GENERALIZED ABSTRACT NONSENSE: CATEGORY THEORY AND ADJUNCTIONS

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Abstract. This paper will move through the basics of category theory, eventually defining natural transformations and adjunctions and showing the equivalence of two dissimilar definitions of adjoint functors, in order to state the Adjoint Functor Theorem. This theorem provides a nice result allowing us to classify exactly which functors have left adjoints. We borrow heavily from Awodey's Category Theory but almost all proofs are independent solutions. Almost no background in math is necessary for understanding.

Contents

1. The Basics

Just as a group is made of a set paired with an operation and a topological space is a set with a topology, a category is made up of objects, arrows, and a rule of composition. One should note that the objects need not be a set, but I will illustrate this point later. For a formal definition:

Definition 1.1. A category, $\mathbf{C}$, consists of:

- Objects: $A, B, C, \ldots$
- Arrows: $f, g, h, \ldots$ such that every arrow, f, has a domain and codomain which are objects in $\mathbf{C}$. We can represent this as:

$$f : A \to B$$

where $A = dom(f), B = cod(f)$.
- Composition: If $f, g \in \mathbf{C}$ and $f : A \to B, g : B \to C$, then $g \circ f : A \to C$.
- Identity: For any object, $A \in \mathbf{C}$, there is an arrow:

$$1_A : A \to A.$$

These must satisfy our usual concepts of:

Date: August 15, 2008.

- Associativity:
$$h \circ (g \circ f) = (h \circ g) \circ f$$
if $cod(f) = dom(g)$ and $cod(g) = dom(h)$.
- Unit:
$$f \circ 1_A = f = 1_B \circ f$$
if $f : A \to B$.

These concepts are not unfamiliar, so let's give two quick examples of categories and move on.

Example 1.2. **Set**: The category of sets and functions.

Example 1.3. **Top**: The category of topological spaces and continuous maps.

Definition 1.4. $\mathrm{Hom}_{\mathbf{C}}(A, B)$, where A, B are objects in a category $\mathbf{C}$, is the set of morphisms, or arrows, from A to B.

Definition 1.5. If $\mathbf{C}$ is a category, then its dual category, $\mathbf{C^{op}}$, is the category with the same objects as $\mathbf{C}$ but with arrows reversed. For instance, if $f : A \to B$ is in the category $\mathbf{C}$, then $f^{\mathbf{op}} : B \to A$ would be an arrow in $\mathbf{C^{op}}$.

The next concept to cover is the idea of functors, which intuitively are like homomorphisms which we can use to map categories to categories, implying that they act on both objects and arrows and preserve composition. For a formal definition:

Definition 1.6. A functor,
$$F : \mathbf{C} \to \mathbf{D}$$
where $\mathbf{C}$ and $\mathbf{D}$ are categories, is a mapping of objects to objects and arrows to arrows with:

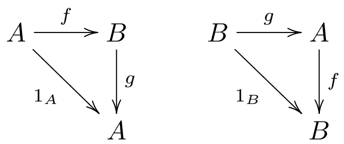
- $F(A) \in \mathbf{D}$ for $A \in \mathbf{C}$
- $F(f) : F(A) \to F(B)$ with $F(f) \in \mathbf{D}$ if $f, A, B \in \mathbf{C}$ and $f : A \to B$
- $F(g \circ f) = F(g) \circ F(f)$
- $F(1_A) = 1_{F(A)}$

Example 1.7. The forgetful functor:
Take **Top** and **Set** and define a functor, $U : \mathbf{Top} \to \mathbf{Set}$, by $U(X, \mathcal{G}) = X$. In other words, U "forgets" the topology on the space X.

2. Isomorphisms and Natural Transformations

Before we can define an adjoint functor we must define isomorphisms and natural transformations.

Definition 2.1. For a category $\mathbf{C}$, an arrow $f : A \to B$, is an isomorphism if there exists an arrow $g : B \to A$ such that:
$$f \circ g = 1_B$$
$$g \circ f = 1_A.$$
In this case we would denote this by $A \cong B$.



This concept of an isomorphism appears identical to the normal definition of a "bijective homomorphism" but does not in fact always coincide. For instance, consider the category **Top** and take the set $X = \{1, 2, 3\}$ and denote the discrete and coarse topologies as $\mathcal{D}$ and $\mathcal{C}$, respectively. Consider the function $f : (X, \mathcal{D}) \to (X, \mathcal{C})$ where $f(1) = 1, f(2) = 2, f(3) = 3$. This is clearly continuous, so it is in our category, and clearly bijective, in that it is one-to-one and onto, but its inverse, $g : (X, \mathcal{C}) \to (X, \mathcal{D})$ sending $g(1) = 1, g(2) = 2, g(3) = 3$ is not continuous because for example, $\{2\}$ is an open set in $(X, \mathcal{D})$ but $g^{-1}(\{2\}) = \{2\}$ is not an open set in $(X, \mathcal{C})$. Thus, g is not in **Top** and so f does not have an inverse and is therefore not an isomorphism.

