. Let $\mathcal{A}, \mathcal{B} \subset 2^X$. We call $\mathcal{B}$ a *refinement* of $\mathcal{A}$ if for every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $B \subset A$

Let us look at an example

Example 7.5. Let X be a metric space. Any open subset U in X is locally finite, as U is already a finite collection of open sets. However, on a more interesting note, any closed $A \subset X$ is a G_δ set.

Lemma 7.6. *Let $\mathcal{A}$ be locally finite in the topological space X. Then the collection $\{\overline{A}\}_{A\in\mathcal{A}}$ is locally finite, and $\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \overline{A}$.*

Proof. We prove the first claim. Let $\mathcal{A}$ be locally finite in the topological space X. Let $x \in X$, and assume for contradiction that all neighborhoods U of x intersect infinitely many elements of $\{\overline{A}\}_{A\in\mathcal{A}}$. However, we know if $U \cap \overline{A} \neq \emptyset$, then

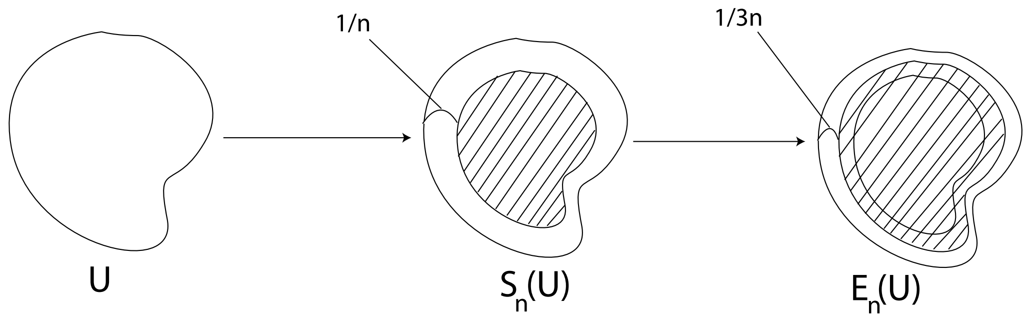

FIGURE 1. An example of $S_n(U)$ and $E_n(U)$ for some $U \in \mathcal{A}$

$x \cap A \neq \emptyset$, and so if U intersects with infinitely many elements of $\{\overline{A}\}_{A\in\mathcal{A}}$ then likewise it must intersect with infinitely many $A \in \mathcal{A}$, which contradicts that $\mathcal{A}$ is locally finite

Now we prove the second claim: Let $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$. Let U be an open neighborhood of x such that U intersects with $A_1, A_2, \ldots, A_n \in \mathcal{A}$. Assume $x \notin \overline{A_k}$ for $1 \leq k \leq n$. This tells us the set $U \setminus \bigcup_{i=1}^{n} \overline{A_i}$ is an open neighborhood of x that does not intersect any $A \in \mathcal{A}$, and hence does not intersect $\bigcup_{A\in\mathcal{A}} A$, which means this neighborhood does not intersect $\overline{\bigcup_{A\in\mathcal{A}} A}$, which contradicts the fact that $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$
Other Direction: Let $x \in \bigcup_{A\in\mathcal{A}} \overline{A}$, so $x \in \overline{A}$ for some $A \in \mathcal{A}$. Hence every neighborhood of x intersects with A, which means it intersects with $\bigcup_{A\in\mathcal{A}} A$, and so $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$ □

Theorem 7.7. *Let X be a metric space with an open cover $\mathcal{A}$. Then there exists an open cover $\mathcal{C}$ that refines $\mathcal{A}$ and is countably locally finite.*

Proof. Fix a metric on X. Let $U \in \mathcal{A}$ be arbitrary. Define $S_n(U) = \{x \in U \mid B_{1/n}(x) \subset U\}$. Next, by the axiom of choice we create a well ordering on the elements of $\mathcal{A}$, which we will call $<$. Define

