CW COMPLEXES OF THE FORM $K(G, n)$

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ABSTRACT. Given a group G we construct CW complexes that are $K(G, n)$ spaces. We begin by constructing a complex with $\pi_1 = F$, a free group. This approach is readily generalized to any group by considering a presentation of it and adding a cell for every relation. We then consider how to construct such a complex for $\pi_n = G$ with $\pi_i = 0$ for all $i < n$ and $n \geq 2$, using a similar approach but different theoretical techniques. In order to control the higher-order homotopy groups of these complexes, we prove a result about adding sequences of higher-dimensional cells. We finally apply this construction to prove a uniqueness theorem about general CW complexes of the form $K(G, n)$.

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1. BACKGROUND

Below are definitions and results from point-set and algebraic topology utilized in the discussion. It is assumed that the reader is familiar with basic group theory and homotopy theory. While sketches of proofs are provided for some of the theorems in this section, full proofs can be found in [1].

Definitions 1.1. Let D^n be the n-dimensional closed unit ball and S^{n-1} the $(n-1)$-dimensional unit sphere (or the boundary of D^n). A **CW complex** is a topological space X and a collection of continuous maps $\phi^n_\alpha : D^n \to X$, called **characteristic maps**, obeying the properties below. $e^n_\alpha = \phi^n_\alpha(\text{Int}(D^n))$ is called an n**-cell** of X.

(1)
$$X = \bigcup_{n \geq 0, \forall \alpha} e^n_\alpha$$

(2) $e^n_\alpha \cap e^m_\beta = \emptyset$ unless $\alpha = \beta$ and $n = m$, and $\phi^n_\alpha|_{\text{Int}(D^n)}$ is a homeomorphism.

(3) Let the n**-skeleton** be
$$X^n = \bigsqcup_{0 \leq i \leq n, \forall \alpha} e^i_\alpha$$

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Then for every $n \geq 1$ and all α, $\phi_\alpha^n(S^{n-1}) \subset X^{n-1}$.

A CW complex also has the following two properties (that explain the name 'CW').

(1) The closure of each cell of X is contained in finitely many other cells; this is known as *closure finiteness*.
(2) $A \subset X$ is open or closed if and only if $A \cap X^n$ is open or closed for all n; this is called the *weak topology*.

A **subcomplex** of a CW complex X is a union A of cells in X such that the closure of each cell is also contained in A; therefore, it is also a CW complex. A **CW pair** (X, A) is simply a pair of a CW complex X and subcomplex A.

Definition 1.2. Given a group G, a topological space X is a $K(G, n)$**, or Eilenberg-MacLane space**, if $\pi_i(X)$ is isomorphic to G for $i = n$ and trivial otherwise.

Example 1.3. S^1 is a $K(\mathbb{Z}, 1)$ since it is connected, $\pi_1(S^1) = \mathbb{Z}$, and the higher homotopy groups are trivial. On the other hand, S^n is not a $K(\mathbb{Z}, n)$ for $n \geq 2$ since its higher homotopy groups are not all trivial.

Theorem 1.4. ***(van Kampen)*** *Let X be a union of path-connected open sets A_α, each containing the basepoint x_0 with the intersections $A_\alpha \cap A_\beta$ and $A_\alpha \cap A_\beta \cap A_\gamma$ path connected. Let $i_{\alpha,\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ be the homomorphism induced by the inclusion map of $A_\alpha \cap A_\beta$ in A_α and let N be the normal subgroup generated by elements of the form $i_{\alpha,\beta}(\omega) i_{\beta,\alpha}^{-1}(\omega)$. Then $\pi_1(X)$ is isomorphic to $*_\alpha \pi_1(A_\alpha)/N$ where $*$ is the free product.*

Sketch of proof. The general motivation behind the proof is to decompose loops in X into a concatenation of loops each contained in some A_α. To prove surjectivity, it suffices to find such a decomposition for every loop in X, which is rather simple. To prove the injectivity, it must be shown that any decomposition is unique up to the given condition, which can be achieved with a rather technical argument involving splitting the decompositions into equivalence classes based on whether adjacent loops can be combined into one loop still in one A_α. □

Definition 1.5. A space X is n**-connected** if $\pi_i(X) = 0$ for all $i \leq n$. Likewise, a pair (X, A) is n-connected if $\pi_i(X, A) = 0$ for all $i \leq n$.

