THE WEAK HOMOTOPY EQUIVALENCE OF S^n AND A SPACE WITH $2n + 2$ POINTS.

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ABSTRACT. Following the lectures [3][4] by J. Peter May in the 2008 University of Chicago Summer Math REU, I intend to go through the meat of the argument for the (surprising!) fact that a space with $2n+2$ points is weakly homotopy equivalent to S^n. I will assume knowledge of basic point-set topology and a basic understanding of homotopy equivalence and homotopy groups.

CONTENTS

1. SUFFICIENT TO CONSIDER T_0-SPACES.

We begin with any arbitrary finite[1] set X. In order to make X interesting to topologists, we ought to make X a *(topological) space* by putting a *topology* on it. After choosing a topology, it will be convenient to take a minimal basis for the topology. We do this as follows:

Following the notation used by May in [3], we consider an element $x \in X$ and consider all open subsets $V \subseteq X$ such that $x \in V$. Define U_x to be the intersection of all such subsets. In other words:

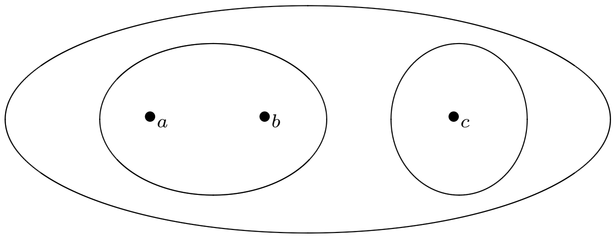
$$U_x = \bigcap_{V \ni x} V. \tag{1.1}$$

Then the basis $\mathcal{B} = \{U_x | x \in X\}$ is our canonical minimal basis. The fact that this is a basis is easy to see, and the minimality of such a basis follows almost entirely from how we have defined it: if $\mathcal{C}$ were another minimal basis for X, then by the definition of U_x we must have that $\mathcal{B}$ is finer than $\mathcal{C}$. The minimality of $\mathcal{C}$ then implies that $\mathcal{B} = \mathcal{C}$, so $\mathcal{B}$ is in fact minimal.

Now that we have a basis, we note that there are only a few things that can happen given points $x, y \in X$: either $U_x \subseteq U_y$, $U_y \subseteq U_x$, or $U_x \cap U_y = \emptyset$. We

[1] For this paper, all spaces will be considered finite unless otherwise stated.

can *almost* define a partial ordering on X based on the order of inclusions of the minimal basis elements, but one problem comes up. Consider the three point space which has the following basis elements:



In this particular basis, we find that it is difficult to tell the difference between the point a and the point b. Since the reader no doubt knows the separation axioms inside-and-out, the reader will not be surprised to find out that this space is *not* T_0[2]. In other words, there is no open set V such that either: ($a \in V$ and $b \notin V$) or ($b \in V$ and $a \notin V$).

So what's the problem? Recall the definition of a partial ordering.

Definition 1.2. A *partial ordering* is an ordering $\leq$ such that the following hold:

(1) $x \leq x$.
(2) $x \leq y$ and $y \leq z$ implies $x \leq z$.
(3) $x \leq y$ and $y \leq x$ implies $x = y$.

In order to have a partial ordering, we need, in particular, the symmetric condition (3). If we try to form a partial ordering on our space by letting $x \leq y$ whenever $U_x \subseteq U_y$, we fail miserably when we try to apply this partial ordering to our space above for exactly the reason that $U_a = U_b$, but $a \neq b$.

At this point, the clever reader will point out that in our above space the points a and b are "essentially the same" as far as the open sets are concerned. This gut feeling can be made formal by the following (not so obvious!) theorem:

Theorem 1.3. *Every finite space X is homotopy equivalent to a T_0-space, X_{T_0}.*

Before diving head-first into the proof, it might be best to briefly describe exactly what we mean by "homotopically equivalent to". For a fuller treatment of homotopies and related topics see [1], [2], or the latter half of [5].

