THE FUNDAMENTAL THEOREM OF MARKOV CHAINS

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ABSTRACT. This paper provides some background for and proves the Fundamental Theorem of Markov Chains. It provides some basic definitions and notation for recursion, periodicity, and stationary distributions.

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1. DEFINITIONS AND BACKGROUND

So what is a Markov Chain, let’s define it.

Definition 1.1. Let \( \{X_0, X_1, \ldots \} \) be a sequence of random variables and \( Z = 0, \pm 1, \pm 2, \ldots \) be the union of the sets of their realizations. Then \( \{X_0, X_1, \ldots \} \) is called a discrete-time Markov Chain with state space \( Z \) if:

\[
P(X_{n+1} = i_{n+1} | X_n = i_n, \ldots, X_1 = i_1) = P(X_{n+1} = i_{n+1} | X_n = i_n)
\]

Now let’s set up some notation for the one-step transition probabilities of the Markov Chain. Let:

\[
p_{ij}(n) = P(X_{n+1} = j | X_n = i); \quad n = 0, 1, \ldots
\]

We will limit ourselves to homogeneous Markov Chains. Or Markov Chains that do not evolve in time.

Definition 1.2. We say that a Markov Chain is homogeneous if its one-step transition probabilities do not depend on \( n \) i.e.

\[
\forall n, m \in \mathbb{N} \text{ and } i, j \in Z \quad p_{ij}(n) = p_{ij}(m)
\]

We then define the \( n \)-step transition probabilities of a homogeneous Markov Chain by

\[
p_{ij}^{(m)} = P(X_{n+m} = j | X_n = i)
\]

Where by convention we define

\[
p_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

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With these definitions we are now ready for a quick theorem: the Chapman-Kolmogorov equation. This serves to relate the transition probabilities of a Markov Chain.

**Theorem 1.3.**

\[ p_{ij}^{(m)} = \sum_{k \in \mathbb{Z}} p_{ik}^{(r)} p_{kj}^{(m-r)}; \quad \forall \quad r \in \mathbb{N} \cup \{0\} \]

**Proof.** Making use of the total probability rule as well as the Markovian property in this proof yields:

By total probability rule:

\[ p_{ij}^{(m)} = P(X_m = j | X_0 = i) = \sum_{k \in \mathbb{Z}} P(X_m = j, X_r = k | X_0 = i) \]

and now by the Markovian property:

\[ = \sum_{k \in \mathbb{Z}} P(X_m = j | X_r = k, X_0 = i)P(X_r = k | X_0 = i) = \sum_{k \in \mathbb{Z}} p_{ik}^{(r)} p_{kj}^{(m-r)} \]

\[ \square \]

The Chapaman-Kolmogrov equation allows us to associate our n-step transition probabilities to a matrix of n-step transition probabilities. Let’s define the matrix:

\[ P^{(m)} = \left( (p_{ij}^{(m)}) \right) \]

**Corollary 1.4.** Observe now that:

\[ P^{(m)} = P^m \]

Where \( P^m \) here denotes the typical matrix multiplication.

**Proof.** We can rewrite the Champan-Kolmogrov equation in matrix form as:

\[ P^{(m)} = P^{(r)} P^{(m-r)}; \quad \forall \quad r \in \mathbb{N} \cup \{0\} \]

Now we can induct on \( m \). It’s apparent that

\[ P^{(1)} = P \]

Giving us:

\[ P^{(m+1)} = P^{(r)} P^{(m+1-r)} \]

\[ \square \]

A Markov Chain’s transition probabilities along with an initial distribution completely determine the chain.

**Definition 1.5.** An initial distribution is a probability distribution \( \{ \pi_i = P(X_0 = i) | i \in \mathbb{Z} \} \sum_{i \in \mathbb{Z}} \pi_i = 1 \)

Such a distribution is said to be stationary if it satisfies

\[ \pi_j = \sum_{i \in \mathbb{Z}} \pi_i p_{ij} \]
Definition 1.6. We say that a subset space state $C \subset \mathbb{Z}$ is closed if
\[
\sum_{j \in C} p_{ij} = 1 \quad \forall \, i \in C
\]
If $\mathbb{Z}$ itself has no proper closed subsets then the Markov Chain is said to be irreducible.

Definition 1.7. We define the period of a state $i$ by
\[
d_i = \gcd(m \in \mathbb{Z} | p_{ii}^{(m)} > 0)
\]
State $i$ is aperiodic if $d_i = 1$

Definition 1.8. A state $i$ is said to be accessible from from a state $j$ if there is an $m \geq 1$ s.t
\[
p_{ij}^{(m)} > 0
\]
If $i$ is accessible from $j$ and $j$ is accessible from $i$ then $i$ and $j$ are said to communicate.

Definition 1.9. We define the first-passage time probabilities by
\[
f_{ij}^{(m)} = P(X_m = j; X_k \neq j, 0 < k < m - 1 | X_0 = i) \quad i, j \in \mathbb{Z}
\]
And we will denote the expected value, the expected return time, of this distribution by
\[
\mu_{ij} = \sum_{m=1}^{\infty} mf_{ij}^{(m)}
\]

Definition 1.10. We say that a state $i$ is recurrent if
\[
\sum_{m=1}^{\infty} f_{ij}^{(m)} = 1
\]
and transient if
\[
\sum_{m=1}^{\infty} f_{ij}^{(m)} < 1
\]
A recurrent state $i$ is positive-recurrent if $\mu_{ii} < \infty$ and null-recurrent if $\mu_{ii} = \infty$

2. Fundamental Theorem of Markov Chains

Theorem 2.1. For any irreducible, aperiodic, positive-recurrent Markov Chain there exist a unique stationary distribution \{\pi_j, j \in \mathbb{Z}\} s.t \forall \, i \in \mathbb{Z}

Proof. Because our chain is irreducible, aperiodic, and positive-recurrent we know that for all $i \in \mathbb{Z}$ \( \pi_j = \lim_{n \to \infty} p_{ij}^{(n)} > 0 \). Likewise as $\pi_j$ is a probability distribution
\[
\sum_{j \in \mathbb{Z}} \pi_j.
\]
We know that for any $m$
\[
\sum_{i=0}^{m} p_{ij}^{(m)} \leq \sum_{i=0}^{\infty} p_{ij}^{(m)} \leq 1
\]
taking the limit

$$\lim_{m \to \infty} \sum_{i=0}^{m} p_{ij}^{(m)} = \sum_{i=0}^{\infty} \pi_j \leq 1$$

which implies that for any $M$

$$\sum_{i=0}^{M} \pi_j \leq 1$$

Now we can use the Chapman-Kolmogorov equation analogously

$$p_{ij}^{(m+1)} = \sum_{i=0}^{\infty} p_{ik}^{(m)} p_{kj} \geq \sum_{i=0}^{M} p_{ik}^{(m)} p_{kj}$$

as before we take the limit as $m, M \to \infty$ yielding

$$\pi_j \geq \sum_{i=0}^{\infty} \pi_k p_{kj}$$

Now we search for contradiction and assume strict inequality holds for at least one state $j$. Summing over these inequalities yields

$$\sum_{j=0}^{\infty} \pi_j > \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \pi_k p_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} p_{kj} = \sum_{k=0}^{\infty} \pi_k$$

But this is a contradiction, hence equality must hold

$$\pi_j = \sum_{k=0}^{\infty} \pi_k p_{kj}$$

Thus a unique stationary distribution exists. \hfill \Box

Further it can be shown that this unique distribution is related to the expected return value of the Markov chain by

$$\pi_j = \frac{1}{\mu_{jj}}$$

REFERENCES