Theorem 1.6. ***(Cellular Approximation)*** *Given two CW complexes X and Y, every map $f : X \to Y$ is homotopic to a map $g : X \to Y$ with the property that $g(X^n) \subset Y^n$ for all n. A map g with this property is called a* ***cellular map****. Moreover, g may be taken to equal f on any subcomplex for which f is already cellular.*

Sketch of proof. The argument here proceeds by induction on the dimension of the skeleton; assuming f is cellular on X^{n-1}, we look at the cell of highest dimension e^k which is in the image of $f|_{X^{n-1} \cup e^n}$ (there are only finitely many cells that meet the image by closure finiteness). A technical argument is used to show that f can be deformed to miss a point y of e^k. But then it can be deformed to miss the entire cell: $D^k - \phi^{-1}(y)$, with ϕ the characteristic map of e^k, deformation retracts to S^{n-1}, and composing this homotopy with ϕ gives the desired deformation. This can be done for all the cells of dimension larger than n, and these homotopies can be performed simultaneously for all e^n in X^n. This homotopy can be extended to the entire space, completing the induction. Performing these homotopies sequentially, the n^{th} step during the time interval $[1 - 1/2^n, 1 - 1/2^{n+1}]$ gives the required deformation. (A subcomplex on which f is already cellular can be left alone the entire process.) □

Remark 1.7. This theorem is easily extended to a map $f : (X, A) \to (Y, B)$ of CW pairs. First deform the restriction $f : A \to B$ to a cellular map. This can be extended to a homotopy $X \to Y$, which in turn can be deformed to cellular map keeping the restriction to A constant. This produces the desired homotopy.

Theorem 1.8. ***(Excision)*** *Let X be a CW complex which can be decomposed as a union of subcomplexes A and B with their intersection C nonempty and connected. Then if (A, C) is m-connected and (B, C) is n-connected, the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < n + m$.*

Theorem 1.9. ***(Whitehead)*** *Let X and Y be CW complexes. If a map $f : X \to Y$ induces isomorphisms in each homotopy group, then it is a homotopy equivalence.*

2. CW complexes with $\pi_1 = G$

Our first goal is to construct a CW complex with a given fundamental group. The arguments here will not generalize to higher homotopy groups. Later arguments will rely heavily on homotopy groups of pairs, which are not groups for $n = 1$. Moreover, the fact that higher homotopy groups are abelian is false for the fundamental group. Thus this special case is indispensable. First we prove a technical lemma which will come in handy later.

Lemma 2.1. *Let X be a 1-dimensional CW complex with basepoint x_0 and Y a CW complex obtained by attaching 2-cells e^2_α to X via maps $\phi_\alpha : S^1 \to X$. Let N be the normal subgroup generated by elements of the form $\gamma_\alpha \phi_\alpha \overline{\gamma_\alpha}$ where γ_α is the path from x_0 to the basepoint of the loop ϕ_α and $\overline{\gamma_\alpha}(t) = \gamma_\alpha(1 - t)$. Then $\pi_1(Y)$ is isomorphic to $\pi_1(X)/N$.*

Proof. We first construct a space Z from Y by attaching for each α a strip $S_\alpha = [0, 1] \times [0, 1]$ as follows: identify $[0, 1] \times \{0\}$ and γ_α, identify $\{1\} \times [0, 1]$ and an arc on e^2_α with $(1, 0)$ the basepoint of the loop ϕ_α and the rest contained in the interior of the cell, and finally identify $\{0\} \times [0, 1]$ with the same edge for all the other S_α. The edge $[0, 1] \times \{1\}$ is left unidentified, and so Z can be deformed to Y by retracting each S_a to the union of its edges $(\{1\} \times [0, 1]) \cup ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$ and then retracting the line $\{0\} \times [0, 1]$ (identified in all S_a) to the basepoint x_0. Choose in each e^2_α a point y_α not on the arc described by the $\{1\} \times [0, 1]$ edge of S_α. Then we set $A = Z - \bigcup_\alpha \{y_\alpha\}$ and $B = Z - X$.

