ALGEBRAICALLY TRIVIAL, BUT TOPOLOGICALLY NON-TRIVIAL MAP

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Abstract. I studied the construction of an algebraically trivial, but topologically non-trivial map by Hopf map $p : S^3 \to S^2$ and a collasping map $q : T^3 \to S^3$.

Contents

1. Introduction

Let $p : S^3 \to S^2$ be the Hopf bundle and let $q : T^3 \to S^3$ be the quotient map collapsing the complement of a ball in the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ to a point. I would like to show that $pq : T^3 \to S^2$ induces the trivial map on H_*, but is not homotopic to a constant map.

I will prove that pq induces the trivial map by calculating homology groups of S^2 and T^3, then prove that pq is not homotopic to a constant map by the homotopy lifting property of fiber bundle.

2. Homology Groups of S^2 and T^3

First we calculate the homology groups of S^2 and T^3.

For S^n in general, we have following proposition.

Proposition 2.1. *$\tilde{H}_n(S^n) = \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$*

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Proof. Take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. The long exact sequence of homology group for the pair (X, A) would be

$$\cdots \longrightarrow \tilde{H}_n(S^{n-1}) \xrightarrow{i_*} \tilde{H}_n(D^n) \xrightarrow{j_*} \tilde{H}_n(S^n) \xrightarrow{\partial}$$

$$\tilde{H}_{n-1}(S^{n-1}) \xrightarrow{i_*} \cdots \longrightarrow \tilde{H}_0(S^n) \longrightarrow 0$$

$\tilde{H}_n(D^n) = 0$ since D^n is contractible.

Exactness of the sequence then implies that the maps $\tilde{H}_n(S^n) \to \tilde{H}_{n-1}(S^{n-1})$ are isomorphisms for all $i > 0$.

Based on the fact that $H_0(X) = \mathbb{Z}$ for any nonempty and path-connected space X, the result follows by induction on n. □

Applying this proposition to S^2,

Corollary 2.2. *$\tilde{H}_2(S^2) = \mathbb{Z}$ and $\tilde{H}_i(S^2) = 0$ for $i \neq 2$*

Now we calculate the homology groups of T^3 by considering the cellcular chian complex.

Proposition 2.3. *$H_i(T^3)$ is $\mathbb{Z}$ for $i = 0, 3$, $\mathbb{Z}^3$ for $i = 1, 2$, and 0 for $i > 3$*

Proof. For T^3 we have a CW structure with one 3-cell, three 2-cells, three 1-cells, and one 0-cell. Thus the cellular chain complexes have the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Having the fact that the cellular boundary maps d_3, d_2 are zero by cellular boundary formula, the result follows. □

3. Inducing Algebraically Trivial Map

Based on these facts, we can determine what kind of map does pq induce on $\tilde{H}_*$.

Proposition 3.1. *The induced map $pq_* : \tilde{H}_n(T^3) \to \tilde{H}_n(S^3) \to \tilde{H}_n(S^2)$ is a zero map.*

Proof. Since $\tilde{H}_n(S^2) = 0$ for $n \neq 2$ by Corollary 2.2., the result follows for the case $n \neq 2$.

For $n = 2$, we have $pq_* : \tilde{H}_2(T^3) \to \tilde{H}_2(S^3) \to \tilde{H}_2(S^2)$. Since $\tilde{H}_2(S^3) = 0$, pq_* must be a zero map. □

4. Fiber Bundle

Definition 4.1. A fiber bundle structure on a space E, with fiber F, consists of a projection map

$$p : E \longrightarrow B$$

such that each point of B has a neighborhood U for which there is a homeomorphism

$$h : p^{-1} \longrightarrow U \times F$$

making the diagram at the below commutes.

$$\begin{array}{ccc} p^{-1} & \xrightarrow{h} & U \times F \\ {\scriptstyle p}\downarrow & \swarrow & \\ U & & \end{array}$$

The map h is called a local trivialization, the Space B is called the base space of the bundle, and E is the total space.

Definition 4.2. A map $p : E \to B$ has homotopy lifting property with respect to a space X if, given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ lifting g_0, i.e. $p\tilde{g}_0 = g_0$, then there exist a homotopy $\tilde{g}_t : X \to E$ lifting g_t.

Proposition 4.3. *A fiber bundle $p : E \to B$ has the homotopy lifting property with respect to all CW pairs (X,A).*

Proof. The homotopy lifting property for CW pair is equivalent to the homotopy lifting property for disks, or equivalently, cubes.

