representation theory of $SL_2$ over a p-adic field: the principal series

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abstract. a result concerning the irreducibility (and reducibility) of the principal series of representations of $SL_2$ over a p-adic field is presented (see theorem 4.16). an overview of the structure of p-adic fields precedes the demonstration, as does the introduction of certain special functions on these fields.

1. introduction

let $K$ be a locally compact hausdorff nondiscrete topological field. the principal series of representations of $SL_2 (K)$ is a collection of continuous unitary representations of $SL_2 (K)$ on $L^2 (K)$. its properties are well understood. we give here a brief analysis of the irreducibility (and reducibility) of this series in the special case where $K$ is a p-adic field and $p$ is odd. the computations and techniques employed follow [1], [2], and [3].

section 2 contains necessary results concerning p-adic fields. the account is terse and proofs are not provided. readers desirous of further information should consult [3].

section 3 treats certain special functions on p-adic fields. proofs can be found in [3].

section 4 introduces the principal series of representations of $SL_2$ over a p-adic field $F$. a convenient unitarily equivalent representation is thoroughly studied. the paper concludes with the main result: theorem 4.16.

2. p-adic fields

let $p$ be an odd prime and $F$ a finite algebraic extension of $\mathbb{Q}_p$. denote the additive and multiplicative groups of $F$ by $F^+$ and $F^\times$, respectively. let $dx$ be a fixed haar measure on $F^+$.

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Definition 2.1. Define $| \cdot | : F \rightarrow \mathbb{R}_{\geq 0}$ such that

(i) $| \cdot |$ is a non-Archimedean norm on $F$;
(ii) For any $a \in F^\times$, $d(ax) = |a| dx$.

Note 2.2. There exists exactly one such function.

Observe that $dx/|x|$ is a Haar measure on $F^\times$. Give $F$ the topology induced by this norm. Define subsets $\mathcal{O}$ and $\mathfrak{p}$ of $F$ by

$$\mathcal{O} = \{ x : |x| \leq 1 \}$$
$$\mathfrak{p} = \{ x : |x| < 1 \}.$$

$\mathcal{O}$ is the maximal compact subring of $F$; $\mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$. Moreover, $\mathfrak{p}$ is principal. Let $\tau$ be a generator for $\mathfrak{p}$. $\mathcal{O}/\mathfrak{p}$ is a field with $q$ elements, where $q$ is some power of $p$. It can be shown that $|\tau| = q^{-1}$ and for all $a \in F^\times$, $|a| = q^n$, for some $n \in \mathbb{Z}$.

Let $U = \{ x : |x| = 1 \}$ be the group of units in $F^\times$. $U$ contains an element $\epsilon$ such that

(i) $\epsilon$ has order $q - 1$
(ii) $F^\times = (F^\times)^2 \bigcup (-\tau)(F^\times)^2 \bigcup (-\epsilon \tau)(F^\times)^2 \bigcup (\epsilon)(F^\times)^2$.

Definition 2.3. Let $F_1 = F^+, F^\times$, or $U$. A character of $F_1$ is a continuous homomorphism

$$\psi : F_1 \rightarrow \mathbb{T},$$

where $\mathbb{T}$ is the group of complex numbers with norm one. Denote the set characters of $F_1$ by $\hat{F}_1$.

Define the following sets:

$$p^n = \{ x \in F : |x| \leq q^{-n} \}, \quad n \in \mathbb{Z}$$
$$U_n = \{ x \in U : |1 - x| \leq q^{-n} \} = 1 + p^n, \quad n \geq 1.$$

If $\chi \in \hat{F}^\times$, there exists $s \in \mathbb{R}$ with

$$\frac{-\pi}{\ln q} < s \leq \frac{\pi}{\ln q}$$

and $\chi^* \in \hat{U}$ such that

$$\chi(x) = |x|^i \chi^*(u)$$

for all $x \in F^\times$, where $|x| = q^{-n}$ and $xq^{-n} = u$. For any nontrivial $\chi^* \in \hat{U}$, there exists $l \geq 1$ such that $\chi^*$ is trivial on $U_l$ and nontrivial on $U_{l-1}$. For any nontrivial $\psi \in \hat{F}^+$, there is an $m \in \mathbb{Z}$ such that $\psi$ is trivial on $p^m$ and nontrivial on $p^{m-1}$.