Definition 1.4. For X, Y topological spaces and two continuous maps $f : X \to Y$ and $g : X \to Y$, a *homotopy* from f to g is defined to be a continuous function $h : X \times [0, 1] \to Y$ such that for all $x \in X$ we have that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. Essentially, a homotopy is a continuous deformation from one map to another.

Definition 1.5. For X, Y topological spaces, if we have continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity on Y (denoted Id_Y) and $g \circ f$ is homotopic to the identity on X (denoted Id_X), then we say that X and Y are *homotopically equivalent* and write $X \simeq Y$.

[2]For more information on the separation axioms, see [5].

With definitions out of the way, we should begin building up some of the machinery we will be using to prove Theorem 1.3. To begin with, we want to be able to make points which are "essentially the same" into "exactly the same". To make this notion precise, we define the equivalence relation $\sim$ by setting $x \sim y$ if and only if $U_x = U_y$. We check to see if this is an equivalence class.

Proposition 1.6. $\sim$ *is an equivalence relation.*

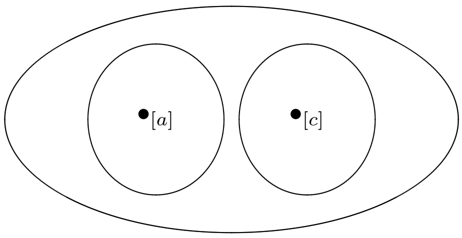
Proof. It should be clear that $x \sim x$ and $x \sim y$ implies $y \sim x$. Lastly, if $x \sim y$ and $y \sim z$, then $U_x = U_y$ and $U_y = U_z$, which implies that $U_x = U_z$ and so $x \sim z$. □

Our equivalence relation $\sim$ allows us to precisely define the notion of "being essentially the same points" that we described above. We now build a new T_0-space X_{T_0} from our old not-necessarily-T_0-space X.

Definition 1.7. The *Kolmogorov Quotient* is defined to be $X/(\sim)$, where $\sim$ is the topological indistinguishability equivalence relation described above. $X/(\sim)$ inherits the quotient topology[3] — that is, a set (of equivalence classes) is open in $X/(\sim)$ if and only if the union of the elements in the equivalence classes is open in X.

In other words, given a set of mutually indistinguishable points (like the points a and b in our picture above) we send all of them to *one* element of $X/(\sim)$, namely the element $[x_0]$, where x_0 happens to be a representative of one of the mutually indistinguishable points. Note that if for some x there does not exist a y such that $U_y = U_x$ but $y \neq x$, then x is sent to the equivalence class $[x]$ which contains only the element x.

Example 1.8. As an example, let's compute the Kolmogorov Quotient for our non-T_0-space above. For sake of notating, let's call that space A. Since $a \sim b$, we have that $[a] = \{a, b\}$ in the Kolmogorov Quotient A_{T_0}. Since c is a distinguishable point in the sense that it has an open set which, for each other point considered, does not contain that other point, we have that the equivalence class $[c] = \{c\}$. Our new space is



which is the same as the two point set with the discrete topology.

With these tools, we are finally able to prove Theorem 1.3.

Proof. (for Theorem 1.3)
Given an arbitrary finite space X, form its Kolmogorov Quotient X_{T_0}. We will show that $X \simeq X_{T_0}$. In order to do this, we must construct two functions, $f : X \to X_{T_0}$ and $g : X_{T_0} \to X$ such that $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_{X_{T_0}}$. Let's try to define f and g in the most naive way possible and hope that it works. Let

[3]For more information on the quotient topology, see [5].

$f(x) = [x]$ for all $x \in X$. Let $g([x_0]) = x_0$, where x_0 is a set chosen representative of each equivalence class.