Let $G : I^n \times I \to B$, $G(x,t) = g_t(x)$, be a homotopy we want to lift, starting with a given lift $\tilde{g}_0$ of g_0.

Choose an open cover $\{U_\alpha\}$ of B with local trivializations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$. Since $I^n \times I$ is compact, we may subdivide I^n into small cubes C and I into intervals $I_j = [t_j, t_{j+1}]$ so that each product $C \times I_j$ is mapped by G into a single U_α.

We may assume by induction on n that $\tilde{g}_t$ has already been constructed over ∂C for each of the subcubes C. To extend this $\tilde{g}_t$ over a cube C, we may proceed in stages, constructing $\tilde{g}_t$ for t in each successive interval I_j. This reduces us to the case that no subdivision of $I^n \times I$ is necessary, so G maps all of $I^n \times I$ to a single U_α.

Then we have $\tilde{G}(I^n \times \{0\} \cup \partial I^n \times I) \subset p^{-1}(U_\alpha)$, and composing $\tilde{G}$ with the local trivialization h_α reduces us to the case of a product bundle $U_\alpha \times F$. In this case the first coordinate of a lift $\tilde{g}_t$ is just the given g_t, so only the second coordinate needs to be constructed. This can be obtained as a composition $I^n \times I \to I^n \times \{0\} \cup \partial I^n \times I \to F$ where the first map is a retraction and the second map is what we are given. ☐

Since fiber bundles have homotopy lifting property, we can have the long exact sequence of homotopy groups by following theorem.

Theorem 4.4. *Suppose $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_*(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$.*

Hence if B is path-connected, there is a long exact sequence

$$\cdots \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow$$

$$\pi_{n-1}(F, x_0) \longrightarrow \cdots \longrightarrow \pi_0(E, x_0) \longrightarrow 0$$

Proof. i) p_* is surjective.

Represent an element of $\pi_n(B, b_0)$ by a map $f : (I^n, \partial I^n) \to (B, b_0)$. The constant map to x_0 provides a lift of f to E over the subspace $J^{n-1} \subset I^n$, so the relative homotopy lifting property for $(I^{n-1}, \partial I^{n-1})$ extends this to a lift $\tilde{f} : I^n \to E$ and this lift satisfies $\tilde{f}(\partial I^n) \subset F$ since $f(\partial I^n) = b_0$.

Then $\tilde{f}$ represents an element of $\pi_n(E, F, x_0)$ with $p_*([\tilde{f}]) = [f]$ since $p\tilde{f} = f$.

ii) p_* is injective.

Given $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ such that $p_*([\tilde{f}_0]) = p_*([\tilde{f}_1])$, let $G : (I^n \times I, \partial I^n \times I) \to (B, b_0)$ be a homotopy from $p\tilde{f}_0$ to $p\tilde{f}_1$.

We have a partial lift $\tilde{G}$ given by $\tilde{f}_0$ on $I^n \times \{0\}$,$\tilde{f}_1$ on $I^n \times \{1\}$, and the constant map to x_0 on $J^{n-1} \times I$.

After permuting the last two coordinates of $I^n \times I$, the relative homotopy lifting property gives an extension of this partial lift to a full lift $\tilde{G} : I^n \times I \to E$.

This is a homotopy $\tilde{f}_t : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ from $\tilde{f}_0$ to $\tilde{f}_1$.
Thus p_* is injective.

iii) The existence of a long exact sequence.

We plug $\pi_n(B, b_0)$ for $\pi_n(E, F, x_0)$ in the long exact sequence for the pair (E,F).

The map $\pi_n(E, x_0) \to \pi_n(E, F, x_0)$ in the exact sequence then becomes the composition $\pi_n(E, x_0) \to \pi_n(E, F, x_0) \to \pi_n(B, b_0)$, which is just $p_* : \pi_n(E, x_0) \to pi_n(B, b_0)$.

The 0 at the end of the sequence, i.e. $\pi_0(F, x_0) \to \pi_0(E, x_0)$ is surjective, comes from the hypothesis that B is path-connected, since a path in E from an arbitrary point $x \in E$ to F can be obtained by lifting a path in B from $p(x)$ to b_0. ☐

Let us consider the fiber bundles given by projective spaces. Over the complex numbers we have a fiber bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$$

Here S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n$ is viewed as the quotient space of S^{2n+1} under the equivalence relation $(z_0, \cdots, z_n) \sim \lambda(z_0, \cdots, z_n)$ for $\lambda \in S^1$ The projection $p : S^{2n+1} \to \mathbb{C}P^n$ sends $(z_0, \cdots, z_n)$ to the equivalence class $[z_0, \cdots, z_n]$, thus the fibers are copies of S^1.