The following definitions are thus sensible.

Definition 2.4. Let $\chi \in \hat{F}^\times$. $\chi$ is said to be unramified if $\chi^*$ is the trivial character, and ramified of degree $l$ otherwise, where $\chi^*$ and $l$ are as above.

Definition 2.5. Let $\psi \in \hat{F}^+$ be nontrivial. Then $p^m$ is said to be the conductor of $\psi$, where $m$ is as above.
The three characters of degree two in \( \hat{F}^\times \) figure prominently below. Denote them by \( \text{sgn}_\epsilon, \text{sgn}_\tau, \) and \( \text{sgn}_{\epsilon\tau}, \) where

\[
\text{sgn}_\epsilon (x) = \begin{cases} 
1 & \text{if } x \in (F^\times)^2 \cup (\epsilon)(F^\times)^2 \\
-1 & \text{otherwise}
\end{cases}
\]

and

\[
\text{sgn}_\theta (x) = \begin{cases} 
1 & \text{if } x \in (F^\times)^2 \cup (-\theta)(F^\times)^2 \\
-1 & \text{otherwise}
\end{cases}
\]

for \( \theta = \tau \) or \( \epsilon\tau. \)

Fix \( \psi \in \hat{F}^+ \) with conductor \( O. \) For all \( u \in F, \) define \( \psi_u \in \hat{F}^+ \) by

\[ \psi_u (x) = \psi (ux) \]

for all \( x \in F. \)

**Definition 2.6.** Let \( f \in L^1(F). \) The Fourier transform of \( f, \mathcal{F} f = \hat{f}, \) is defined by

\[ \hat{f} (u) = \int_F f(x) \psi_u (x) \, dx \]

for all \( u \in F. \)

\( \mathcal{F} \) restricted to \( L^1(F) \cap L^2(F) \) extends to an isometry of \( L^2(F). \) Denote this extension by \( \mathcal{F} \) as well. Without loss of generality, assume that \( dx \) is normalized so that \( \hat{\hat{f}} (x) = f (-x) \) for all \( f \in L^2(F) \) and \( x \in F. \)

**Definition 2.7.** The Schwarz-Bruhat space of \( F, \mathcal{S}, \) is the set of all complex-valued, compactly supported, locally constant functions on \( F. \)

**Theorem 2.8.** \( \mathcal{S} \) is dense in \( L^p(F), \) for \( 1 \leq p < \infty. \)

**Theorem 2.9.** The map defined by \( \varphi \mapsto \hat{\varphi} \) for all \( \varphi \in \mathcal{S} \) is a bijection of \( \mathcal{S} \) onto itself.

3. **Special Functions on \( p \)-adic Fields**

**Note 3.1.** The special functions below are vital to section 4. This section characterizes them more fully than required there, as they are also of independent interest.

**Definition 3.2.** Let \( f : F \rightarrow \mathbb{C} \) be locally integrable, except (possibly) at 0. For all \( n \geq 0, [f]_n : F \rightarrow \mathbb{C} \) is defined by

\[
[f]_n (x) = \begin{cases} 
f(x) & \text{if } q^{-n} \leq x \leq q^n \\
0 & \text{otherwise}
\end{cases}
\]

If the limit in (*) exists, define the principal value integral of \( f \) by

\[
P.V. \int_F f(x) \, dx = \lim_{n \rightarrow \infty} \int_F [f]_n (x) \, dx.
\]

**Theorem 3.3.** Let \( f \in L^2(F). \) Suppose

\[
P.V. \int_F f(x) \, \psi_u (x) \, dx
\]

exists for almost all \( u \in F. \) Then

\[
\hat{f} (u) = P.V. \int_F f(x) \, \psi_u (x) \, dx
\]

for almost all \( u \in F. \)
Note 3.4. This is part of a result known as Plancherel’s theorem.