$$T_n(U) = S_n(U) \setminus \bigcup_{V<U} V$$

.
Now we show that $T_n(U)$ is disjoint from $T_n(V)$ for $U, V \in \mathcal{A}$ distinct: Because $<$ is a total ordering over $\mathcal{A}$, either $V < U$ or $U < V$. Assume WLOG that $V < U$. Then, for $u \in T_n(U)$, by definition $T_n(U) = S_n(U) \setminus \bigcup_{V<U} V$, and as $V < U$, $u \notin V$. But, we know that for all $v \in T_n(V)$, $v \in V$, hence $T_n(U) \cap T_n(V) = \emptyset$. It is useful to put a bound on exactly how close the two sets $T_n(U)$ and $T_n(V)$ can be. Let $x \in T_n(U)$ and $y \in T_n(V)$. Once again we assume WLOG that $V < U$. Then $x \in T_n(U)$ implies $x \in S_n(U) \setminus V$. Also, $x \in S_n(U)$ means $B_{1/n}(x) \subset U$. So $d(x, y) \geq 1/n$, as $y \in V$.

Now define $E_n(U) = \{B_{1/3n}(x) \mid x \in T_n(U)\}$. Once again, we try to find a bound on how close together the sets $E_n(U)$ and $E_n(V)$ can be. Using the triangle inequality combined with the construction of each E_n, we see that the distance

between $E_n(U)$ and $E_n(V)$ is at least $1/3n$ for distinct $U, V \in \mathcal{A}$.
Next define $\mathcal{C}_n = \{E_n(U) \mid U \in \mathcal{A}\}$. We first note that $\mathcal{C}_n$ is a refinement of $\mathcal{A}$, as $E_n(U) \subset U$ for all $U \in \mathcal{A}$. Also note that as E_n is a union of open balls, each E_n is open. $\mathcal{C}_n$ is locally finite, because the distance between $E_n(V)$ and $E_n(U)$ is at least $1/3n$ for distinct U and V, $B_{1/6n}(x)$ intersects only one $E_n(U) \in \mathcal{C}_n$.

We now let $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$, and we claim that $\mathcal{C}$ is the open refinement of $\mathcal{A}$ that is countably locally finite. As we have already shown that each $\mathcal{C}_n$ is locally finite, we have shown that $\mathcal{C}$ is countably locally finite. All that is left to show is that $\mathcal{C}$ is a cover of X.
Showing $\mathcal{C}$ is a Cover of X: Let $x \in X$. As $\mathcal{A}$ is an open cover of X, there exists $U \in X$ such that $x \in U$, which tells us $S = \{U \in \mathcal{A} \mid x \in U\}$ is non-empty. As $<$ is a well ordering on $\mathcal{A}$, let V denote the least element of S in the well ordering $<$.Choose $n \in \mathbb{N}$ such that $B_{1/n}(x) \subset V$. This tells us $x \in S_n(V)$. As V is the first element of $\mathcal{A}$ containing x we have that $x \in T_n(U) \subset E_n(U) \in \mathcal{C}_n \in \mathcal{C}$. Therefore, we have shown that $\mathcal{C}$ is a countably locally finite refinement of the open cover $\mathcal{A}$ □

Now we prove a relevant theorem

Theorem 7.8. *Let X be a regular topological space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and all closed subsets of X are G_δ sets.*

Proof. Let X be a regular topological space with a countably locally finite basis $\mathcal{B}$. We will first show that for any open set T in X there is a countable collection $\{U_n\}$ of open sets such that $T = \bigcup U_n = \bigcup \overline{U_n}$, which we will then use to show closed sets are G_α sets, and then use to help us prove the normality condition on X.

Subclaim: If $T \subset X$ is open, then $T = \bigcup U_n = \bigcup \overline{U_n}$
Let $T \subset X$ be open. By hypothesis, we know the basis for X, $\mathcal{B}$, is countably locally finite, and hence $\mathcal{B} = \bigcup_{n\in\mathbb{N}} B_n$, where each B_n is a locally finite collection of subsets of X. Define

$$\mathcal{C}_n = \{B \in B_n \mid \overline{B} \subset T\}$$

Now, let $U_n = \bigcup_{B\in\mathcal{C}_n} B$. Since U_n is a union of open sets, we know it must also be open. Also, as $\mathcal{C}_n$ is a sub-collection of B_n, we know it must also be locally finite. Moreover, as $\mathcal{C}_n$ is locally finite, we use lemma 7.6 to give us $\overline{U_n} = \bigcup_{B\in\mathcal{C}_n} \overline{B}$. Therefore as each $\overline{B} \subset \mathcal{C}_n \subset T$, we have