We then have that A deformation retracts onto X by the homotopy described above followed by the linear deformation retraction of $e^2_\alpha - y_\alpha$ to the loop ϕ_α for each α. B, on the other hand, is contractible. $A \cup B$ covers Z, so by van Kampen's theorem, $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(A) = \pi_1(X)$ with the normal subgroup which is the image of the homomorphism induced by the inclusion $A \cap B \to A$.

Consider next the cover of $A \cap B$ by the sets

$$A_\alpha = A \cap B - \bigcup_{\beta \neq \alpha} e^2_\beta.$$

Each of these consists of a union of S_β with the edge $\{0\} \times [0, 1]$ identified and all the others free, which all retract to the $\{0\} \times [0, 1]$ edge of S_α, which in turn

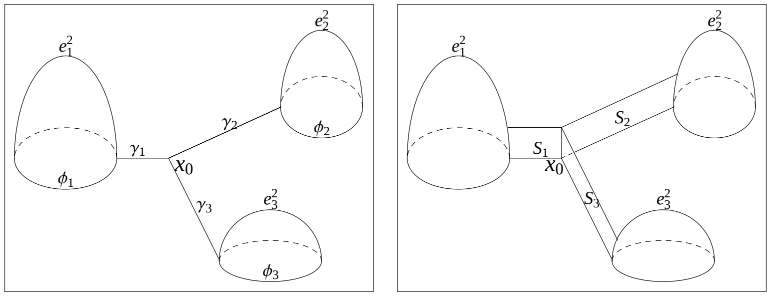

FIGURE 1. The left diagram illustrates the relevant paths in the space Y while the right one shows how the S_α are added to Y to create Z in the proof of Lemma 2.1.

retracts to the arc on $e^2_\alpha - y_\alpha$. But $e^2_\alpha - y_\alpha$ retracts to a loop, and so $\pi_1(A_\alpha) = \mathbb{Z}$ and is generated by the loop ϕ_α. Applying van Kampen's theorem again, we have any intersection of several of the A_α is contractible (just a union of the S_α with one edge identified) and so

$$\pi_1(A \cap B) = \langle \gamma_\alpha \phi_\alpha \overline{\gamma_\alpha} \rangle = N.$$

The homomorphism induced by inclusion of $A \cap B$ in A is injective, so it follows that $\pi_1(Y) = \pi_1(X)/N$. □

We proceed by constructing, using van Kampen's theorem, a CW complex with fundamental group the free group on generators g_α.

Construction 2.2. Take $\bigvee_\alpha (S^1_\alpha, x_\alpha)$, the disjoint union of circles S^1_α with basepoints x_α identified. This is a CW complex; in particular, it is obtained by the characteristic maps

$$\phi^0(D^0) = \{x_0\}, \phi^1_\alpha(\mathrm{Int}(D^1)) = S^1_\alpha - \{x_0\}, \phi^1_\alpha(\partial D^1) = \{x_0\}$$

Theorem 2.3. *$\pi_1(\bigvee_\alpha S^1_\alpha) = \langle g_\alpha \rangle$, the free group on generators g_α.*

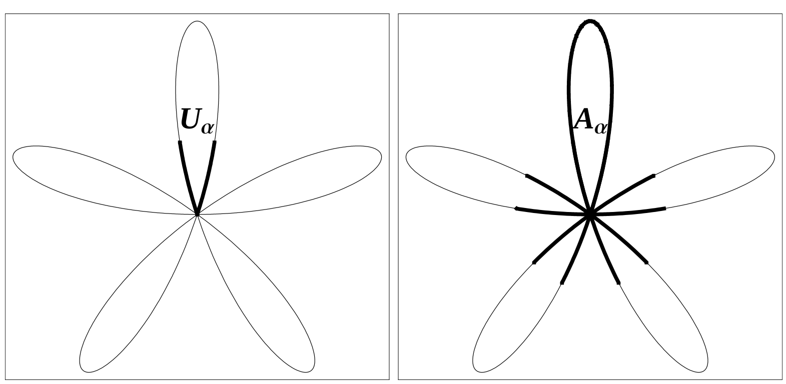

FIGURE 2. These show the definitions of U_α and A_α in the proof of Theorem 2.3.