Definition 3.5. Let $\chi$ be a nontrivial character of $F^\times$. Then
$$\Gamma (\chi) = \Gamma (\chi^* | \cdot |^s) = \Gamma_{\chi^*} (s)$$
is defined as follows:
(i) if $\chi$ is ramified,
$$\Gamma (\chi) = \Gamma_{\chi^*} (s) = \text{P.V.} \int_F \psi (x) \chi (x) \frac{dx}{|x|}.$$  
(ii) if $\chi$ is unramified and $\Re (s) > 0$,
$$\Gamma (\chi) = \Gamma_1 (s) = \text{P.V.} \int_F \psi (x) \chi (x) \frac{dx}{|x|}. \quad (**)$$
(iii) if $\chi$ is unramified and $\Re (s) \leq 0$,
$$\Gamma (\chi) = \Gamma_1 (s)$$
is given by the analytic continuation of (**)..

Note 3.6. The above is well-defined. It is known as the gamma function. See [3].

Definition 3.7. Define $q' \in \mathbb{R}$ by
$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Theorem 3.8. Let $\chi = \chi^* | \cdot |^s$ be a nontrivial multiplicative character on $F^\times$.
(i) If $\chi$ is ramified of degree $h \geq 1$,
$$\Gamma (\chi) = \Gamma_{\chi^*} (s) = C_{\chi^*} q^{h(s-1/2)},$$
where
$$C_{\chi^*} = \Gamma_{\chi^*} (1/2).$$
Note that
$$|C_{\chi^*}| = 1$$
and
$$C_{(\chi^*)^{-1} \chi^*} = \chi^* (-1).$$
(ii) If $\chi$ is unramified,
$$\Gamma (\chi) = \Gamma_1 (s) = \frac{1 - q^{s-1}}{1 - q^{-s}}.$$  
$\Gamma_1 (s)$ has a simple pole at $s = 0$ with residue
$$\frac{1}{q' \ln q},$$
$1/ (\Gamma_1 (s))$ has a simple pole at $s = 1$ with residue
$$\frac{-1}{q' \ln q}.$$  
The only singularity of $\Gamma_1 (s)$ occurs at $s = 0$; the only zero occurs at $s = 1$. 
(iii) If \( \chi(x) \neq |x| \), then
\[
\Gamma_{\chi^*}(s) = \chi^*(-1) \Gamma_{(\chi^*)^{-1}}(\overline{s})
\]
\[
\Gamma_{\chi^*}(s) \Gamma_{(\chi^*)^{-1}}(1-s) = \chi^*(-1).
\]
Hence,
\[
\Gamma_{\chi^*}(s) \Gamma_{(\chi^*)^{-1}}(1-\overline{s}) = 1.
\]

**Definition 3.9.** For \( \chi \in \hat{F}^\times \) and \( u, v \in F^\times \), define the **Bessel function** \( J_\chi(u, v) \) as follows:
\[
J_\chi(u, v) = \text{P.V.} \int_F \psi \left( ux + \frac{v}{x} \right) \chi(x) |x|^{-1} \, dx.
\]

**Note 3.10.** The Bessel function is well-defined. See [3].

**Lemma 3.11.** Let \( u, v \in F^\times \). Then
(i) \( J_\chi(u, v) = J_{\chi^{-1}}(v, u) \).
(ii) \( \chi(u) J_\chi(u, v) = \chi(v) J_\chi(v, u) \).
(iii) \( J_\chi(u, v) = J_{\chi^{-1}}(-u, -v) = \chi(-1) J_{\chi^{-1}}(u, v) \).
(iv) If \( \chi(-1) = 1 \) (resp. \(-1\)), then \( J_\chi(u, u) \) is real-valued (resp. pure imaginary-valued).