$$\bigcup U_n \subset \bigcup \overline{U_n} \subset T$$

We now show the other inclusion. Let $x \in T$. Using the regularity of X, by theorem 4.4 there exists a basic open set $B \in \mathcal{B}$ such that $x \in \overline{B} \subset T$. Because $\mathcal{B}$ is countably locally finite, there exists some B_n such that $x \in B_n$ with B_n locally finite. Now, we have that $x \in \overline{B} \subset T$, and by definition of $\mathcal{C}_n$, we have that $B \in \mathcal{C}_n$, and hence $x \in \bigcup_{B\in\mathcal{C}_n} B = U_n$, and therefore $x \in \bigcup U_n$, and so we have shown $\bigcup U_n = \bigcup \overline{U_n} = T$

Now we prove that all closed sets are G_δ sets in X: Let $A \in X$ be closed. Then $X \setminus A$ is open, and by the first part of this proof we have that $X \setminus A = \bigcup \overline{U_n}$ for U_n open. This tells us $A = X \setminus \bigcup \overline{U_n} = \bigcap X \setminus \overline{U_n}$ by DeMorgan's Law. And as each

$\overline{U_n}$ is closed, we have that $X \setminus \overline{U_n}$ is open, and we have written A as a countable intersection of open sets, making A a G_δ set

Proof of Normality of X: Let A, B be disjoint closed subsets of X. As A is closed, we use part 1 of this proof to give us that $X \setminus A = \bigcup U_n = \bigcup \overline{U_n}$. Because B is disjoint from A, we know that $B \subset \bigcup U_n$. We do the same steps to create the open cover $\{V_n\}_{n\in\mathbb{N}}$ of A. However, while the collection of open sets $\{U_n\}$ and $\{V_n\}$ do form an open cover of B and A respectively, they are not necessarily disjoint. Nonetheless, we perform the same trick we did in theorem 4.4 to ensure these sets are disjoint. Define

$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i} \text{ and } V_n' = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}$$

Just as in theorem 6.6, it readily follows that $\{U_n'\}_{n\in\mathbb{N}}$ and $\{V_n'\}_{n\in\mathbb{N}}$ are disjoint open sets covering B and A respectively, and so X is normal as required □

Next we prove a theorem that we use explicitly in the next section

Corollary 7.9. *Let X be a normal topological space, and let A be a closed G_δ set in X. Then there exists a continuous map $f : X \to [0,1]$ such that $f(a) = 0$ for all $a \in A$ and $f(y) > 0$ for all $y \in X \setminus A$*

Proof. Let X be a topological space with $A \subset X$ such that A is closed and G_δ. Because A is G_δ we have that $A = \cap_{n\in\mathbb{N}} U_n$ for U_n open in X. Since U_n is open, we know $X \setminus U_n$ is closed and disjoint from A, hence by Urysohn's lemma we define the continuous map $f_n : X \to [0,1]$ such that $f_n(a) = 0$ for all $a \in A$ and $f_n(x) = 1$ for all $x \in X \setminus U_n$. Now define the map $f : X \to [0,1]$ such that $f(x) = \sum_{i=1}^{\infty} f_i(x)/2^i$. For each $n \in \mathbb{N}$ we have that $0 \leq f_n(x) \leq 1$, and so $f(x) = \sum_{i=1}^{\infty} f_i(x)/2^i \leq \sum_{i=1}^{\infty} 1/2^i$, and hence $f(x) = \sum_{i=1}^{n} f_n(x)/2^n$ converges uniformly to f by the comparison test, and as each f_n is continuous, this tells us f is continuous. Now we check our hypotheses, if $x \in A$, then we have that $f_n(x) = 0$ for all $n \in \mathbb{N}$, and so $f(x)$ is simply an infinite sum of zeros, which is zero. If $x \notin A$, then $x \notin U_N$ for some $n \in \mathbb{N}$, and hence for all $n > N$, $f_n(x) > 0$, and so $f(x) > 0$ □

8. Nagata-Smirnov Metrization Theorem

Theorem 8.1. *(Nagata-Smirnov Metrization Theorem) Let X be a topological space. Then X is metrizable iff X is regular and has a countably locally finite basis $\mathcal{B}$.*