It is clear that $(f \circ g) \simeq Id_{X_{T_0}}$. Now let's see if $g \circ f \simeq Id_X$. Given some $x \in X$, $f(x) = [x]$, and then $g([x]) = x_0$ where x_0 is a representative of the group $[x]$. Now we define a homotopy to show that $(g \circ f) \simeq Id_X$:

$$h(x,t) = \left\{ \begin{array}{ll} x_0 = (g \circ f)(x) & \text{if t} < 1 \\ x = Id_X(x) & \text{if t} = 1 \end{array} \right\}. \tag{1.9}$$

First, let's show that $h : X \times I \to X$ is continuous. Take V open in X. Since X is a finite space, V is also a finite space, so we have that $V = \{y_1, y_2, \ldots, y_m\}$. We claim that $y_j \in V$ if and only if U_{y_j} is in V. It is clear that if $U_{y_j} \subseteq V$, then $y_j \in V$. Conversely, if $y_j \in V$, then we have that V contains a neighborhood of y_j, but since U_{y_j} is contained in every neighborhood containing y_j, we have that $U_{y_j} \subseteq V$. As V is the union of the minimal open sets U_x and if $x \in [x_0]$, we have that $(g \circ f)(x) = x_0$, so it suffices to prove that $h^{-1}(V)$ is open for $V = U_{x_0}$.

Suppose that $h(x,t) \in U_{x_0}$. If $t = 1$, then $h(x,1) = x$, so $x \in U_{x_0}$; if $t < 1$, then $h(x,t) = (g \circ f)(x) = x_0 \in U_{x_0}$. It is clear that $(g \circ f)(y) \in U_{x_0}$ if only if $x \in U_{x_0}$, and so $h^{-1}(U_{x_0}) = U_{x_0} \times [0,1]$, which is open. Hence, h is continuous. Therefore, X is homotopy equivalent to X_{T_0}. □

To sum up what we've done so far, the process should go something like this: Given an arbitrary set X we first form the Kolmorogov Quotient X_{T_0}, and then let the ordering be $[x] \leq [y]$ if and only if $U_{[x]} \subseteq U_{[y]}$. The reader can check that this ordering is, in fact, a partial ordering.

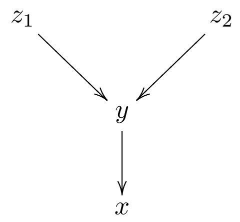
As a final note, since every space X is homotopy equivalent to a T_0-space, we will henceforth assume that all spaces are T_0. Accordingly, we shall no longer need to use Kolmogorov Quotients or equivalence classes.

2. Sufficient to Consider Minimal T_0-Spaces.

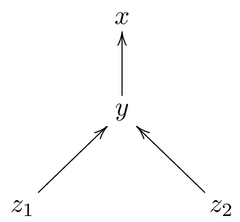
Say we're given a finite T_0-space X. We hinted at before (but never explicitly said) that a minimal basis for the topology on X also induces a partial ordering on X. But we also have that a partial order on a set X induces a topology for which the minimal basis satisfies $x \leq y$ if and only if $U_x \subseteq U_y$. Since we only care about spaces up to homotopy, we might as well toss out any and all points which don't affect the homotopy properties of the space.

Definition 2.1. We say that a point $x \in X$ is *upbeat* if there exists a point $y \in X$ such that $x < y$ and $y \leq z$ for all $z \in X$ satisfying $x < z$. A point which is *downbeat* is similarly defined by replacing each $\geq$ with a $\leq$.

Perhaps it is not immediately clear exactly *what* an upbeat or downbeat point is, or even why we should care about such a point. An illuminating picture can be drawn to illustrate what an upbeat or downbeat point is:



Here we have that the arrows mean "is greater than or equal to", with x, y, z_1 and z_2 satisfying the criteria above. In this picture, x is an upbeat point. Notice that for all of the points greater than it (namely y, z_1 and z_2) we have that y is less than or equal to each of these. Hence, x is an upbeat point by our definition. A simple inversion of the diagram



gives us an example of when x is a downbeat point. The argument for this is similar.

