Dependance Rank

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Let $\phi : V \to W$ be a linear transformation where $n = \dim(V)$ and $m = \dim(W)$, and let M be the matrix corresponding to ϕ written using the basis vectors $\{\hat{v}_i\}$ for V, and $\{\hat{w}_i\}$ for W.

I would like in this paper to present a method for determining the dimension d of the Kernel of ϕ by looking at M.
One idea for how to count to d is to count the number of linearly dependent row vectors in M.
The problem with this idea is that a set of row vectors a and a superset of a, b, could be linearly dependent for the same reason, motivating the following counterexample:

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$$

If we let $a = \{M_0, M_1\}$ and $b = \{M_0, M_1, M_2\}$, then we would count two linearly dependent sets of row vectors, but all elements of the Kernel of M are of the form,

$$r = \begin{bmatrix} 2c \\ -c \\ 0 \end{bmatrix}$$

and are thus one dimensional.
This problem of overlapping sets of linearly dependant vectors motivates the following definition:

Definition. A set of vectors S is a minimal linearly dependent set iff no proper subset of S is linearly dependent.

The idea behind minimal linear dependence is that a minimal linearly dependent set of row vectors of M represents a reason why the Kernel of M should have an extra dimension, without any meaningless garbage rows included by accident. Indeed, were we to count the minimal linearly dependent sets of M in our previous example, we would count only a, and properly compute 1 to be the dimension of the Kernel of M.
Counting minimal linearly dependent sets is still problematic for the following

reason (among others). Let M be a $7 \cdot 1$ matrix containing all 1s. Each of the $\binom{7}{2} = 21$ pairs of row vectors is minimal linearly dependent, which implies a 21 dimensional Kernel, yet the dimension of V is only 7, and the Kernel of M is a subspace of V. The extra counting that we did in this case seems to be a result of the high degree of overlapping of our minimal linearly dependent sets. Motivating the following, and final, method for counting to d:

Theorem 1. *Count minimal linearly dependent sets of row vectors of* M $(S_0, S_1 \ldots S_l)$ *such that* $(\forall j)\,(S_j \cap \bigcup_i S_i \neq S_j)$. *Then* $l = d$.

To address this theorem we will need a few definitions.

Definition. If $S = \{\hat{s}_1, \hat{s}_2, \cdots\}$ is a set of row vectors of matrix $M = \{\hat{m_1}, \hat{m_2}, \cdots\}$, then a vector $\hat{v} \neq \hat{0}$ is a dependency in S iff $M \cdot \hat{v} = 0$ and $\hat{m_i} \notin S \Rightarrow \hat{v(i)} = \hat{0}$.

Definition. If $S = \{\hat{s_1}, \hat{s_2}, \cdots\}$ is a set of row vectors of matrix $M = \{\hat{m_1}, \hat{m_2}, \cdots\}$, then the dependence multiplicity of S, (dep-mult(S)) is the rank of its dependencies.

Definition. If v is a vector of R^n, then v^{supp} is a vector containing only those elements in the support of v, indexed in the same order.

Thus if

$$v = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 0 \\ 8 \end{bmatrix} \tag{1}$$

Then

$$v^{supp} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \tag{2}$$

First we will show that every minimal linearly dependent set counted has a single dependency.

Theorem 2. *If* $S = \{\hat{s}_1, \hat{s}_2, \cdots\}$ *is a set of row vectors of* $n \times n$ *matrix* $M = \{\hat{m}_1, \hat{m}_2, \cdots\}$*, and* S *is minimal linearly dependent, then* dep-mult(S) = 1

Proof. Let $\hat{v}, \hat{w}$ be two dependencies of S
To show that dep-mult(S) = 1 is suffices to show that $\hat{v}$ and $\hat{w}$ are linearly dependent.
We know that

$$0 = \sum_{i=0}^{n-1} \hat{v}^{supp}(i) \cdot \hat{s}_i = \hat{v}^{supp}(0) \cdot \hat{s}_0 + \sum_{i=1}^{n-1} \hat{v}^{supp}(i) \cdot \hat{s}_i \tag{3}$$

Also,

$$0 = \sum_{i=0}^{n-1} \hat{w}^{supp}(i) \cdot \hat{s}_i = \hat{w}^{supp}(0) \cdot \hat{s}_0 + \sum_{i=1}^{n-1} \hat{w}^{supp}(i) \cdot \hat{s}_i \tag{4}$$

Thus, multiplying (4) by $\frac{\hat{v}^{supp}(0)}{\hat{w}supp(0)}$

$$\frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w}^{supp}(0) \cdot \hat{s}_0 + \sum_{i=1}^{n-1} \frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w}^{supp}(i) \cdot \hat{s}_i = \hat{v}^{supp}(0) \cdot \hat{s}_0 + \sum_{i=1}^{n-1} \frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w}^{supp}(i) \cdot \hat{s}_i = 0 \tag{5}$$