Proposition 2.2. *Inverses are unique.*

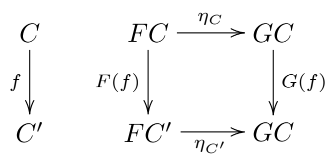
Proof. Take an arrow $f : A \to B$ and suppose there are inverses $h, g : B \to A$ such that $f \circ h = f \circ g = 1_B$ and $g \circ f = h \circ f = 1_A$. Then:

$$h \circ (f \circ g) = (h \circ f) \circ g$$
$$h \circ 1_B = 1_A \circ g$$
$$h = g$$

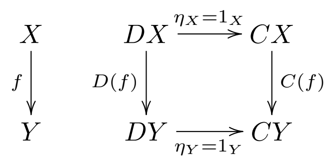
□

Thus, we can write the inverse of f as f^{-1}.

Definition 2.3. Given categories **C, D** and functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformation $\eta : F \to G$, is a family of arrows $\eta_C : FC \to GC$ for $C \in \mathbf{C}$ such that for any $f : C \to C'$, with $C, C' \in \mathbf{C}$, the following diagram commutes:



Example 2.4. Let $D : \mathbf{Set} \to \mathbf{Top}$ be the functor which applies the discrete topology (all sets are open) and let $C : \mathbf{Set} \to \mathbf{Top}$ be the functor which takes a set and applies the coarse topology (only the entire set and the empty set are open), then we see that the natural transformation $\eta : D \to C$ could be the family of arrows mapping every $x \in X \in \mathbf{Top}$ to itself, giving us, for any f:



This commutes by our requirement on the identity in Definition 1.1 (Since D, C do not change where f maps elements of X, they simply make topologies such that f

is continuous). We can see this more explicitly by taking $x \in X$ and:

$$\begin{aligned}(C(f) \circ \eta_X)(x) &= Cf(x) = f(x)\\ &= Df(x)\\ &= (\eta_Y \circ Df)(x)\end{aligned}$$

3. Adjoint Functors: Two Definitions and Their Equivalence

Definition 3.1. Given categories **C, D** and functors $F : \mathbf{C} \to \mathbf{D}, U : \mathbf{D} \to \mathbf{C}$, a family of arrows, $\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \to \mathrm{Hom}_{\mathbf{C}}(C, UD)$, is natural in C if given any $h : C' \to C$ the following diagram commutes:

$$\begin{array}{ccc} Hom_{\mathbf{D}}(FC, D) & \xrightarrow{\phi_{C,D}} & Hom_{\mathbf{C}}(C, UD) \\ {\scriptstyle (Fh)^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ Hom_{\mathbf{D}}(FC', D) & \xrightarrow[\phi_{C',D}]{} & Hom_{\mathbf{C}}(C', UD) \end{array}$$

(Where h^* is simply h composed on the right.) In other words, for any $f : FC \to D$, we have:

$$\phi_{C,D}(f) \circ h = \phi_{C',D}(f \circ Fh)$$

ϕ is said to be natural in D if given any $g : D \to D'$ the following diagram commutes:

$$\begin{array}{ccc} Hom_{\mathbf{D}}(FC, D) & \xrightarrow{\phi_{C,D}} & Hom_{\mathbf{C}}(C, UD) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (Ug)_*} \\ Hom_{\mathbf{D}}(FC, D') & \xrightarrow[\phi_{C,D'}]{} & Hom_{\mathbf{C}}(C, UD') \end{array}$$

(Where g_* is simply g composed on the left.) In other words, for any $f : FC \to D$, we have:

$$\phi_{C,D'}(g \circ f) = U(g) \circ \phi_{C,D}(f)$$

A more sophisticated definition of what it means to be natural in C or D requires the use of covariant and contravariant representable functors, a topic discussed in section five, but the definition above will suffice as we only need it to define adjoint functors.

We will now give two different definitions for adjoint functors, show in an example how they match up, and then prove their equivalence.

Definition 3.2. Hom-Set: Given categories **C, D** and functors $F : \mathbf{C} \to \mathbf{D}, U : \mathbf{D} \to \mathbf{C}$, then F is left adjoint to U iff for any $C \in \mathbf{C}, D \in \mathbf{D}$, there is an isomorphism, $\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \to \mathrm{Hom}_{\mathbf{C}}(C, UD)$ that is natural in both C and D.

Definition 3.3. Unit: Given categories **C, D** and functors $F : \mathbf{C} \to \mathbf{D}, U : \mathbf{D} \to \mathbf{C}$, then F is left adjoint to U iff there is a natural transformation $\eta : 1_{\mathbf{C}} \to U \circ F$ such that for any $C \in \mathbf{C}, D \in \mathbf{D}$ and $f : C \to U(D)$ there exists a unique $g : FC \to D$ so that the following diagram commutes:

$$\begin{array}{ccccc} FC & & U \circ F(C) & \xleftarrow{\eta_C} & C \\ {\scriptstyle g}\downarrow & & {\scriptstyle U(g)}\downarrow & \swarrow{\scriptstyle f} & \\ D & & U(D) & & \end{array}$$

In other words: $f = U(g) \circ \eta_C$. We call η the unit.

