TOPOLOGICAL PROOFS OF THE EXTREME AND INTERMEDIATE VALUE THEOREMS

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Abstract. In this paper, I will present some elementary definitions in Topology. In particular, I will explain topological spaces, continuous functions in a topological space, and the notions of connectedness and compactness defined in a topological setting. Using these ideas, I will present proofs for the Intermediate and Extreme Value Theorems.

Contents

1. Introductory Definitions

Definition 1.1. A *topology* on a set X is a set of subsets, called the open sets, which satisfies the following conditions. (i) The empty set and the set X are open. (ii) Any finite intersection of open sets is open. (iii) Any union of open sets is open.

A *neighborhood* of a point $x \in X$ is an open set U such that $x \in U$. *Open sets* are the sets that meet these criterion. We sometimes write $\mathcal{T}$ for the set of open sets defining a topology, and write (X,$\mathcal{T}$) for the set X with the topology $\mathcal{T}$. More usually, when the topology $\mathcal{T}$ is understood, we just say that X is a topological space. The following definition is a clarification of the terms just discussed.

Definition 1.2. A set A is called *open* if for every $x \in A$ there exists a neighborhood of x in A. An open interval in $\mathbb{R}$ is denoted with parenthesis, so the open interval from a to b with $a < b$ is denoted (a, b)

Topologies can be created in a variety of ways. A distance function, or metric, gives rise to a topology on a set. Consider a metric on $\mathbb{R}$ defined by $\delta(x, y) := |x - y|$, with $x, y \in \mathbb{R}$. This creates a topology for the real line. Since it creates a collection of open intervals (x,y), that cover the real line. In a general metric on a set X, take all the balls produced by the metric to be open. The, a set $A \subset X$ is open if, given $x \in A$, we can find a ball in A that contains x. Topologies can can also be applied to abstracts spaces. Consider the set A= {1,2,3,}. Then for $\mathcal{T}$= {0} and

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{1,2,3} form a topology on A. This is called a *trivial topology*, however, since it is just the empty set and the set itself.

Definition 1.3. A set B is called *closed* if its complement is open. A closed interval from a to b in $\mathbb{R}$ with $a < b$ is denoted $[a, b]$.

Definition 1.4. Let S be a subset of a topological space X. A point p in X is called a *limit point* of S if every open set containing p also contains a point of S other than p itself. Equivalently, every neighborhood of x contains a point of S other than p itself.

These definitions can be used to reach a strong conclusion about closed sets, which will be useful in alter sections. In fact, the conclusion of the following theorem is a very useful way to characterize closed sets in general, since its definition as the complement of an open set is not always useful.

Theorem 1.5. *A set is closed if and only if it contains all its limit points.*

Proof. If A is closed, its complement $X - A$ is open. Since an open set is a neighborhood of each of its points, no point of $X - A$ can be a limit point of A. therefore, A contains all its limit points. Conversely, suppose A contains all its limit points, and let x$\in X - A$. Since x is not a limit point of A we can find a neighborhood N of x which does not meet in A. So N is inside $X - A$, showing $X - A$ to be a neighborhood of each of its points and thus open. Therefore, A is closed. □

Definition 1.6. The union of A and its limit points is called the *closure* of A, and is denoted $\bar{A}$.

The closure of A is the smallest closed set containing A. A simples illustration of a closure uses an interval in $\mathbb{R}$. For an open interval (a, b), its closure is $[a, b]$, since a and b are both limit points of (a, b). However, the closure of $[a, b]$ is itself, since it contains all its limit points.

We can easily formulate the idea of a continuous function with regards to topological spaces. However the definition of continuity in a topological setting will need to be far more abstract than the standard $\delta - \epsilon$ definition of calculus. Since topological spaces can be abstract spaces where the concept of a defined numerical value is nonsensical, we need a broader, more abstract definition.

Definition 1.7. Let X and Y be topological spaces. A function f from X to Y is *continuous* if and only if the inverse image of each open set of Y is open in X.

We are now prepared to define to critical concepts for characterizing topological spaces: compactness and connectedness. Our proofs of the Extreme Value Theorem and the Intermediate Value Theorem will require these concepts.

2. Compactness and Connectedness

Having defined some of the fundamental concepts in point-set topology, we can now introduce to important notions regarding topological spaces: compactness and connectedness.

Compactness has many definitions that are equivalent in different circumstances. In particular, we will later prove a different characterization of compactness in $\mathbb{R}$ is equivalent to the following definition. However, the following definition is the most abstract and the most general; it applies to both spaces like $\mathbb{R}^n$ and more abstract spaces. Thus, it will be our standard definition. However, both this definition and the later definition in $\mathbb{R}$ will be used in out main proofs.

Our definition of connectedness is both general and applicable for all of our later proofs.

Definition 2.1. A topological space X is compact if and only if every open cover of X has a finite subcover.