To check the local triviality condition, let $U_i \subset \mathbb{C}P^n$ be the open set of equivalence classes $[z_0, \cdots, z_n]$ with $z_i \neq 0$ and define $h_i : p^{-1}(U_i) \to U_i \times S^1$ by

$$h_i(z_0, \cdots, z_n) = ([z_0, \cdots, z_n], z_i/|z_i|) \tag{4.5}$$

The inverse of this map is

$$([z_0, \cdots, z_n], \lambda]) \mapsto \lambda |z_i| z_i^{-1} (z_0, \cdots, z_n) \tag{4.6}$$

Thus h_i is a homeomorphism.

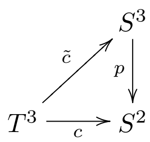
Definition 4.7. The Hopf bundle can be consider as the case $n = 1$ of the fiber bundle given by projective spaces, $S^1 \to S^3 \to \mathbb{C}P^1 = S^2$

The fiber, total space, and base space are all spheres in Hopf bundle. Since Hopf bundle is a fiber bundle, we can apply homotopy lifting property to our map $p : S^3 \to S^2$.

5. Topologically Non-Trivial

Now we prove that pq is not homotopic to a constant map.

Assume that pq is homotopic to a constant map $c : T^3 \to S^3$. Applying the homotopy lifting property, for given map q lifting pq, there exists $\tilde{c} : T^3 \to S^3$ lifting c which is represented by following diagram.



Thus q and $\tilde{c}$ must be homotopic.

Proposition 5.1. *$\tilde{c}_* : H_3(T^3) \to H_3(S^3)$ is a zero map.*

Proof. Since c is a constant map, we can define an element $a \in S^2$ where $c(t) = a$ for all $t \in T^3$.

Since $p\tilde{c} = c$, $p\tilde{c}(t) = a$. Then $\tilde{c}(t) \in p^{-1}(a)$ which is a fiber of the Hopf bundle.

The fibers of the Hopf bundle are circles. Thus the image of $\tilde{c}$ lies inside circle.

Then the map $\tilde{c}$ can be decomposed in following sense:

$$\tilde{c} : T^3 \to S^1 \hookrightarrow S^3 \tag{5.2}$$

which induces

$$\tilde{c}_* : H_3(T^3) \to H_3(S^1) \hookrightarrow H_3(S^3) \tag{5.3}$$

However, since $H_3(S^1) = 0$, $\tilde{c}_*$ must be a zero map. □

Proposition 5.4. *If M is $\mathbb{Z}$-orientable, the map $H_n(M; \mathbb{Z}) \to H_n(M, M - \{x\}; \mathbb{Z}) \approx \mathbb{Z}$ is an isomorphism for all $x \in M$*

To prove this proposition we need some definitions and a lemma.

Definition 5.5. Every manifold M has an orientable two-sheeted covering space $\tilde{M}$ which is constructed in general,

$\tilde{M} = \{\mu_x | x \in M$ and μ_x is a local orientation of M at $x\}$

Definition 5.6. The covering space $\tilde{M} \to M$ can be embedded in a larger covering space $M_{\mathbb{Z}} \to M$ where $M_{\mathbb{Z}}$ consists of all elements $\alpha_x \in H_n(M, M-\{x\})$ as x ranges over M.

We topologize $M_{\mathbb{Z}}$ via the basis of sets $U(\alpha_B)$ consisting of α_x's with $x \in B$ and α_x the image of an element $\alpha_B \in H_n(M, M-B)$ under the map $H_n(M, M-B) \to H_n(M, M-\{x\})$. The covering space $M_{\mathbb{Z}} \to M$ is infinite-sheeted since for fixed $x \in M$, the α_x's range over the infinite cyclic group $H_n(M, M-\{x\})$. Restricting α_x to be zero, we get a copy M_0 of M in $M_{\mathbb{Z}}$. The rest of $M_{\mathbb{Z}}$ consists of an infinite sequence of copies M_k of $\tilde{M}$, $k = 1, 2, \cdots$, where M_k consists of the α_x's that are k times either generator of $H_n(M, M-\{x\})$.

Definition 5.7. A continuous map $M \to M_{\mathbb{Z}}$ of the form $x \mapsto \alpha_x \in H_n(M, M-\{x\})$ is called a section of the covering space.