Why should we care about this? Intuitively, if we are just looking at spaces which are homotopically equivalent to X, we "don't need" an upbeat or downbeat point because we have another point that acts very similar to it. Think of our first drawing where x was an upbeat point: we don't "need" x, since y has nearly the same properties as x, except for the fact that $y > x$! The homotopic properties of the space should be unchanged if we were to, say, yank x out of the space all together, since, with x gone, y can basically "take x's place". Let's stop beating around the bush and make this notion precise.

Proposition 2.2. *If X is a finite T_0-space and x_0 is an upbeat or downbeat point, then the inclusion $i : X - \{x\} \hookrightarrow X$ is a homotopy equivalence.*

Proof. Clearly it suffices to prove this when x_0 is an upbeat point, since the proof is nearly identical for the case when x_0 is a downbeat point. We need to find a map $g : X \to X - \{x_0\}$ such that $(i \circ g)$ and $(g \circ i)$ are homotopic to the identity. Define

$$g(x) = \left\{ \begin{array}{ll} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{array} \right\}. \tag{2.3}$$

where y is the same y which "take's x's place" as in the paragraph above (that is, it is the $y \in X$ such that $x_0 < y$ such that if we have $z \in X$ such that $x_0 < z$ then $y \leq z$. We notice that $(g \circ i)(x) = x$ for all $x \in X - \{x_0\}$, so this is clearly homotopic to the identity. We have that for all $x \neq x_0$, that $(f \circ g)(x) = x$, and $(f \circ g)(x_0) = y$. Since this is not exactly the identity, let's spell out the homotopy:

$$h(x,t) = \left\{ \begin{array}{ll} x & \text{t} < 1 \text{ and x} \neq x_0 \\ x & \text{t} = 1 \text{ and x} \neq x_0 \\ y & \text{t} < 1 \text{ and x} = x_0 \\ x_0 & \text{t} = 1 \text{ and x} = x_0 \end{array} \right\}. \tag{2.4}$$

Which is continuous, and proves that $i \circ g$ is homotopic to the identity, which implies that $i : X - \{x\} \hookrightarrow X$ is a homotopy equivalence. □

Definition 2.5. We call a finite T_0-space X *minimal* if it contains no upbeat or downbeat points.

Corollary 2.6. *Given a finite T_0-space, X, there exists $X_{min} \simeq X$ such that X_{min} is minimal.*

Proof. We simply repeat the process in Proposition 2.2 a finite number of times. It is clear that after this process is completed, we obtain a minimal space which is homotopically equivalent to X. □

As a philosophical corollary to all of this hard work we note that it suffices to consider an even smaller class of finite spaces if we only care about properties up to homotopy: if we are given a space X, we can take its Kolmogorov Quotient to obtain X_{T_0}, a T_0-space which is homotopically equivalent to X, and then remove upbeat and downbeat points to obtain $X_{T_0,min}$ which is also homotopically equivalent to X. We write this down as a theorem just to record our achievements.

Theorem 2.7. *Given a finite space, X, there exists a finite minimal T_0-space, $X_{T_0,min}$, which is homotopically equivalent to X.*

In fact, this minimal space is unique up to homeomorphism as shown by [3] but we won't need such a strong result.

3. Weak Homotopy Equivalence

At this point, the reader who has not taken an Algebraic Topology class or who has not seen homotopy groups may become quickly lost in the forest of jargon-filled mathematics that's quickly approaching. Fear not, reader! Though I will not go into homotopy groups (it would take far too much space and would be bothersome for those readers who are already familiar with the topic), there are many good books which go over the subject in great depth[4]. I *highly* encourage the curious reader who knows nothing of homotopy groups to go forth and study them until they seem like old friends to you.