As an aside, there is a dual definition using a counit, or a natural transformation from $U \circ F$ to $1_{\mathbf{D}}$, but as its equivalence is proved similarly we will do no more than mention it.

We can write that F is left adjoint to U (or equivalently that U is right adjoint to F) as $F \dashv U$.

Example 3.4. Consider the functors, $U : \mathbf{Top} \to \mathbf{Set}$ and $F : \mathbf{Set} \to \mathbf{Top}$, where U is the forgetful functor and F is the functor which applies the discrete topology, denoted by $\mathcal{D}$. (For instance, if we have a set X, then $FX = (X, \mathcal{D})$). We will show that $F \dashv U$.
Let's start with the unit definition: Take any objects $X \in \mathbf{Set}, Y \in \mathbf{Top}$, and $f : X \to UY$, let $\eta_X = 1_X$, and the following commutes:

$$\begin{array}{ccccc} FX & \qquad & U \circ F(X) & \xleftarrow{\eta_X} & X \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle U(f)} & \swarrow{\scriptstyle f} & \\ Y & & U(Y) & & \end{array}$$

By definition of the identity and since $Uf(x) = f(x)$ for all $x \in X$, f is unique. (Noting that since all subsets of $FX = (X, \mathcal{D})$ are open, any function with FX as the domain is continuous, so $f : FX \to Y$ is continuous and thus $f \in \mathbf{Top}$.)

Now for the hom-set definition: Take $X \in \mathbf{Set}$ and $Y \in \mathbf{Top}$. Let $\phi_{X,Y}$: $\mathrm{Hom}_{\mathbf{Top}}(FX, Y) \to \mathrm{Hom}_{\mathbf{Set}}(X, UY)$ be the function $\phi_{X,Y} = U(g)$, where g : $FX \to Y$ (since $U \circ F(X) = X$). We now need to show that the following diagram commutes, for any $f : X' \to X$ in $\mathbf{S}et$:

$$\begin{array}{ccc} Hom_{\mathbf{Top}}(FX, Y) & \xrightarrow{\phi(X,Y)} & Hom_{\mathbf{Set}}(X, UY) \\ \Big\downarrow{\scriptstyle (Ff)^*} & & \Big\downarrow{\scriptstyle f^*} \\ Hom_{\mathbf{Top}}(FX', Y) & \xrightarrow[\phi_{X',Y}]{} & Hom_{\mathbf{Set}}(X', UY) \end{array}$$

$$\begin{aligned} f^*(\phi_{X,Y}(g)) &= U(g) \circ f \\ &= U(g) \circ UF(f) \\ &= \phi_{X',Y}((Ff)^*(g)) \end{aligned}$$

Thus, ϕ is natural in X. For naturality in Y we need to check that the following commutes for $h : Y \to Y'$:

$$\begin{array}{ccc} Hom_{\mathbf{Top}}(FX, Y) & \xrightarrow{\phi(X,Y)} & Hom_{\mathbf{Set}}(X, UY) \\ \Big\downarrow{\scriptstyle h_*} & & \Big\downarrow{\scriptstyle (Uh)_*} \\ Hom_{\mathbf{Top}}(FX, Y') & \xrightarrow[\phi_{X,Y'}]{} & Hom_{\mathbf{Set}}(X, UY') \end{array}$$

To check this:

$$\begin{aligned}(Uh)_*(\phi_{X,Y}(g)) &= Uh \circ U(g)\\ &= U(h \circ g)\\ &= \phi_{X,Y'}(h_*(g))\end{aligned}$$

Thus ϕ is natural in Y. Now we need only check that ϕ is in fact an isomorphism. Let $\psi_{X,Y} : \mathrm{Hom}_{\mathbf{Set}}(X, UY) \to \mathrm{Hom}_{\mathbf{Top}}(FX, Y)$ take any function in $\mathbf{S}et$ and change its domain to FX and its range to Y, without changing where it sends individual elements of the set. We can do this because any function with FX as its domain is continuous. Then it is easy to check that $\psi_{X,Y} \circ \phi_{X,Y} = 1_{Hom_{\mathbf{Top}}(FX,Y)}$ and $\phi_{X,Y} \circ \psi_{X,Y} = 1_{Hom_{\mathbf{Set}}(X,UY)}$ and thus, ϕ is an isomorphism.