Proof. For each α, let U_α be an open neighborhood of x_α in S^1_α and $A_\alpha = S^1_\alpha \bigvee_{j \neq \alpha} U_j$. Then each A_α is open under the weak topology and the intersections of two or more A_α is just $\bigvee_\alpha U_\alpha$ and so is path connected. Moreover, any loop based at x_0 in $\bigvee_\alpha U_\alpha$ deformation retracts to the identity loop, meaning that in the van Kampen theorem $i_{\alpha,\beta}(\omega)$ is trivial for any ω and so $N = 0$. Then $\pi_1(X)$ is isomorphic to $*_\alpha \pi_1(A_\alpha)$. But A_α deformation retracts to S^1_α, which has the fundamental group $\mathbb{Z}$; the free product gives the free group $\langle g_\alpha \rangle$. □

We now proceed to generalize this result to any group G by adding a 2-cell for each relation on G and applying the lemma.

Construction 2.4. Let $\langle g_\alpha | r_\beta \rangle$ be a presentation of G. First, let X be the CW complex from Construction 2.2 with $\pi_1(X)$ the free group on generators g_α. Each r_β is a reduced word of the g_α's, so let $\|r_\beta\|$ be the length of the word. Then let Y be the union of Y with the 2-cells e^2_β attached via the maps $\psi_\beta : S^1 \to X$ created by first parametrizing S_1 with the map $s \mapsto (\cos(2\pi s), \sin(2\pi s))$. Now let ψ_β be the map which sends each interval $[\frac{n}{\|r_\beta\|}, \frac{n+1}{\|r_\beta\|}]$ onto the loop corresponding to the $(n+1)$st letter in r_β, with the endpoints of the interval sent to x_0.

For example, take the group $\langle \alpha_1, \alpha_2, \alpha_3 | \alpha_1\alpha_2\alpha_3 \rangle$. Applying Construction 2.4, we get the attaching map in Figure 3. The resulting CW complex will be a wedge of three circles with a single disk having its boundary identified according to the attaching map.

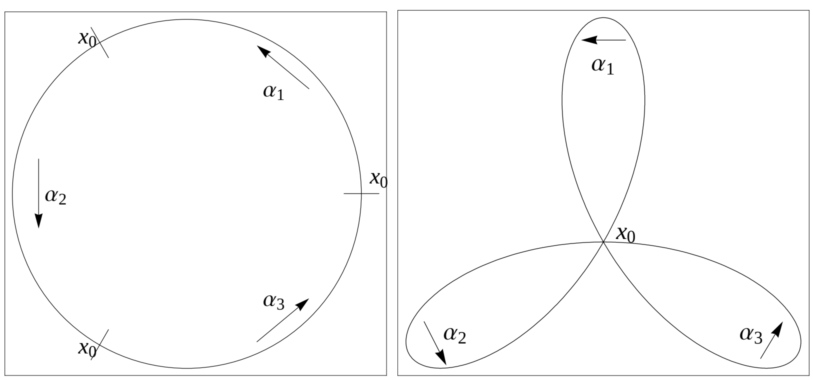

FIGURE 3. This gives an attaching map of a disk (left) to a wedge of three circles (right).

Theorem 2.5. *The CW complex Y has fundamental group G.*

Proof. By Lemma 2.1, $\pi_1(Y)$ is the quotient of the free group on the g_α's by N. But N is the smallest normal subgroup generated by the words r_β, which means, by definition of presentation, $\pi_1(Y) = G$. □

3. CW COMPLEXES WITH $\pi_n = G$, $n \geq 2$

We now turn our attention to constructing CW complexes with $G = \pi_n$, $n \geq 2$, which are $(n-1)$-connected. Note first that G must be abelian, as π_n is always abelian. We will use the cellular approximation theorem to provide a construction analogous to the above. First we prove a different technical result which will end up being used in place of Lemma 2.1.