**Definition 3.12.** Let \( k \in \mathbb{Z}_{>0}, \chi \in \hat{F}^\times \), and \( v \in F^\times \). Then
\[
F_\chi(k, v) = \int_{|x|=q^k} \psi(x) \psi \left( \frac{v}{x} \right) \chi(x) |x|^{-1} \, dx.
\]

**Lemma 3.13.** Suppose that \( |v| = q^m \) and \( 1 \leq k < m \).
(i) If \( \chi \) is unramified, then \( F_\chi(k, v) \neq 0 \) if and only if \( m \) is even and \( k = m/2 \).
(ii) If \( \chi \) is ramified of degree \( h \geq 1 \), then \( F_\chi(k, v) \neq 0 \) if and only if one of the following holds:
   (a) \( m \) is even, \( m \geq h \), and \( k = m/2 \)
   (b) \( m < 2h < 2m \) and \( k = h \) or \( k = m - h \).

**Theorem 3.14.** If \( \chi \in \hat{F}^\times \) is unramified, \( \chi \neq 1 \), and \( u, v \in F^\times \), then
\[
J_\chi(u, v) = \begin{cases} 
\chi(v) \Gamma(\chi^{-1}) + \chi^{-1}(u) \Gamma(\chi) & |uv| \leq q \\
\chi^{-1}(u) F_\chi \left( \frac{m}{2}, uv \right) & |uv| = q^m, m > 1, m \text{ even} \\
0 & |uv| = q^m, m > 1, m \text{ odd}
\end{cases}
\]
If \( \chi \equiv 1 \), then the first case becomes
\[
J_1(u, v) = \frac{m + 1}{q^2} - \frac{2}{q} = \frac{1}{q^2} \left[ -\ln |uv| + 1 \right] - \frac{2}{q}
\]
for \( |uv| = q^{-m} \leq q \). The other cases remain valid as stated.

**Theorem 3.15.** If \( \chi \in \hat{F}^\times \) is ramified of degree \( h \geq 1 \), and \( u, v \in F^\times \), then
\[
J_\chi(u, v) = \begin{cases} 
\chi(v) \Gamma(\chi^{-1}) + \chi^{-1}(u) \Gamma(\chi) & |uv| \leq q^h \\
\chi^{-1}(u) F_\chi(h, uv) + F_\chi(m-h, uv) & |uv| = q^m, h < m < 2h \\
\chi^{-1}(u) F_\chi \left( \frac{m}{2}, uv \right) & |uv| = q^m, m \geq 2h, m \text{ even} \\
0 & |uv| = q^m, m > 2h, m \text{ odd}
\end{cases}
\]
4. The Principal Series of Representations of $SL_2(F)$

**Theorem 4.1.** Let $\chi \in \hat{\mathbb{F}}^\times$, $f \in L^2(F)$, $x \in F$, and
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(F).
\]
If
\[
\pi_\chi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f (x) = \chi (\beta x + \delta) |\beta x + \delta|^{-1} f \left( \frac{\alpha x + \gamma}{\beta x + \delta} \right)
\]
for all $f \in L^2(F)$ and (almost all) $x \in F$, then $\pi_\chi$ is a continuous unitary representation of $SL_2(F)$ on $L^2(F)$.

**Proof.** See [1] or [3].

**Definition 4.2.** The collection of representations
\[
\{ \pi_\chi : \chi \in \hat{\mathbb{F}}^\times \}
\]
is called the principal series of representations of $SL_2(F)$.