Theorem 3.5. *Definitions 3.2 and 3.3 are equivalent.*

Proof. **Definition 3.3 implies Definition 3.2:**
Define $\phi_{C,D}(g) = U(g) \circ \eta_C$ (where $g : FC \to D$ and η_C is as defined in Def. 3.3. We can see immediately that $\phi_{C,D} : \mathrm{Hom}_{\mathbf{D}}(FC, D) \to \mathrm{Hom}_{\mathbf{C}}(C, UD)$. That $\phi_{C,D}$ is an isomorphism follows from the uniqueness of g, given $f : C \to UD$, such that $f = U(g) \circ \eta_C$ (i.e. because for any $f : C \to UD$ there is a unique $g : FC \to D$ such that $f = U(g) \circ \eta_C$, the function $\psi_{C,D}(f) = g$ is an inverse to $\phi_{C,D}$). So we must show naturality in C and D.

For naturality in C, take $f : C' \to C$ and $h : FC \to D$:

$$\begin{aligned}f^*(\phi_{C,D}(g) &= (U(h) \circ \eta_C) \circ f\\ &= U(h) \circ UF(f) \circ \eta_{C'}\end{aligned}$$

(from the commutative diagram for natural transformations)

$$\begin{aligned}&= U(h \circ F(f)) \circ \eta_{C'}\\ &= \phi_{C',D}((Ff)^*(h))\end{aligned}$$

Thus, ϕ is natural in C.

For naturality in D examine $g : D \to D'$

$$\begin{aligned}U(g)_*(\phi_{C,D}(h)) &= U(g) \circ (U(h) \circ \eta_C)\\ &= U(g \circ h) \circ \eta_C\\ &= \phi_{C,D'}(g \circ h)\\ &= \phi_{C,D'}(g_*(h))\end{aligned}$$

Thus ϕ is natural in D.

Now: **Definition 3.2 implies Definition 3.3**
Consider any $f : C \to UD$ and note that $\phi^{-1}_{C,D}(f) : FC \to D$. Let $\eta_C : C \to UFC$ be defined by $\eta_C = \phi_{C,FC}(1_{FC})$. To show that $\eta : 1_C \to U \circ F$ is a natural

transformation, take $\gamma : C \to C'$ and consider the following commutative diagrams:

$$\begin{CD}
Hom_{\mathbf{D}}(FC, FC) @>{\phi_{C,FC}}>> Hom_{\mathbf{C}}(C, UFC) \\
@V{(F\gamma)_*}VV @VV{(UF\gamma)_*}V \\
Hom_{\mathbf{D}}(FC, FC') @>>{\phi_{C,FC'}}> Hom_{\mathbf{C}}(C, UFC')
\end{CD}
\qquad
\begin{CD}
Hom_{\mathbf{D}}(FC', FC') @>{\phi_{C',FC'}}>> Hom_{\mathbf{C}}(F', UFC') \\
@V{(F\gamma)^*}VV @VV{\gamma^*}V \\
Hom_{\mathbf{D}}(FC, FC') @>>{\phi_{C,FC'}}> Hom_{\mathbf{C}}(C, UFC')
\end{CD}$$

Naturality in D **Naturality in C**

Naturality in D gives us that:

$$UF(\gamma) \circ \phi_{C,FC}(1_{FC}) = \phi_{C,FC'}(F\gamma \circ 1_{FC})$$

Naturality in C gives us that:

$$\phi_{C',FC'}(1_{FC'}) \circ \gamma = \phi_{C,FC'}(1_{FC'} \circ F\gamma)$$

Working from one to the other:

$$\begin{aligned}
UF(\gamma) \circ \phi_{C,FC}(1_{FC}) &= \phi_{C,FC'}(F\gamma \circ 1_{FC}) \\
&= \phi_{C,FC'}(F\gamma) \\
&= \phi_{C,FC'}(1_{FC'} \circ F\gamma) \\
&= \phi_{C',FC'}(1_{FC'}) \circ \gamma \\
\eta_{C'} \circ \gamma &= UF(\gamma) \circ \eta_C
\end{aligned}$$

Thus since the following diagram commutes, η is a natural transformation.

$$\begin{CD}
C @>{\eta_C}>> UFC \\
@V{\gamma}VV @VV{UF(\gamma)}V \\
C' @>>{\eta_{C'}}> UFC'
\end{CD}$$

Now we notice from the commutative diagram given by Def. 3.2:

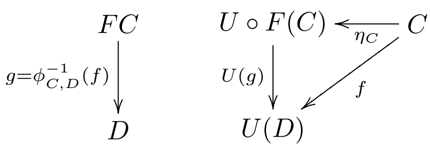
$$\begin{CD}
Hom_{\mathbf{D}}(FC, FC) @>{\phi_{C,FC}}>> Hom_{\mathbf{C}}(C, UFC) \\
@V{(\phi^{-1}_{C,D}(f))_*}VV @VV{(U\phi^{-1}_{C,D}(f))_*}V \\
Hom_{\mathbf{D}}(FC, D) @>>{\phi_{C,D}}> Hom_{\mathbf{C}}(C, UD)
\end{CD}$$

$$\begin{aligned}
U\phi^{-1}_{C,D}(f) \circ \phi_{C,FC} &= \phi_{C,D}(\phi^{-1}_{C,D}(f)) \\
&= \phi_{C,D}(\phi^{-1}_{C,D}(f)
\end{aligned}$$