Proof. Before we begin the proof we define the uniform metric. Let J be a set. Define the uniform metric on $\Pi_{\alpha\in J}\mathbb{R}_\alpha$ as $\rho(x,y) = \sup\{|x_\alpha - y_\alpha|, 1\}_{\alpha\in J}$

We begin with the harder direction. Let X be a space with a countably locally finite basis $\mathcal{B}$. This tells us $\mathcal{B} = \bigcup_{n\in\mathbb{N}} B_n$ where each B_n is locally finite. Let B be a basis element in B_n. Define

$$f_{n,B} : X \to [0, 1/n]$$

where $f_{n,B}(y) > 0$ for all $y \in B$ and $f_{n,B}(z) = 0$ for all $z \in X \setminus B$. We can define such a function because B is an open set, and hence $X \setminus A$ is a closed set, and as

X is regular with a countably locally finite basis, it is also a G_δ set by theorem 7.8, and hence we apply corollary 7.9 to construct our function $f_{n,B}$.

Next, let K be the set consisting of the pairs $K = (n, B)$ such that B is a basis element contained in the locally finite set B_n. Construct the function

$$F : X \to [0,1]^J : F(x) = (f_{n,B}(x))_{(n,B)\in K}$$

In remark 6.2 the *Imbedding Theorem* is mentioned, and we use it here. We have a sequence of functions $\{f_{n,B}\}_{(n,B)\in K}$ with the property that for each $x \in X$ and each neighborhood U of X, there is some $(m, B) \in K$ such that $f_{m,B}(x) > 0$ and $f_{m,B}(y) = 0$ for all $y \in X \setminus U$, hence $F : X \to [0,1]^J$ as defined above is an imbedding relative to the product topology on $[0,1]^J$.

We now have an imbedding from X into $[0,1]^J$ in the product topology. Next, we give $[0,1]^J$ the uniform metric, defined above, and show that F is still an imbedding. With the uniform metric, it is clear that any set that is open in the product topology on $[0,1]^J$ will also be open in the metric topology. Therefore, as F is an imbedding relative to the product topology on $[0,1]^J$, it must take open sets in X to open sets in the product topology on $[0,1]^J$, which by our argument are also open in the uniform metric topology. Hence all we need to show to prove F is an imbedding relative to the uniform metric on $[0,1]^J$ is to show that F is continuous, as the previous sentence shows it preserves open sets in the forward direction.

Showing F is continuous relative to the uniform metric: To prove this claim, what we must show is that given $\epsilon > 0$ and any $x \in X$ there exists some neighborhood U of x such that for all $y \in U$ we have $\rho(F(x), F(y)) < \epsilon$.

Let $x \in X$ and $\epsilon > 0$ be given. Let $n \in \mathbb{N}$ be fixed. Because B_n is locally finite, there exists some neighborhood U_n of X such that U_n intersects only finitely many elements in B_n. This means there are only finitely many basis elements $B \in B_n$ such that $f_{n,B}(x) \neq 0$, for by construction we know $f_{n,B}(x) = 0$ if $x \notin B$.

Let $G_n = \{B \in B_n \mid f_{n,B}(U_n) \neq \{0\}\}$. For each $B \in G$ we know that $f_{n,B} : X \to [0, 1/n]$ is a continuous map, and so for each $B \in G$ we choose a neighborhood Q_B of x such that for all $y \in Q$ we have that $\rho(f_{n,B}(x), f_{n,B}(y)) < \epsilon/2$. Let $W_n = \bigcap_{B\in G_n} Q_B$, which is an open neighborhood of x because G_n is finite.

Inductively, we know that we have such an open interval W_n of x for each $n \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that $1/N < \epsilon/2$, and let $W = W_1 \cap W_2 \cap \cdots \cap W_N$, and we claim that W is the neighborhood that will prove F is continuous relative to the uniform metric on $[0,1]^J$. Let $y \in W$. We will show that $|f_{n,B}(x) - f_{n,B}(y)| < \epsilon$ for all $n \in N$, hence showing that $\rho(F(x), F(y)) < \epsilon$.
If $n \leq N$ then either $f_{n,B}(x) = f_{n,B}(y) = 0$ or $|f_{n,B}(x) - f_{n,B}(y)| \leq \epsilon/2$ by the definition of W.
If $n > N$ then as $f_{n,B} : X \to [0, 1/n]$ it is clear that $|f_{n,B}(x) - f_{n,B}(y)| \leq 1/n < \epsilon/2$ by choice of N.
Therefore $|f_{n,B}(x) - f_{n,B}(y)| < \epsilon$ for all $n \in N$ and all $y \in W$, and so:

$$\rho(F(x), F(y)) = \sup\{|f_{n,B}(x) - f_{n,B}(y)|\}_{(n,B)\in K} < \epsilon$$

So F is continuous, and by earlier arguments, an imbedding, thereby making X metrizable.