**Definition 4.3.** For all $\chi \in \hat{\mathbb{F}}^\times$ and $g \in SL_2(F)$, set
\[
\hat{\pi}_\chi (g) = \mathcal{F} \pi_\chi (g) \mathcal{F}^{-1}.
\]

**Note 4.4.** Let $\chi \in \hat{\mathbb{F}}^\times$. $\hat{\pi}_\chi$ is a representation of $SL_2(F)$ on $L^2(F)$. It is unitarily equivalent to $\pi_\chi$ and more tractable, computationally.

**Lemma 4.5.** Let $\gamma \in F^\times$, $\varphi \in \mathcal{S}$, and $\chi \in \hat{\mathbb{F}}^\times$. Then for all $x \in F$,
\[
\hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \varphi (x) = \psi (-\gamma x) \varphi (x).
\]

**Proof.** Suppose $\varphi = \hat{f}$. For all $x \in F$,
\[
\pi_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mathcal{F}^{-1} \varphi (x) = \mathcal{F}^{-1} \varphi (x + \gamma).
\]
Hence,
\[
\hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \varphi (x) = \mathcal{F} \pi_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mathcal{F}^{-1} \varphi (x)
\]
\[
= \int_F \pi_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mathcal{F}^{-1} \varphi (y) \psi_x (y) \, dy
\]
\[
= \int_F f (y + \gamma) \psi_x (y) \, dy
\]
\[
= \psi (-\gamma) \hat{f} (x)
\]
\[
= \psi (-\gamma x) \varphi (x)
\]
for all $x \in F$.

**Lemma 4.6.** Let $\alpha \in F^\times$, $\varphi \in \mathcal{S}$, and $\chi \in \hat{\mathbb{F}}^\times$. Then for all $x \in F$,
\[
\hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \varphi (x) = \chi (\alpha) |\alpha| \varphi (\alpha^2 x).
\]
Proof. Suppose $\varphi = \hat{f}$. For all $x \in F$,

$$\pi_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \varphi(x) = \chi(\alpha) |\alpha|^{-1} \varphi(\alpha^{-2}x)$$

$$= \chi(\alpha) |\alpha|^{-1} f(\alpha^{-2}x).$$

Hence,

$$\hat{\pi}_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \varphi(x) = \mathcal{F} \pi_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \mathcal{F}^{-1} \varphi(x)$$

$$= \int_F \pi_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \mathcal{F}^{-1} \varphi(y) \psi_x(y) \, dy$$

$$= \chi(\alpha) |\alpha|^{-1} \int_F f(\alpha^{-2}y) \psi_x(y) \, dy$$

$$= \chi(\alpha) |\alpha| \int_F f(y) \psi_x(\alpha^2 y) \, dy$$

$$= \chi(\alpha) |\alpha| \varphi(\alpha^2 x)$$

for all $x \in F$. □

**Lemma 4.7.** Suppose $\mathcal{L} : L^2(F) \longrightarrow L^2(F)$ is a bounded linear operator such that for all $\gamma \in F$, $\alpha \in F^\times$, and $\chi \in \hat{F}^\times$,

$$\mathcal{L} \hat{\pi}_x \left( \begin{array}{cc} 1 & 0 \\ \gamma & 1 \end{array} \right) = \hat{\pi}_x \left( \begin{array}{cc} 1 & 0 \\ \gamma & 1 \end{array} \right) \mathcal{L}$$

and

$$\mathcal{L} \hat{\pi}_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) = \hat{\pi}_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \mathcal{L}.$$Then there exists $m \in L^\infty(F)$ such that for all $f \in L^2(F)$,

$$\mathcal{L} f(x) = m(x) f(x)$$

for almost all $x \in F$ and $m$ is almost everywhere constant on the cosets of $(F^\times)^2$ in $F^\times$.