Plugging in 1_{FC}:

$$\begin{aligned}
U\phi^{-1}_{C,D}(f) \circ \phi_{C,FC}(1_{FC}) &= \phi_{C,D}(\phi^{-1}_{C,D}(f(1_{FC}))) \\
&= \phi_{C,D}(\phi^{-1}_{C,D}(f)) \\
&= f
\end{aligned}$$

Thus, for $f : C \to UD$ there exists $g : FC \to D$ (where $g = \phi^{-1}_{C,D}(f)$) such that the following commutes:



Now we need show that such a g is unique. Since ϕ is an isomorphism, the uniqueness of g requires only that we show $\phi_{C,D}(g) = U(g) \circ \eta_C$.

$$\begin{aligned} U(g) \circ \eta_C &= U(g) \circ \phi_{C,FC}(1_{FC}) \\ &= \phi_{C,D}(g \circ 1_{FC}) \\ &= \phi_{C,D}(g) \end{aligned}$$

(The second line comes from naturality in D.) □

4. Limits

Definition 4.1. A terminal object in a category $\mathbf{C}$ is an object $1 \in \mathbf{C}$ such that for any $C \in \mathbf{C}$, there is a unique morphism

$$C \to 1.$$

Proposition 4.2. *Terminal objects are unique up to unique isomorphism.*

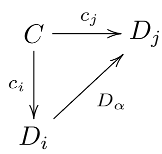
Proof. Take two terminal objects, 1 and $1'$, and find the unique maps

$$1 \xrightarrow{\alpha} 1' \xrightarrow{\beta} 1$$

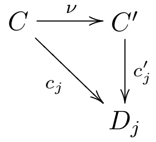
Since 1 is a terminal object then the only arrow from 1 to itself is its identity, 1_1. Since $\beta \circ \alpha$ is a map from 1 to itself then it must be that $1_1 = \beta \circ \alpha$. Similarly, $1_{1'} = \alpha \circ \beta$. Thus, $1 \cong 1'$. Since 1 and $1'$ are terminal objects, then α and β are the unique maps for which this holds. □

Definition 4.3. Let $\mathbf{J}$ and $\mathbf{C}$ be categories. A diagram of type $\mathbf{J}$ in $\mathbf{C}$ is a functor $D : \mathbf{J} \to \mathbf{C}$. We write objects in the "index category", $\mathbf{J}$ as $i, j, \ldots$ and values of the functor D as $D_i, D_j, \ldots$.

Definition 4.4. A cone to a diagram D consists of an object $C \in \mathbf{C}$ and a family of arrows $c_i : C \to D_i$ in $\mathbf{C}$ such that for each $\alpha : i \to j$ in $\mathbf{J}$ the following commutes:



With this we can define a category $\mathbf{Cone}(D)$ of cones to the diagram D, where a morphism of cones $\nu : (C, c_j) \to (C', c'_j)$ is an arrow $\nu \in \mathbf{C}$ such that the following commutes for all $j \in \mathbf{J}$:



Definition 4.5. A limit for a diagram $D : \mathbf{J} \to \mathbf{C}$ is a terminal object in $\mathbf{Cone}(D)$. A finite limit is a limit for a diagram on a finite index category.

We denote a limit as

$$p_i : \varprojlim_j D_j \to D_i$$

noting that the definition says that given any cone (C, c_i) to D, there is a unique

$$u : C \to \varprojlim_j D_j$$

such that for all i

$$p_i \circ u = c_i.$$

Example 4.6. Take $\mathbf{0}$ to be the empty category. Then there is only one diagram $D : \mathbf{0} \to \mathbf{C}$. Then cones are simply objects in $\mathbf{C}$ and so the limit is just a terminal object in $\mathbf{C}$:

$$\varprojlim_{j \in \mathbf{0}} D_j \cong 1$$

5. Preservation of Limits

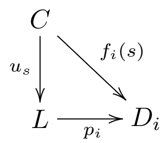
Definition 5.1. A functor $F : \mathbf{C} \to \mathbf{D}$ preserves limits of type $\mathbf{J}$ iff whenever $p_j : L \to D_j$ is a limit for a diagram $D : \mathbf{J} \to \mathbf{C}$, the cone $Fp_j : FL \to FD_j$ is a limit for the diagram $FD : \mathbf{J} \to \mathbf{D}$. We can also write this as:

$$F(\varprojlim D_j) \cong \varprojlim F(D_j)$$

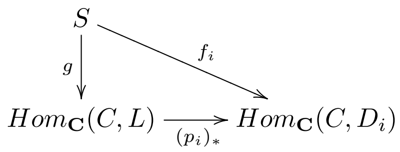
For an example of this we look at representable funtors. A representable functor, denoted $\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Set}$ for some object $C \in \mathbf{C}$, takes an object $X \in \mathbf{C}$ to the set $\mathrm{Hom}_{\mathbf{C}}(C, X)$ and takes an arrow, $f : X \to Y$, to $\mathrm{Hom}_{\mathbf{C}}(C, f) = f_* : \mathrm{Hom}_{\mathbf{C}}(C, X) \to \mathrm{Hom}_{\mathbf{C}}(C, Y)$ defined by $f_*(g) = f \circ g$ where $g : C \to X$ in $\mathbf{C}$. It is easy to check that this satisfies all the properties of functors. (Note: we can only use categories $\mathbf{C}$ considered *locally small*, meaning that for any $X, Y \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is small enough to constitute a set. A *small* category would be one where the objects can be considered a set and the morphisms can be considered a set.) Similarly, we also have contravariant representable functors, denoted $\mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{\mathbf{op}} \to \mathbf{Set}$ which takes any $f : X \to Y$ to $f^* : \mathrm{Hom}_{\mathbf{C}}(Y, C) \to \mathrm{Hom}_{\mathbf{C}}(X, C)$ such that for $g : X \to C$, $f^*(g) = g \circ f$.

Lemma 5.2. *Representable functors preserve limits.*

Proof. Take any diagram $D : \mathbf{J} \to \mathbf{C}$ for some index category $\mathbf{J}$ and assume it has a limit L. Then for any cone (A, c_i) to D there is a unique arrow $u : A \to L$ such that for all i, $p_i \circ u = c_i$. Now take any cone (S, f_i) to $\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Set}$. Given any element $s \in S$ we have a map and a cone (since that L is a limit):



Note that u_s is unique by the definition of limits. Now define a map $g : S \to \text{Hom}_{\mathbf{C}}(C, L)$ defined by $g(s) = u_s$ as given before. Then the following diagram commutes:



Uniqueness of g follows from the uniqueness of each element u_s. $\square$

Definition 5.3. The Yoneda embedding is the functor $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C^{op}}}$ which takes $C \in \mathbf{C}$ to the contravariant representable functor (5.4) and takes $f : C \to D$ to the natural transformation (5.5):

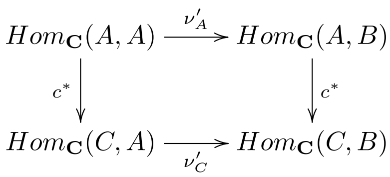
$$yC = Hom_{\mathbf{C}}(-, C) : \mathbf{C^{op}} \to \mathbf{Set} \tag{5.4}$$

$$yf = Hom_{\mathbf{C}}(-, f) : Hom_{\mathbf{C}}(-, C) \to Hom_{\mathbf{C}}(-, D) \tag{5.5}$$

Before we start the next proposition, we introduce the notation $\mathbf{C^D}$ for the category with functors taking $\mathbf{D}$ to $\mathbf{C}$ as objects and natural transformations between such functors as arrows.

Proposition 5.6. *Given objects A and B in a locally small category, $\mathbf{C}$, if $yA \cong yB$, then $A \cong B$.*

Proof. Let ν be the natural isomorphism from yA to yB. We will now show that the function $g : \text{Hom}_{\mathbf{C}}(A, B) \to \text{Hom}_{\mathbf{Set}^{\mathbf{C^{op}}}}(yA, yB)$, defined by $g(f) = yf$ for any $f : A \to B$, is an isomorphism. First, take any $\nu' : yA \to yB$ and define a function $\phi : A \to B$ as $\phi = \nu'_A(1_A)$. From the commutative diagram following we see that $\nu' = \text{Hom}_{\mathbf{C}}(-, \nu'_A(1_A))$:



where $c : C \to A$. Applying these functions to 1_A, we get that:

$$\begin{aligned} \nu'_C(c^*(1_A)) &= c^*(\nu'_A(1_A)) \\ \nu'_C(1_A \circ c) &= \phi \circ c \\ \nu'_C(c) &= \phi \circ c \end{aligned}$$

Because of this, we can define an arrow $h : \text{Hom}_{\mathbf{Set}^{\mathbf{C^{op}}}}(yA, yB) \to \text{Hom}_{\mathbf{C}}(A, B)$ by $h(\nu') = \nu'_A(1_A)$. Now, to check that h and g are mutually inverse we first note that for any $f : A \to B$, $f = \text{Hom}_{\mathbf{C}}(A, f)(1_A)$. Also, for any $\nu' : yA \to yB$, we have $\nu' = \text{Hom}_{\mathbf{C}}(-, \nu'_A(1_A))$, as shown above. Therefore:

$$\begin{aligned} g(h(\nu)) &= y(\nu_A(1_A)) \\ &= Hom_{\mathbf{C}}(-, \nu_A(1_A)) \\ &= \nu \end{aligned}$$

and also:

$$\begin{aligned} h(y(f)) &= h(Hom_{\mathbf{C}}(-, f)) \\ &= Hom_{\mathbf{C}}(A, f)(1_A) \\ &= f \end{aligned}$$