Lemma 3.1. *Let (X, A) be an r-connected CW pair with A an s-connected subcomplex of X. Then $\pi_n(X, A) \cong \pi_n(X/A)$ for $n \leq r + s$.*

Proof. Construct $X \cup CA$ by attaching the cone CA along A to X. We now divide the proof into three steps:

(1) Note that CA is contractible, so the map $X \cup CA \to (X \cup CA)/CA = X/A$ is a homotopy equivalence and induces an isomorphism in π_n.

(2) Next, consider the long exact sequence for $(X \cup CA, CA)$:

$$\pi_n(CA) \xrightarrow{i_*} \pi_n(X \cup CA) \xrightarrow{j_*} \pi_n(X \cup CA, CA) \xrightarrow{\partial} \pi_{n-1}(CA)$$

with i_* and j_* the maps induced by inclusions and ∂ the boundary map. But CA is contractible, so the first and last groups are trivial. Then

$$0 = \text{Im } i_* = \text{Ker } j_*,$$

meaning j_* is injective, and

$$\text{Im } j_* = \text{Ker } \partial = \pi_n(X \cup CA, CA),$$

meaning j_* is an isomorphism.

(3) Consider the long exact sequence for (CA, A):

$$0 = \pi_{k+1}(CA) \xrightarrow{j_*} \pi_{k+1}(CA, A) \xrightarrow{\partial} \pi_k(A) \xrightarrow{i_*} \pi_k(CA) = 0$$

The outside groups are trivial since CA is contractible. We thus have an isomorphism between $\pi_{k+1}(CA, A)$ and $\pi_k(A)$, meaning that as A is s-connected, (CA, A) is $(s + 1)$-connected. It is given that (X, A) is r-connected. We can then apply the excision theorem on the inclusion

$$(X, A) \to (X \cup CA, CA)$$

to get $\pi_n(X, A) \cong \pi_n(X \cup CA, CA)$ for $n \leq r + s$.

Now we have the following groups isomorphic:

$$\pi_n(X/A) \overset{(1)}{\cong} \pi_n(X \cup CA) \overset{(2)}{\cong} \pi_n(X \cup CA, CA) \overset{(3)}{\cong} \pi_n(X, A)$$

Which completes the proof. □

Now for the actual construction. In parallel with the previous section, we begin with free abelian groups and then move on to any presented abelian group.

Construction 3.2. Let $X = \bigvee_\alpha S^n_\alpha$, $n \geq 2$, with basepoints x_α identified to x_0. This is clearly an n-dimensional CW complex.

Theorem 3.3. *Construction 3.2 has $\pi_n(X)$ the free abelian group on free generators the homotopy classes of the inclusions $S^n_\alpha \to X$ and is $(n-1)$-connected.*

Proof. Any map $h : S^i \to X$ can be deformed to a cellular map $h' : S^i \to X^i$. But $X^i = x_0$ for $i < n$, meaning that $\pi_i(X)$ is trivial and so X is $(n - 1)$-connected.

Consider first the case of finitely many n-cells in X. We can regard X as the n-skeleton of $\prod_\alpha S^n_\alpha = Y$, for Y has one 0-cell, an n-cell for each α, and then more cells of dimensions that are multiples of n. From this it is also follows that (Y, X) is $(2n - 1)$-connected: take any map

$$f : (D^k, S^{k-1}, s_0) \to (Y, X, x_0).$$

By cellular approximation this is homotopic to a cellular map

$$g : (D^k, S^{k-1}, s_0) \to (Y^k, X^{k-1}, x_0).$$

But for $k \leq 2n-1$ $Y^k \subset X$, so by the definition of a relative homotopy group,

$$0 = \pi_k(Y^k, X, x_0) \cong \pi_k(Y, X, x_0).$$

Now consider the exact sequence of (Y, X):

$$\pi_{n+1}(Y, X) \xrightarrow{\partial} \pi_n(X) \xrightarrow{i_*} \pi_n(Y) \xrightarrow{j_*} \pi_n(Y, X)$$