And subtracting (5) from (3) we have,
$0 \; = \; {}^{supp}(0) \cdot \hat{s}_0 - \hat{v}^{supp}(0) \cdot \hat{s}_0 + \sum_{i=1}^{n-1} \frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w}^{supp}(i) \cdot \hat{s}_i - \hat{v}^{supp}(i) \cdot \hat{s}_i$
$0 = \sum_{i=1}^{n-1} \frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w}^{supp}(i) - \hat{v}^{supp}(i) \cdot \hat{s}_i$
And since $\{\hat{s}_1, \cdots, \hat{s}_{n-1}\}$ is linearly independent, we know that

$$\forall i \frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w}_i - \hat{v}_i = 0 \tag{5}$$

$$\forall i \frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w}_i = \hat{v}_i \tag{6}$$

$$\hat{v} = \frac{\hat{v}^{supp}(0)}{\hat{w}^{supp}(0)} \cdot \hat{w} \tag{7}$$

$$\tag{8}$$

And thus $\hat{v}$ and $\hat{w}$ are linearly dependent. □

We can also prove that by including extra row vectors in a minimal linearly dependent set S, we increase the dependence rank of S only if the new set has more minimal linearly dependent subsets than the old one.

Theorem 3. *If $S = \{\hat{s}_1, \hat{s}_2, \cdots\}$ is a minimal linearly dependent set of row vectors of $n \times n$ matrix $M = \{\hat{m}_1, \hat{m}_2, \cdots\}$, and $\hat{k} = \hat{m}_k$, and $\forall s$ a proper subset of S, $\{\hat{k}\} \cup S$ is linearly independent, and $K = \{\hat{k} \cup S\}$ then* dep-mult$(K) = 1$

Proof. Clearly, there is a one dimensional set of dependencies of K, namely the dependencies of S.
Assume that $\hat{v}$ is a dependency of K, then we need to show that $\hat{v}(k) = 0$.
We have

$$0 = \hat{k} \cdot \hat{v}^{supp}(k) + \hat{s}_1 \cdot \hat{v}^{supp}(1) + \sum_{\substack{i \neq k \\ i \neq 1}} \hat{s}_i \cdot \hat{v}^{supp}(i) \tag{9}$$

Also, for some $\hat{w}$, a dependency of S, which we can choose such that $\hat{w}^{supp}(1) = \hat{v^{supp}}(1)$,

$$0 = \hat{v}^{supp}(1) \cdot \hat{s}_1 + \sum_{\substack{i \neq 1 \\ i \neq k}} \hat{w}^{supp}(i) \cdot \hat{s}_i \tag{10}$$

Subtracting (10) from (9) we get,

$$0 = 0 + \hat{k} \cdot \hat{v}(k) + \sum_{\substack{i \neq 1 \\ i \neq k}} (\hat{v}(i) - \hat{w}(i)) \cdot \hat{s}_i \tag{11}$$

Since we have $|S|$ terms in this sum, the vectors $(\hat{v}_i - \hat{w}_i) \cup \hat{k}$ must be linearly independent and $\hat{v}(k) = 0$. □

Theorem 4. *If S is a set of row vectors and we count minimal linearly depedent subsets $(s_1, s_2, \ldots s_l)$ of it as described above, then the dependencies of the s_i, namely $(\hat{v}_1, \hat{v}_2, \ldots \hat{v}_l)$ are linearly independent.*

Proof. Assume that for some (a_i) we have

$$0 = \sum a_i \cdot \hat{v}_i^{supp} \tag{12}$$

And now we wish to show that $(\forall j)\, a_j = 0$.
We know because of the way in which the s_i were generated that there is some w such that $\hat{m_w}$ is an element of s_i if and only if $i = j$. This is true because if it were not the case, then s_j would be a subset of the union of the other s_i, and we have prohibited this in our construction of the s_i. Let A be a matrix whose w row is equal to that of M, but contains 0 everywhere else. Thus:

$$0 = 0 * A = a_j \cdot \hat{v}_j^{supp} \cdot A + \sum_{i \neq j} a_i \cdot \hat{v}_i^{supp} \cdot A \tag{13}$$

$\hat{v}_j \cdot A \neq 0$ because otherwise $\{\hat{m_w}\} \subset s_j$ would be a linearly dependent set, and thus s_i would not be a minimal linearly dependent set.
Also, for $i \neq j$ we have $\hat{v}_i \cdot A = 0$ because $\hat{v}_i(w) = 0$ and all the non-w rows of A are 0 vectors.
Thus we can conclude that $a_j = 0$, and this must be true for all j.

□

Now proving our original theorem is trivial. We can start off considering any minimal linearly dependent set S in M, and start extending it with more rows of M. We know that every time we extend S, its dependence rank would be equal to the number of minimal linear dependent subsets of S that we would count. Thus by something equivalent to induction, the number of minimal linear dependent subsets of M that we would count would be equal to the dependence rank of M which is equal to the dimension of the Kernel of ϕ.