Now, reader, I will assume you know all there is to know about the basics of homotopy groups. In fact, we will not need much: we will only be using the notion of a *weak homotopy equivalence*. In general, as the name suggests, a weak homotopy equivalence is weaker than a homotopy equivalence; this suggests that we could find even "nicer" spaces which are weak homotopy equivalent to our original, nasty, not-necessarily-even-T_0-space X. So, without further ado:

[4][1],[2], and the latter half of [5] are just a few of the many books which cover homotopy groups.

Definition 3.1. A *weak homotopy equivalence* between two topological spaces[5] is a continuous map $f : X \to Y$ which induces an isomorphism

$$f_* : \pi_n(X) \to \pi_n(Y)$$

for all $n \geq 1$.

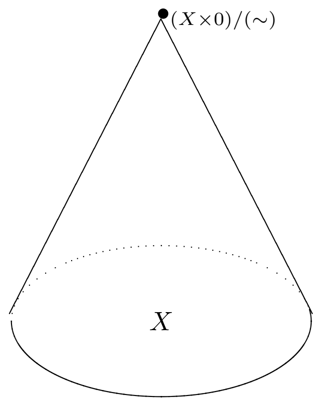
It is difficult to understand the idea of a weak homotopy conceptually at first, but from a formal viewpoint it should be clear how to prove two spaces are weakly homotopy equivalent. Note that weak homotopy is much stronger than all of the homotopy groups of two spaces being isomorphic — weak homotopy implies that there is a continuous map which induces *all* of these isomorphisms.

4. Cones and Suspensions.

We're close to proving our main result, but before we do, we need to define a particular construction.

Definition 4.1. The *cone* on X is the quotient space defined by $CX := (X \times I)/(X \times \{0\})$ with the quotient topology.

It's much more instructive to just see what the cone on X is if, say, X looks like the circle:



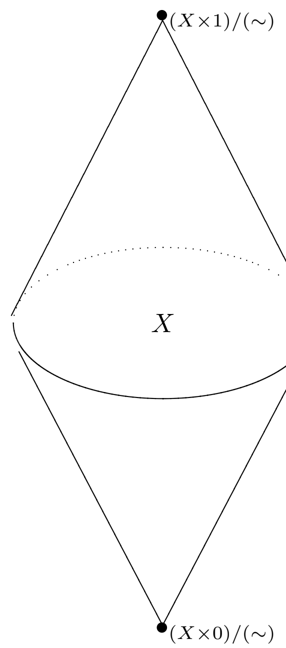
so the cone on X is literally a cone when X looks like a circle. It is difficult to think of a cone when X is more complicated, but there are, fortunately, other methods of calculation which can give us information about CX without having to try to visualize CX itself — we will not go into these methods, but any of the algebraic topology books that have been referred to should contain information for the curious reader.

Now, a related construction is the *suspension* on X.

Definition 4.2. The *suspension* on X is the quotient space defined as $SX := (X \times I)/(\sim)$ where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$.

[5]For simplicity, in this paper we assume that all of our spaces are path-connected and based.

It may be difficult for the reader who has never seen the definition of SX before to imagine what it could possibly look like. The suspension of the circle S^1, denoted SS^1 confusingly enough, is simply a circle with two cones — one going "up" and one going "down". So for spaces which look like S^1, the suspension looks like:



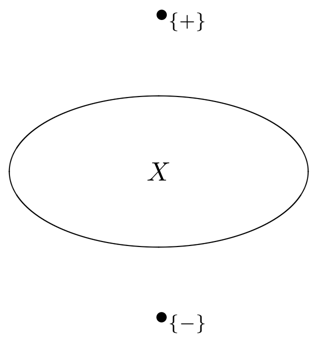
The last two constructions are much easier to define now that we have these basic constructions out of the way.