But the first and fourth groups are trivial by the above, so we have

$$\pi_n(X) \cong \pi_n(Y) \cong \bigoplus_\alpha \pi_n(S^n_\alpha),$$

which is the free abelian group with generators the homotopy classes of the inclusions $S^n_\alpha \to Y$, which are by cellular approximation the same as the homotopy classes of the inclusions $S^n_\alpha \to X$.

Now consider the general case. Let $\Phi : \bigoplus_\alpha \pi_n(S^n_\alpha) \to \pi_n(X)$ be the homomorphism induced by the inclusions $S^n_\alpha \to X$. Now, take any $f : S^n_\alpha \to X$. Then the compact image of f is contained in the wedge of finitely many of S^n_α (by closure finiteness of CW complexes), so by applying the finite case the homotopy class of f is in $\bigoplus_\alpha \pi_n(S^n_\alpha)$ and Φ is surjective. On the other hand, the compact image of a nullhomotopy is also contained in the wedge of finitely many of S^n_α, and so the finite case implies that Φ is an injection and so an isomorphism. □

Construction 3.4. Given an abelian group G with presentation $\langle g_\alpha | r_\beta \rangle$, construct the complex C as follows: take C^n to be, from Construction 3.2, $\bigvee_\alpha S^n_\alpha$, with one n-cell for each generator and with basepoint x_0. Then for each relation, let $g_{\beta,1} g_{\beta,2} ... g_{\beta,k}$ be the trivial word it describes. Let $U_{\beta,1}, U_{\beta,2}, ..., U_{\beta,k}$ be open sets of the form

$$U_{\beta,i} = \{x \in S^n | \|x - x_i\| < \epsilon_i, x_i \in S^n\},$$

choosing x_i and ϵ_i to make them disjoint, $\gamma_{\beta,i} : U_{\beta,i}/\partial U_{\beta,i} \to S^n$ homotopy equivalences, and $\psi_{\beta,i} : S^n_{\beta,i} \to C^n$ characteristic maps. Then add $(n+1)$-cells by the following attaching maps $\varphi_\beta : S^n \to C^n$:

$$\varphi_\beta(y) = \begin{cases} x_0 & y \in S^n - \bigcup_{i=1}^k U_{\beta,i} \\ \psi_{\beta,i}\left(\gamma_{\beta,i}(y)\right) & y \in U_{\beta,i} \end{cases}$$

These are well-defined since the sets $U_{\beta,i}$ are disjoint. C, then, is the union of C^n with these $(n+1)$-cells.

For example, take the abelian group $\langle \alpha_1, \alpha_2 | \alpha_1 \alpha_2^2 \rangle$. The complex created as in Construction 3.4 for $n = 2$ consists of a wedge of two 2-spheres and a 2-ball glued on with the attaching map shown in Figure 4. Note that unlike Construction 2.4, the order in which the attachment is performed is no longer specified with arrows since the group is abelian.

Theorem 3.5. *The CW complex C from Construction 3.4 is $(n-1)$-connected and $\pi_n(C) = G$.*

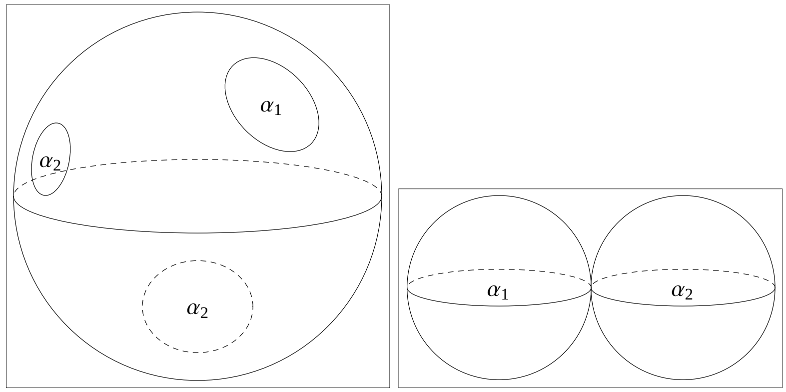

FIGURE 4. This is an attaching map of a 2-ball (left) to a wedge of two 2-spheres. The region of the ball which is unmarked is mapped to the basepoint of the wedge.