Other Direction: Now we show if X is metrizable, then X has a countably locally finite basis. Let X be metrizable. As X is metrizable, X must be regular. Let $\mathcal{A}_m = \{B_{1/m}(x) \mid x \in X\}$. By theorem 7.7 there exists $\mathcal{B}_m$ such that $\mathcal{B}_m$ is a countably locally finite open refinement of $\mathcal{A}_m$. Let $\mathcal{B} = \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$. Then for any each $B \in \mathcal{B}_m$, we know that the greatest the diameter of B can be, by our construction in theorem 7.7 is $2/m$. Now we show that $\mathcal{B}$ is a basis for X.
Let $x \in X$ and $\epsilon > 0$ be given. Let $m \in \mathbb{N}$ such that $1/m < \epsilon/2$. As $\mathcal{B}_m$ is an open cover of X there exists $B \in \mathcal{B}_m$ such that $x \in B$. Because $\mathcal{B}_m$ is a refinement of $\mathcal{A}_m$, we have that each $B \in \mathcal{B}_m$ is contained in some $A \in \mathcal{A}$, hence for any $B \in \mathcal{B}_m$ we know the diameter of B is at most $2 \cdot 1/m = 2/m$, because $\mathcal{A}$ consists of balls of radius $1/m$. Moreover, as $1/m < \epsilon/2$ we have that $2/m < \epsilon$ and as the diameter of $B < 2/m$, we have that $B \subset B_\epsilon(x)$

□

Example 8.2. The space $\mathbb{R}'^\omega = \Pi_{i\in\mathbb{N}}\mathbb{R}$, where $\mathbb{R}$ has the discrete topology, is metrizable under the product topology

Proof. We first see that this space does not have a countable basis, as each of its coordinates $\mathbb{R}$ are discreet. Consequently, theorem 6.1 does not apply, so we use theorem 8.1.
We first show that this space is regular. Let $x \in \mathbb{R}'^\omega$ and $B \subseteq \mathbb{R}'^\omega$ closed under the above topology with $x \notin B$. This tells us $x_n \notin B_n$ for some $n \in \mathbb{N}$. It follows that $B' = \Pi_{i\in\mathbb{N}}U_i$ where $U_i = \mathbb{R}$ except when $i = n$, in which case $U_n = B_n$, and $A = \Pi_{i\in\mathbb{N}}U_i$ where $U_i = \mathbb{R}$ except when $i = n$, where $U_n = x$, are disjoint open sets containing B and x respectively.
Now we show $\mathbb{R}'^\omega$ has a countably locally finite basis. We know the sets $A_{m_x} = \Pi_{i\in N}U_i$ with $U_i = \mathbb{R}$ for all but $i = i_1, i_1, \ldots, i_n = m$, where $U_{i_j} = \{x\}$ or $\mathbb{R}$ for $x \in \mathbb{R}$ and $1 \leq j \leq n$, form a basis for $\mathbb{R}'^\omega$. Define

$$\mathcal{F}_n = \{A_{m_x} \mid 1 \leq m \leq n\}$$

Note $\mathcal{F}_n$ is nothing more than the collection of all basis elements with all but the first n coordinates being $\mathbb{R}$. We now show that $\mathcal{F}_n$ is locally finite.
Let $x \in \mathbb{R}'^\omega$. The basic open neighborhood $x_1 \times x_2 \times \cdots \times x_n \times \mathbb{R} \times \mathbb{R} \ldots$ intersects only itself in $\mathcal{F}_n$, hence $\mathcal{F}_n$ is locally finite. It follows that $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is a basis for $\mathbb{R}'^\omega$.
Therefore, we have shown that $\mathbb{R}'^\omega$ is a regular space with a countably locally finite basis, making it metrizable by theorem 8.1.

□

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