**Proof.** Lemma 4.5 implies the first half. The required details are intricate. See [1] or [3]. To obtain the second half, take the first as given and note that for all $\alpha \in F^\times$, $\varphi \in S$ and almost all $x \in F$,

$$\left[ \mathcal{L} \hat{\pi}_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \varphi \right](x) = m(x) \chi(\alpha) |\alpha| \varphi(\alpha^2 x),$$

while

$$\left[ \hat{\pi}_x \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \mathcal{L} \varphi \right](x) = \chi(\alpha) |\alpha| m(\alpha^2 x) \varphi(\alpha^2 x)$$

by Lemma 4.6. □

**Note 4.8.** Examination of the algebra of bounded linear operators on $L^2(F)$ commuting with $\hat{\pi}_x$, for $\chi \in \hat{F}^\times$ not of order 2, ultimately yields the proof of the irreducibility of $\hat{\pi}_x$.

Recall the following from [4]:
Theorem 4.9. Let $\chi \in \hat{F}^\times$, $x \in F^\times$, and $\varphi \in \mathcal{S}$. The principal value integral

$$P.V. \int_{F} \psi \left( \frac{x}{y} \right) \chi(y) |y|^{-1} \varphi(y) \, dy$$

exists. Moreover,

$$P.V. \int_{F} \psi \left( \frac{x}{y} \right) \chi(y) |y|^{-1} \hat{\varphi}(y) \, dy = \int_{F} \varphi(u) J_{\chi}(u, x) \, du.$$

Note 4.10. $f \in \mathcal{S}$ implies that

$$\int_{F} f(u) J_{\chi}(u, x) \, du$$

is absolutely convergent for all $\chi \in \hat{F}^\times$ and $x \in F^\times$. See [4].

Lemma 4.11. Let $\varphi \in \mathcal{S}$ and $\chi \in \hat{F}^\times$. For almost all $x \in F$,

$$\hat{\pi}_{\chi} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \varphi(x) = \int_{F} \varphi(u) J_{\chi}(x, u) \, du$$

Proof. Suppose $\varphi = \hat{f}$. By Theorems 2.9 and 4.9,

$$P.V. \int_{F} \chi \left( \frac{1}{y} \right) f(-y) \psi \left( \frac{x}{y} \right) |y|^{-1} \, dy$$

exists for all $x \in F^\times$. Clearly,

$$P.V. \int_{F} \chi \left( \frac{1}{y} \right) f(-y) \psi \left( \frac{x}{y} \right) |y|^{-1} \, dy = P.V. \int_{F} \chi(y) f(-y^{-1}) \psi(xy) |y|^{-1} \, dy.$$

Theorem 3.3 implies

$$\mathcal{F} \left( \pi_{\chi} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) f \right)(x) = P.V. \int_{F} \chi(y) f(-y^{-1}) \psi(xy) |y|^{-1} \, dy$$

for almost all $x \in F$. Theorem 4.9 and Lemma 3.11 give

$$P.V. \int_{F} \chi \left( \frac{1}{y} \right) f(-y) \psi \left( \frac{x}{y} \right) |y|^{-1} \, dy$$

for all $x \in F^\times$. Hence,

$$\hat{\pi}_{\chi} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \varphi(x) = \int_{F} \varphi(u) J_{\chi}(x, u) \, du$$

for almost all $x \in F$.

Lemma 4.12. Let $\chi \in \hat{F}^\times$ have order different from 2. If $C_{1}$ and $C_{2}$ are distinct cosets of $(F^\times)^{2}$ in $F^\times$, then there exist sets $A_{1} \subset C_{1}$ and $A_{2} \subset C_{2}$ of positive measure such that $x \in A_{1}$ and $u \in A_{2}$ implies $J_{\chi}(x, u) \neq 0$. 
Proof. Take $C_1 = (F^\times)^2$ and $C_2 = (\epsilon)(F^\times)^2$. The other cases are similar.