Proof. The $(n-1)$-connectedness is obvious from cellular approximation, as C contains no cells of dimensions between 0 and n.

First note that (C, C^n) is n-connected by the same argument as in the proof of Theorem 3.2. C^n is $(n-1)$-connected since X is. Then, by Lemma 3.1, $\pi_{n+1}(C, C^n) \cong \pi_{n+1}(C/C^n)$. Then consider the following exact sequence:

$$\pi_{n+1}(C, C^n) \xrightarrow{\partial} \pi_n(C^n) \xrightarrow{i_*} \pi_n(C) \xrightarrow{j_*} \pi_n(C, C^n) = 0$$

Now, $\pi_n(C^n) = \langle g_\alpha \rangle$ by definition. On the other hand,

$$\pi_{n+1}(C/C^n) = \pi_{n+1}(\bigvee_\beta S_\beta^{n+1}),$$

which, by Theorem 3.2, is the free group generated by the homotopy classes of the characteristic maps $S_\beta^{n+1} \to C/C^n$. But the boundary map ∂ takes each of these to the homotopy class of the corresponding attaching map φ_β, which is by its definition $r_\beta = g_{\beta,1}...g_{\beta,k}$. Then we have that $\text{Im } i_* = \text{Ker } j_* = \pi_n(C)$ and $\text{Ker } i_* = \text{Im } \partial \cong \langle r_\beta \rangle$, meaning $\pi_n(C) \cong \langle g_\alpha | r_\beta \rangle = G$. □

4. Constructing CW Complexes of the form $K(G, n)$

It only remains to be shown, given an $(n+1)$-dimensional CW complex X, how to add cells of dimension greater than $n+1$ to X to cancel out $\pi_i(X)$ for $i > n$ without affecting the lower homotopy groups. We do this in the following construction by using the cellular approximation theorem repeatedly.

Construction 4.1. Given an $(n+1)$-dimensional CW complex X, let $\varphi_\alpha : S^{n+1} \to X$ be maps whose homotopy classes generate $\pi_{n+1}(X)$ (from a technical perspective, these can be defined analogously to the maps φ_β in Construction 3.4). Let Y_{n+1} be created by adding $(n+2)$- cells via attaching maps φ_α to X. Proceeding inductively, define Y_{n+k+1} by substituting Y_{n+k} for X in the above. Let $Y = \bigcup_{k=1}^{\infty} Y_{n+k}$.

Theorem 4.2. *In Construction 4.1, $\pi_i(Y)$ is trivial for $i > n$ and isomorphic to $\pi_i(X)$ for $i \leq n$.*

Proof. Begin by considering Y_{n+1}. For $i \leq n$, the inclusion map $X \to Y_{n+1}$ induces the isomorphisms $\pi_i(Y_{n+1}) \cong \pi_i(X)$; to see this, consider the following exact sequence:

$$\pi_{i+1}(Y_{n+1}, X) \xrightarrow{\partial} \pi_i(X) \xrightarrow{i_*} \pi_i(Y_{n+1}) \xrightarrow{j_*} \pi_i(Y_{n+1}, X)$$

In the proof of Theorem 3.3, we showed that (C, C^n) is n-connected by cellular approximation of pairs, and as X is the $n+1$ skeleton of Y_{n+1} the first and fourth groups of the sequence are trivial. The induced isomorphism follows.

Now take any map $S^{n+1} \to Y_{n+1}$; by cellular approximation this is homotopic to a cellular map $S^{n+1} \to X$. But the homotopy class of the latter is generated by the homotopy classes of the φ_α, each of which is trivial by construction. Thus $\pi_{n+1}(Y_{n+1}) = 0$.