We also write g and h for the analogous isomorphisms between $\mathrm{Hom}_{\mathbf{C}}(B, A)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{op}}}(yB, yA)$. Now, to show that $A \cong B$ we observe, by the commutative diagram for naturality of ν:

$$\begin{aligned} \nu_A(1_A \circ \nu_A^{-1}(1_B)) &= \nu_A(1_A) \circ \nu_A^{-1}(1_B) \\ \nu_A(\nu_A^{-1}(1_B) &= h(\nu)) \circ h(\nu^{-1}) \\ 1_B &= h(\nu) \circ h(\nu^{-1}) \end{aligned}$$

and similarly,

$$\begin{aligned} \nu_A^{-1}(1_B \circ \nu_A(1_A)) &= \nu_A^{-1}(1_B) \circ \nu_A(1_A) \\ \nu_A^{-1}(\nu_A(1_A)) &= h(\nu^{-1}) \circ h(\nu) \\ 1_A &= h(\nu^{-1}) \circ h(\nu) \end{aligned}$$

Thus, $A \cong B$. □

One consequence of Proposition 5.6 is that if we have some $\phi : \mathrm{Hom}_{\mathbf{C}}(X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, Z)$ that is natural in X, then $Y \cong Z$.

Lemma 5.7. *If we have a diagram $D : \mathbf{J} \to \mathbf{C}$ and if:*

$$\phi : Hom_{\mathbf{C}}(X, L) \cong \varprojlim Hom_{\mathbf{C}}(X, D_j)$$

which is natural in X, then L is a limit of the diagram D.

Proof. First note that from our proof for Proposition 5.6, we can denote any natural transformation from $\mathrm{Hom}_{\mathbf{C}}(-, L)$ to $\mathrm{Hom}_{\mathbf{C}}(-, D_i)$ as $(g_i)_*$ since it corresponds to some $g_i : L \to D_i$. Notice that as we vary X, $p_{X_i} \circ \phi_X$ forms a natural transformation $\mathrm{Hom}_{\mathbf{C}}(-, L)$ to $\mathrm{Hom}_{\mathbf{C}}(-, D_i)$ which we denote $(k_i)_*$ for some $k_i : L \to D_i$.

We need to show that (L, k_i) is, in fact, a cone. Take a map $\alpha : i \to j$, where $i, j \in \mathbf{J}$. Since $(\lim \mathrm{Hom}_{\mathbf{C}}(X, D_j), p_{X_i})$ is a cone (Diagram 1), then we have that:

$$\begin{aligned} p_{X_j} &= (D\alpha)_* \circ p_{X_i} \\ p_{X_j} \circ \phi_X &= (D\alpha)_* \circ p_{X_i} \circ \phi_X \\ k_j &= D\alpha \circ k_j \end{aligned}$$

Thus, (L, k_i) is a cone (Diagram 2).

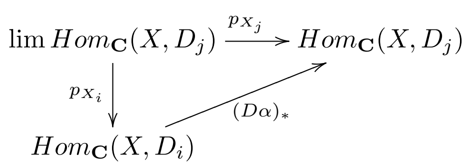


Diagram 1

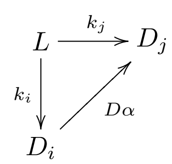


Diagram 2

Now, take any cone (X, x_i) to D and we will show that there is a map u such that the following commutes:

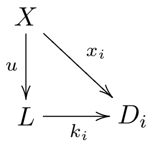


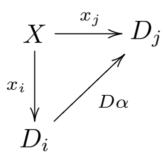
Diagram 3

Take any one element set, $1 = \{\star\}$, (which particular set is irrelevant, since any one element set is isomorphic to any other one element set) and define a map $h_i : 1 \to \mathrm{Hom}_{\mathbf{C}}(X, D_i)$ such that $h_i(\star) = x_i$. To show that $(1, h_i)$ is a cone, we again take the arrow $(D\alpha)_*$, and since (X, x_i) is a cone (Diagram 4):

$$x_j = D\alpha \circ x_i$$
$$h_j(\star) = (D\alpha)_* \circ h_i(\star)$$

since star is the only element of 1, then:

$$h_j = (D\alpha)_* \circ h_i$$



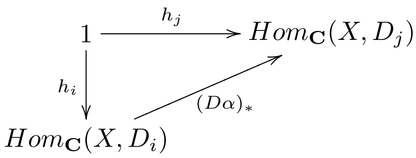
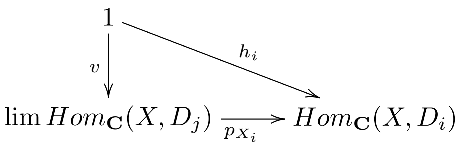


Diagram 4 **Diagram 5**

Thus, $(1, h_i)$ is a cone (Diagram 5), and there is some unique $v : 1 \to \lim \mathrm{Hom}_{\mathbf{C}}(X, D_j)$ such that the following commutes:



Thus:

$$\begin{aligned} p_{X_i}(v(\star)) &= h_i(\star) \\ &= x_i \end{aligned}$$

Define a map $u : X \to L$ by $u = \phi_X^{-1}(v(\star))$. To check that Diagram 3 commutes:

$$\begin{aligned} p_{X_i}(v(\star)) &= x_i \\ p_{X_i} \circ \phi_X \circ \phi_X^{-1}(v(\star))) &= x_i \\ (k_i)_*(u) &= x_i \\ k_i \circ u &= x_i \end{aligned}$$

Now, we need to show that such a map u is unique, and if it is, we are finished with the proof. Since $(k_i)_* = p_{X_i} \circ \phi_X$, to show that our arrow u is uniquely determined by its composites with all the k_i, we need only show that the functions $p_{X_i} \circ \phi_X$ are jointly one-to-one. Essentially, since ϕ_X is an isomorphism, what we need to show is that p_{X_i} are jointly one-to-one. Take the commutative diagram below:

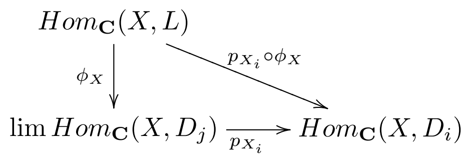


Diagram 6

Suppose p_{X_i} is not jointly one-to-one, or in other words, suppose that for some a and b in $\mathrm{Hom}_{\mathbf{C}}(X, L)$, $p_{X_i} \circ \phi_X(a) = p_{X_i} \circ \phi_X(b)$ for all i. Then define:

$$m(x) = \begin{cases} \phi_X(a) & \text{if } x = b \\ \phi_X(b) & \text{if } x = a \\ \phi_X(x) & \text{otherwise} \end{cases}$$

Clearly, $p_{X_i} \circ \phi_X = p_{X_i} \circ m$ even though $\phi_X \neq m$, so ϕ_X is not unique for Diagram 6. Since this contradicts our definition of limit, p_{X_i} must be one-to-one. Thus, our arrow u is unique. Thus, $k_i : L \to D_i$ is a limit of the diagram D. □

We have already shown that representable functors preserve limits, and now we will use Lemma 5.7 to show that right adjoints preserve limits as well.

Proposition 5.8. *Right adjoints preserve limits, or more formally, if* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{C}$ *are functors and* $F \dashv G$*, then if there exists a diagram* $D : \mathbf{J} \to \mathbf{D}$ *with a limit, we have that the diagram* $GD : \mathbf{J} \to \mathbf{C}$ *has a limit and that:*

$$G(\varprojlim D_j) = \varprojlim GD_j$$

Proof. Since $F \dashv G$, then for any $X \in \mathbf{C}$ we have:

$$Hom_{\mathbf{C}}(X, G(\varprojlim D_j)) \cong Hom_{\mathbf{D}}(FX, \varprojlim D_j)$$

Since representable functors preserve limits:

$$\cong \varprojlim Hom_{\mathbf{D}}(FX, D_j)$$
$$\cong \varprojlim Hom_{\mathbf{C}}(X, GD_j)$$

By Lemma 5.7, this implies that $G(\lim D_j)$ is a limit of the diagram G. In other words:

$$G(\varprojlim D_j) \cong \varprojlim GD_j$$

In other words, G preserves limits. □

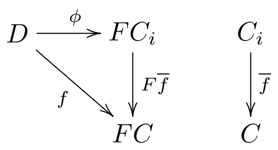
6. Completeness and the Adjoint Functor Theorem

Definition 6.1. A category, $\mathbf{C}$, is complete iff for any small category $\mathbf{J}$ and diagram $D : \mathbf{J} \to \mathbf{C}$, there is a limit in $\mathbf{C}$.

We are now at the point where we can state the Adjoint Functor Theorem, an important tool because it tells us a necessary and sufficient condition for a functor to have a left adjoint.

Theorem 6.2. Adjoint Functor Theorem (Freyd) *Let* $\mathbf{C}$ *be locally small and complete. Given any category* $\mathbf{D}$ *and a functor* $F : \mathbf{C} \to \mathbf{D}$ *between them, then the following are equivalent:*

- F *has a left adjoint.*
- F *preserves limits, and for each object* $D \in \mathbf{D}$ *the functor* F *satisfies the following condition: There exists a set of objects* $(C_i)_{i \in I} \in \mathbf{C}$ *such that for any* $C \in \mathbf{C}$ *and any arrow* $f : D \to FC$*, there exists an* $i \in I$ *and arrows* $\phi : D \to FC_i$ *and* $\overline{f} : C_i \to C$ *such that the following diagram commutes:*



$$f = F(\overline{f}) \circ \phi$$

7. Acknowledgements

I would like to thank Peter May and the REU program for providing a motivation for this paper and mentors to guide it, Michael Shulman for providing guidance and putting up with numerous drafts and last minute questions, Emily Riehl and Claire Tomesch for being stand-in mentors early in the process, and Steve Awodey for writing the clearest, most coherent text on category theory.

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