Definition 2.2. A space X is connected if whenever it is decomposed as the union of $A \cup B$ of two nonempty subsets, then $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.

At this point, all of the necessary background has been provied for us to move to the bulk of the paper: topological proofs of the extreme and intermediate value theorems.

3. The Extreme Value Theorem

Before proving the extreme value theorem, some lemmas are required.

Lemma 3.1. *The image of a compact space under a continuous function is compact.*

Proof. Take f:continuous, f: $X \to Y$, where X is a compact space. Without loss of generality, we can assume that f is surjective. Let $\mathcal{F}$ be an open cover of Y. Then for any $\mathcal{O} \in \mathcal{F}$, $f^{-1}(\mathcal{O})$ is an open subset of X by the definition of a continuous function. So, we have the collection $\mathcal{F}^{-1} = \{f^{-1}(\mathcal{O}) \mid \mathcal{O} \in \mathcal{F}\}$, which is an open cover of X. Since X is compact, there exists a finite subset of $\mathcal{F}^{-1}$ that covers X. Thus, we have $X = f^{-1}(\mathcal{O}_1) \cup f^{-1}(\mathcal{O}_2) \bigcup ... \cup f^{-1}(\mathcal{O}_k)$, where k $\in \mathbb{N}$. Since f is one-one, $f(f^{-1}(\mathcal{O}_i)) = \mathcal{O}_i$ for 1$\leq$ i $\leq$ k. Thus, $f(X) = Y = \mathcal{O}_1 \cup \mathcal{O}_2 \cup ...\ \mathcal{O}_k$. Thus, there exists a finite subset of our open cover $\mathcal{F}$ that covers Y. This imples Y is compact. □

Lemma 3.2. *The closed subset of compact set is compact.*

Proof. For the right to left implication, we must first prove that a closed subset of a compact space is compact. To show this, let X be a compact space, C be a closed subset of X, and $\mathcal{F}$ a set of open subsets of X such that $C \subseteq \bigcup \mathcal{F}$. If we add the open set $X - C$ to $\mathcal{F}$, we obtain an open cover of X. Since X is compact, we know this open cover has a finite subcover. Therefore, we can find $\mathcal{O}_1$, $\mathcal{O}_2$,...$\mathcal{O}_k$ $\in \mathcal{F}$ such that $\mathcal{O}_1\cup$ $\mathcal{O}_2\cup$ $\mathcal{O}_k\cup$ $(X - C)$=X. This gives $C \subseteq \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_k$ and the open sets $\mathcal{O}_1$, $\mathcal{O}_2$,...$\mathcal{O}_k$ are the required subfamily of $\mathcal{F}$. Thus, a closed subset of a compact set is compact. □

Theorem 3.3 (Heine-Borel Theorem)**.** *A subset of* $\mathbb{R}$ *is compact if and only if it is closed and bounded.*

Proof. For the left to right implication, assume a subset of $\mathbb{R}$ is not bounded. Take open intervals of length 1 that cover the space. Any finite subset of this open cover cannot cover the space, since it is not bounded. Thus, it cannot be compact. Therefore, a compact set is bounded. Now, assume a subset of R is not closed. Take a limit point x for this set that is not in the set. Cover the set with the infinite open cover of the complements of the closed balls B[x, 1/n], where n $\in \mathbb{N}$. Then any finite subset of this cover will contain some smallest value of 1/n, which means that the interval (x−1/n, x+1/n) will be uncovered, and since x is a limit point of the set, we can find some point a in our subset of $\mathbb{R}$ such that x−1/n < a < x, meaning that this finite subset of our infinite cover cannot cover the interval in question. Thus, a subset of $\mathbb{R}$ cannot be compact if it is not closed. Thus, a compact space is closed.

Now, take an interval [a,b]. Define $\mathcal{T}$ to be an open cover of [a,b]. Then define a subset X of [a,b] by X = {x ∈ [a,b] | [a,x] is contained in the union of a finite subset of $\mathcal{T}$}. Then X is nonempty, since $a \in X$, and it is bounded above by b. It therefore contains a least upper bound s. Let $\mathcal{O}$ be the element of $\mathcal{T}$ which contains s. Since $\mathcal{O}$ is open, we can choose $\epsilon > 0$ such that $(s-\ \epsilon,\ s] \subseteq \mathcal{O}$, and if $s< b$, we can have $(s - \epsilon,\ s + \epsilon) \subseteq \mathcal{O}$. Since s is the least upper bound of X and , we have $s-\epsilon/2 \in X$. By the definition of X, the interval $[a,\ s-\epsilon/2]$ is contained in the union of some finite subset $\mathcal{T}$' of $\mathcal{T}$. Adding $\mathcal{O}$ to $\mathcal{T}$', we obtain a finite subset of $\mathcal{T}$ whose union contains s. Therefore, $s \in X$. If $s < b$, then $\bigcup \mathcal{T}' \cup \mathcal{O}$ contains $[a,\ s + \epsilon/2]$, giving $s + \epsilon/2 \in X$, contradicting the notion that s is an upper bound for X. Thus, $s = b$, and all of $[a, b]$ is contained in $\bigcup \mathcal{T}' \cup\ \mathcal{O}$. Since $\bigcup \mathcal{T}' \cup\ \mathcal{O}$ is finite, [a,b] is compact. Since any closed and bounded subset of R is a subset of an interval in R, by $Lemma$3.2, it is also compact. □

We are now prepared to prove the Extreme Value Theorem.