Suppose $\chi \equiv 1$. Let $N = \max \left(0, \frac{2q}{q-1} \right)$. Set

$$A_1 = \left\{ y \in F : |y| \leq q^{-N} \right\} \cap (F^\times)^2$$

and

$$A_2 = \left\{ y \in F : |y| \leq q^{-N} \right\} \cap (\epsilon)(F^\times)^2.$$

Theorem 3.14 gives the result.

Suppose $\chi \not\equiv 1$ is unramified. Set

$$A_1 = \left\{ y \in F : |y| \geq q \right\} \cap (F^\times)^2$$

and

$$A_2 = \left\{ y \in F : |y| \geq q \right\} \cap (\epsilon)(F^\times)^2.$$

Lemma 3.13 and Theorem 3.14 finish this case.

Suppose $\chi$ is ramified of degree $h \geq 1$. Set

$$A_1 = \left\{ y \in F : |y| \geq q^h \right\} \cap (F^\times)^2$$

and

$$A_2 = \left\{ y \in F : |y| \geq q^h \right\} \cap (\epsilon)(F^\times)^2.$$

Lemma 3.13 and Theorem 3.15 complete the proof. □

Note 4.13. The result above requires the hypothesis concerning the order of $\chi$.

Take $C_1 = (F^\times)^2$, $C_2 = (-\tau)(F^\times)^2$, and $\chi = \text{sgn}_\epsilon$ to observe this. The computation is easy.

Theorem 4.14. Let $\chi \in \hat{F^\times}$. Suppose that $\chi$ is not of order 2. Then $\hat{\pi}_\chi$ is irreducible.

Proof. Suppose $\mathcal{L} : L^2(F) \rightarrow L^2(F)$ is a bounded linear operator such that $\mathcal{L}$ commutes with

$$\hat{\pi}_\chi \left( \begin{array}{cc} 1 & 0 \\ \gamma & 1 \end{array} \right), \quad \hat{\pi}_\chi \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right), \quad \text{and} \quad \hat{\pi}_\chi \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

for all $\gamma \in F$ and $\alpha \in F^\times$. By Lemma 4.7, there exists $m \in L^\infty(F)$ such that for all $f \in L^2(F)$,

$$\mathcal{L} f(x) = m(x) f(x)$$

for almost all $x \in F$ and $m$ is almost everywhere constant on the cosets of $(F^\times)^2$ in $F^\times$. It suffices to show that $m$ is almost everywhere constant on $F$.

For $\varphi \in \mathcal{S}$ and almost all $x \in F$,

$$\int_F \varphi(u) m(u) J_\chi(x,u) \, du = \int_F \varphi(u) m(x) J_\chi(x,u) \, du$$

by Lemma 4.11. Hence,

$$m(u) J_\chi(x,u) = m(x) J_\chi(x,u)$$

for almost all $x,u \in F$. The result follows by Lemma 4.12. □

Theorem 4.15. Let $\chi \in \hat{F^\times}$ have order 2. Then $\hat{\pi}_\chi$ is reducible.
Proof. The characters of order 2 on $F$ are $\text{sgn}_\epsilon$, $\text{sgn}_\tau$, and $\text{sgn}_{\epsilon \tau}$. $\text{sgn}_\epsilon$ and $\text{sgn}_\tau$ are ramified of degree 1. $\text{sgn}_\epsilon$ is unramified. Let $\theta = \epsilon, \tau,$ or $\epsilon \tau$. Let $\varphi \in \mathcal{S}$. Theorems 3.14 and 3.15 and Lemma 4.11 give

$$
\hat{\pi}_X \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \varphi (x) = \int_F \varphi (u) J_X (x, u) \, du 
= \int_{|xu| \leq q} \varphi (u) \Gamma (\text{sgn}_\theta) [\text{sgn}_\theta (u) + \text{sgn}_\theta (x)] \, du 
+ \sum_{m > 0, m \text{ even}} \int_{|xu| = q^m} \varphi (u) \text{sgn}_\theta (x) F_{\text{sgn}_\theta} \left( \frac{m}{2}, xu \right) \, du
$$

for almost all $x \in F$.