Lemma 5.8. *Let M be a manifold of dimension n and let $A \subset M$ be a compact subset. Then :*
(a) If $x \mapsto \alpha_x$ is a section of the covering space $M_{\mathbb{Z}} \to M$, then there is a unique class $\alpha_A \in H_n(M, M-A; \mathbb{Z})$ whose image in $H_n(M, M-\{x\}; \mathbb{Z})$ is α_x for all $x \in A$.
(b) $H_i(M, M-A; \mathbb{Z}) = 0$ for $i > n$.

Proof. In this paper, we only need the case when A is union of convex compact sets, while the actual proof contains the case for an arbitrary compact set. Here I'll only present the sketch of three steps of the proof.

(1) We check that if the lemma is true for compact sets A, B, and $A \cap B$, then it is true for $A \cup B$ by considering the Mayer-Vietoris sequence $0 \to H_n(M, M-(A \cup B)) \to H_n(M, M-A) \oplus H_n(M, M-B) \to H_n(M, M-(A \cap B))$.

(2) We reduce to the case $M = \mathbb{R}^n$. A compact set $A \subset M$ can be written as the union of finitely many compact sets $A_1, \cdots, A_m$ each contained in an open $\mathbb{R}^n \subset M$. We apply the result in (1) to $A_1 \cup, \cdots, \cup A_{m-1}$ and A_m. The intersection of these two sets is $(A_1 \hat{A}_m) \cup, \cdots, \cup (A_{m-1} \hat{A}_m)$, a union of $m-1$ compact sets each contained in an open $\mathbb{R}^n \subset M$. By induction on m this gives a reduction to the case $m = 1/$ When $m = 1$, excision allows us to replace M by the neighborhood $\mathbb{R} \subset M$.

(3)When $M = \mathbb{R}^n$ and A is a union of convex compact sets $A_1, \cdots, A_m$, an inductive argument as in (2) reduces to the case that A itself is convex. When A is convex the result is evident since the map $H_i(\mathbb{R}^n, \mathbb{R}^n - A) \to H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ is an isomorphism for any $x \in A$, as both $\mathbb{R}^n - A$ and $\mathbb{R}^n - \{x\}$ deformation retract onto a sphere centered at x.

□

Proof. (of Proposition 5.4) Choose $A = M$, a compact set by assumption.

Let $\Gamma_{\mathbb{Z}}(M)$ be the set of sections of $M_{\mathbb{Z}} \to M$. The sum of two sections is a section, and a scalar multiple of a section is a section, so $\Gamma_{\mathbb{Z}}(M)$ is an $\mathbb{Z}$-module.

There is a homomorphism $H_n(M;\mathbb{Z}) \to \Gamma_{\mathbb{Z}}(M)$ sending a class α to the section $x \mapsto \alpha_x$, where α_x is the image of α under the map $H_n(M;\mathbb{Z}) \to H_n(M, M-\{x\};\mathbb{Z})$.

Part(a) of the lemma asserts that this homomorphism is an isomorphism.If M is connected, each section is uniquely determined by its value at one point, so the proposition is apparent from the structure of $M_{\mathbb{Z}}$. □

Proposition 5.9. *$q_* : H_3(T^3, \mathbb{Z}) \to H_3(S^3, \mathbb{Z})$ is an isomorphism.*

Proof. We know that $H_n(T^n, \mathbb{Z}) \approx \mathbb{Z}$ and $H_n(S^n, \mathbb{Z}) \approx \mathbb{Z}$

Applying Proposition 5.4 to our case, let $M = T^3$

$H_3(T^3, T^3 - \{x\}; \mathbb{Z}) \approx H_3(\mathbb{R}^3, \mathbb{R}^3 - \{0\}; \mathbb{Z})$ by excision

$\approx \tilde{H}_2(\mathbb{R}^3 - \{0\}; \mathbb{Z})$ since $\mathbb{R}^n$ is contractible

$\approx \tilde{H}_2(S^{n-1}; \mathbb{Z})$ since $\mathbb{R}^3 - \{0\} \simeq S^{n-1}$

$\approx H_3(S^3; \mathbb{Z})$

Thus $q_* : H_3(T^3, Z) \to H_3(T^3, T^3 - \{x\}; \mathbb{Z}) \approx H_3(S^3, \mathbb{Z})$ is an isomorphism. □

Since the two maps q_* and $\tilde{c}_*$ are different maps, the map q cannot be homotopic to $\tilde{c}$.

Thus by contradiction, pq is not homotopic to a constant map.

References

[1] A. Hatcher Algebraic Topology. Cambridge University Press. 2001.