Theorem 3.4. (The Extreme Value Theorem) *If* $f : X \to \mathbb{R}$ *is real valued function from a compact space to the real numbers, then* f *attains a greatest value, that is there is an* $x \in X$ *such that* $f(x) \geq f(y)$ *for all* $y \in X$*.*

Proof. Let $f : X \to \mathbb{R}$. Since X is compact, the image is compact, and since its image is compact, it is closed and bounded. Let α be the least upper bound of the image. By the definition of a least upper bound, it is a limit point of Imf, and since Imf is closed, $\alpha \in Imf$. Since α is in the image, there exists $x \in X$ such that $f(x) = y$. Thus, $f(x)$ is the maximum value that f can attain. □

4. The Intermediate Value Theorem

Whereas our proof for the Extreme Value Theorem relied on the notion of compactness, the proof for the Intermediate Value theorem rests on connectedness. Again, we need a few lemmas before we can proceed.

Lemma 4.1. *The real line is a connected space.*

Proof. Suppose $\mathbb{R} = A \cup B$, where A and B are nonempty and $A \cap B = \emptyset$. We will show that some point of A is a limit point of B, or that some point of B is a limit point of A. Choose $a \in A$, $b \in B$, and suppose (without loss of generality) that a<b. Let X consist of those points of A which are less than b and let $s = \sup X$. This s may or may not be in A. By the definition of the supremum, $s \in \bar{A}$. If $s \in A$, then s< b, and since s is an upper bound for X, all the point between s and b are in B. Therefore, s is a limit point of B. If $s \notin A$, the s lies in B, since $A \cup B = \mathbb{R}$. We noted that in this case s is a limit point of A. Thus, if $s \in A$ it is a limit point of B, and if $s \in B$, s is a limit point of A. Therefore, either $\bar{A} \cup B \neq \emptyset$, or $\bar{B} \cup A \neq \emptyset$. □

Corollary 4.2. *A nonempty subset of the real line is connected if and only if it is an interval.*

Proof. The proof of our first lemma shows that any interval of the real line is connected. Now, take X, a non-empty subset of the real line that is not can interval. Then we can find $a, b \in X$ and $p \notin X$ but $a < p < b$. Let A be be the subset of X consisting of those points less than p, and define $B = X - A$. Since $p \notin X$, every point of the closure of A in X is less than p, and every point of the closure of B in X is greater than p. Therefore $\bar{A} \cap B$ and $\bar{B} \cap A$ are both empty and X is therefore not connected. Thus, a nonempty connected subset of the real line is an interval. □

Lemma 4.3. *Let X and Y be topological spaces. Let A be a connected subset of X, and let $f{:}X \to Y$ be a continuous function. Then $f(A)$ is a connected subset of Y.*

Proof. The proof is by contradiction. If $f(A)$ is not connected, then $f(A) = P \cup Q$ with P and Q separated. Let $P_1 = f^{-1}(P) \cap A$ and $Q_1 = f^{-1}(Q) \cap A$; then we find $P_1 \neq \emptyset \neq Q_1$ and $A = P_1 \cup Q_1$. Since f is continuous, $f^{-1}(\bar{P})$ is a closed set in X which has P_1 as a subset. Thus, $\bar{P_1} \subset f^{-1}(\bar{P})$ and $\bar{P_1} \cap Q_1 \subset f^{-1}(\bar{P}) \cap f^{-1}(Q) = f^{-1}(\bar{P} \cap Q) = f^{-1}(\emptyset) = \emptyset$. Similarly, $P_1 \cap \bar{Q_1} = \emptyset$. This contradicts A being connected. Thus, $f(A)$ is connected. □

Theorem 4.4. (The Intermediate Value Theorem) *If f([a,b])$\to \mathbb{R}$ is a function such that $f(a) < 0$ and $f(b) > 0$, then there exists $c \in$ [a,b] such that f(c)=0.*

Proof. Since [a,b] is an interval, it is connected. Thus, its image $f([a, b])$ is connected, since f is a continuous function. This means $f([a, b])$ is an interval, since it is compact. Thus, if $f(a) < 0$ and $f(b) > 0$, there is some point α in $f([a, b])$ such that $\alpha = 0$, since $f([a, b])$ is an interval. Since α is in the image of f with domain $[a, b]$, there is some $c \in [a, b]$ such that $f(c) = \alpha = 0$. □

References

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