Finally, apply the above inductively, setting $X = Y_{n+k}$. Then Y_{n+k+1} has $\pi_{n+k+1}(Y_{n+k+1})$ trivial, and by taking infinite union we cancel all the homotopy groups higher than n. □

We are now in a position to attack the central question. Combining the previous constructions, we can create a CW complex of the form $K(G, n)$ for any G and n.

Construction 4.3. Given group G and $n \geq 1$, construct the CW complex X using Construction 3.4 or 2.4, depending on whether $n = 1$. Then apply Construction 4.1 to X to get a CW complex Y. Then Y has, by theorems 4.2 and 3.4 or 2.4, only one nontrivial homotopy group; specifically, $\pi_n(Y) = G$.

5. Uniqueness

A consequence of the construction is that CW complexes of the form $K(G, n)$ turn out to be unique in a rather strong sense, as we show below using Whitehead's theorem.

Theorem 5.1. *Two CW complexes X and Y of the form $K(G, n)$ for the same G and n, $n \geq 1$, are homotopy equivalent.*

Proof. We assume without loss of generality that X is created via Construction 4.3, for homotopy equivalence is an equivalence relation and by transitivity the more general case will follow.

We first show that for any homomorphism $\phi : \pi_n(X^{n+1}) \to \pi_n(Y)$, there exists a map $f : X^{n+1} \to Y$ that induces ϕ. Note that X^{n+1} is of the form

$$\bigvee_\alpha S^n_\alpha \cup \bigcup_\beta (D^{n+1}_\beta - \partial D^{n+1}_\beta)$$

(i.e., of the form of Construction 3.5 or 2.5). Let $f(x_0) = y_0$ where x_0 is the natural basepoint of X^{n+1} and y_0 is some basepoint in Y (Y is path-connected since it is a $K(G, n)$). Next, for each α let f map S^n_α to Y via a map $f_\alpha : S^n \to Y$ in the homotopy class $\phi([i_\alpha])$, where i_α is the inclusion $S^n_\alpha \to X^{n+1}$. Then we have $f_*([i_\alpha]) = \phi([i_\alpha])$, and since by Theorem 3.3 (or 2.3) $[i_\alpha]$ generate $\pi_n(X^n)$, $f_*([\sigma]) = \phi([\sigma])$ for all $\sigma : S^n \to X^n$.

For each $(n+1)$ cell we consider the composition of the attaching map $\varphi_\beta : S^n \to X^n$ with f on X^n as defined above. Now, φ_β is nullhomotopic in X^{n+1}, so we need to show that $f \circ \varphi_\beta$ is nullhomotopic in Y. By the preceeding paragraph, $[f \circ \varphi_\beta] = f_*([\varphi_\beta]) = \phi([\varphi_\beta])$. However, $[\varphi_\beta] = 0$, so $\phi([\varphi_\beta]) = 0$ since ϕ is a homomorphism. Thus $f \circ \varphi_\beta$ is indeed nullhomotopic in Y. We can then extend f

to each cell of X^{n+1}, obtaining the extension $f : X^{n+1} \to Y$. We then have $f_* = \phi$ as the $[i_\alpha]$ generate $\pi_n(X^{n+1})$ by cellular approximation.

Setting ϕ to be an isomorphism between $\pi_n(X^{n+1}) \cong \pi_n(X) \cong G$ and $Y \cong G$, we get $f : X^{n+1} \to Y$ that induces said isomorphism. We extend it inductively using the same argument as provided above for the $(n+1)$ cells: extending f to each cell of X^{n+k+1} is possible because the attaching map of the $(n+k+1)$-cell composed with f on X^{n+k} is nullhomotopic as $\pi_{n+k}(Y) = 0$. Then $f : X \to Y$ induces an isomorphism in every homotopy group and by Whitehead's theorem is a homotopy equivalence. □

References

[1] A. Hatcher. Algebraic Topology. Cambridge University Press. 2002.