Suppose that $\varphi$ is supported on $\ker (\text{sgn}_\theta)$. Then

$$
\int_{|xu| \leq q} \varphi (u) \Gamma (\text{sgn}_\theta) [\text{sgn}_\theta (u) + \text{sgn}_\theta (x)] \, du 
= \Gamma (\text{sgn}_\theta) [1 + \text{sgn}_\theta (x)] \int_A \varphi (u) \, du \quad (\ast)
$$

where

$$A = \ker (\text{sgn}_\theta) \cap \{ u \in F : |xu| \leq q \}.$$ 

If $x \notin \ker (\text{sgn}_\theta)$, $(\ast)$ vanishes.

Let $n > 0$, $n$ even. Fix $u \in \ker (\text{sgn}_\theta)$ and suppose $|xu| = q^n$, where $x \in F^\times$.

Then

$$
\int_{|y| = q^{n/2}} \psi (y) \psi \left( \frac{xy}{y} \right) \text{sgn}_\theta (y) \frac{dy}{|y|} 
= \int_{|y| = q^{n/2}} \psi \left( \frac{xy}{y} \right) \psi (y) \text{sgn}_\theta \left( \frac{xy}{y} \right) \frac{dy}{|y|} 
= \text{sgn}_\theta (x) \int_{|y| = q^{n/2}} \psi (y) \psi \left( \frac{xy}{y} \right) \text{sgn}_\theta (y) \frac{dy}{|y|}.
$$

Hence, if $x \notin \ker (\text{sgn}_\theta)$,

$$
\int_{|y| = q^{n/2}} \psi (y) \psi \left( \frac{xy}{y} \right) \text{sgn}_\theta (y) \frac{dy}{|y|} = 0.
$$

But

$$
\int_{|xu| = q^n} \varphi (u) \text{sgn}_\theta (x) F_{\text{sgn}_\theta} \left( \frac{n}{2}, xu \right) \, du 
= \int_{|xu| = q^n} \varphi (u) \text{sgn}_\theta (x) \left( \int_{|y| = q^{n/2}} \psi (y) \psi \left( \frac{xy}{y} \right) \text{sgn}_\theta (y) \frac{dy}{|y|} \right) \, du.
$$

Hence, $x \notin \ker (\text{sgn}_\theta)$, implies that

$$
\sum_{m > 0, m \text{ even}} \int_{|xu| = q^m} \varphi (u) \text{sgn}_\theta (x) F_{\text{sgn}_\theta} \left( \frac{m}{2}, xu \right) \, du = 0,
$$

so

$$L^2 (\ker (\text{sgn}_\theta)).$$
is invariant under the action of
\[ \hat{\pi}_\chi \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \]
by Theorem 2.8.

Theorem 2.8 and Lemmas 4.5 and 4.6 indicate that
\[ \hat{\pi}_\chi \left( \begin{array}{cc} 1 & 0 \\ \gamma & 1 \end{array} \right) \quad \text{and} \quad \hat{\pi}_\chi \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right) \]
also fix this space. Since matrices of the form
\[ \left( \begin{array}{cc} 1 & 0 \\ \gamma & 1 \end{array} \right), \quad \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \alpha \end{array} \right), \quad \text{and} \quad \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \]
where \( \gamma \in F \) and \( \alpha \in F^\times \), generate \( SL_2(F) \), the result follows. \( \square \)

**Theorem 4.16.** Let \( \chi \in \hat{F}^\times \). If \( \chi \) has order 2, then \( \pi_\chi \) is reducible. Otherwise, \( \pi_\chi \) is irreducible.

**Proof.** \( \hat{\pi}_\chi \) is unitarily equivalent to \( \pi_\chi \). Cite Theorems 4.14 and 4.15. \( \square \)

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6. **REFERENCES**