Definition 4.3. The *non-Hausdorff cone* on X is defined to be $\mathcal{C}X := X \amalg \{+\}$ where $\{+\}$ is just a point we introduce which is disjoint from X. We let all of the proper open sets of $\mathcal{C}X$ be just the open subsets of X along with the set $X \cup \{+\}$.

and similarly

Definition 4.4. The *non-Hausdorff suspension* on X is defined to be $\mathcal{S}X \equiv X \amalg \{+\} \amalg \{-\}$ where $\{+\}$ and $\{-\}$ are points we introduce which are disjoint from X. We let all of the proper open sets of $\mathcal{S}X$ be just the open subsets of X along with the sets $X \cup \{+\}$ and $X \cup \{-\}$.

For clarity's sake, the non-Hausdorff suspension of our space X looks like this:



The non-Hausdorff cone is exactly the same, without the point $\{-\}$ at the bottom. We won't discuss the cone or the non-Hausdorff cone, but we will talk more about the properties of the suspension and the non-Hausdorff suspension. It turns out that this seemingly silly notion of "adding two points" is what gives us the "$2n + 2$" part of our weak homotopy between S^n (the n-sphere, which we will formally define a bit later) and the space with $2n + 2$ points.

5. Properties of the Suspension and the Non-Hausdorff Suspension.

Let's once again state the definition of the non-Hausdorff suspension.

Definition 5.1. The *non-Hausdorff suspension* on X is defined to be $\mathcal{S}X := X \amalg \{+\} \amalg \{-\}$ where $\{+\}$ and $\{-\}$ are points we introduce which are disjoint from X. We let all of the proper open sets of $\mathcal{S}X$ be just the open subsets of X along with the sets $X \cup \{+\}$ and $X \cup \{-\}$.

Notice the last part of the definition: the *only* open set which contains the points $\{+\}$ and $\{-\}$ is the entire space $\mathcal{S}X$. This observation opens a few doors for us; for example, given a map $f : X \to Y$, it's easy to construct a map $\mathcal{S}f : \mathcal{S}X \to \mathcal{S}Y$ by defining

$$\mathcal{S}f(x) = \left\{ \begin{array}{ll} f(x) & \text{if } x \in X \\ + & \text{if } x = + \\ - & \text{if } x = - \end{array} \right\}. \tag{5.2}$$

The reader can check that this will be a continuous mapping if f is continuous. We also note, at this point, that if X is a minimal T_0-space, then $\mathcal{S}X$ is also minimal and T_0. This should be clear, since we are only adding one point which has the whole space as its only open set.

Now we have a notion of suspension on a space and a notion of non-Hausdorff suspension on a space, but how are these two notions similar? Let's define a map to allow us to translate back and forth between our regular suspension and our non-Hausdorff suspension.

Definition 5.3. Define $\gamma : SX \to \mathcal{S}X$ by

$$\gamma(x,t) = \left\{ \begin{array}{ll} x & \text{if } 0 < t < 1 \\ + & \text{if } t = 1 \\ - & \text{if } t = 0 \end{array} \right\}. \tag{5.4}$$

Similarly, we can take our spaces X and Y to be SX and $\mathcal{S}X$, respectively, to obtain $\gamma^2 : SSX \to \mathcal{SS}Y$. Inductively, we can define $\gamma^n : S^nX \to \mathcal{S}^nY$ in the same way.

Now comes the big theorem of the paper. After this theorem, the rest of the paper will be devoted to explaining the specific case where $X = S^1$. Since it's often easier to try to knock down a building brick by brick instead of all at once, we split this theorem into four parts: three lemmas and then the theorem itself. The first lemma we will not prove, but rather cite from [4] (where it is Theorem 1.6).

Lemma 5.5. *If $p : E \to B$ is a continuous map, and B has an open cover $\mathcal{O}$ such that*

(1) If x is in the intersection of sets U and V in $\mathcal{O}$, then there is some $W \in \mathcal{O}$ with $x \in W \subseteq U \cap V$.
(2) For each $U \in \mathcal{O}$ the restriction $p : p^{-1}(U) \to U$ is a weak homotopy equivalence.

then we have that p is a weak homotopy equivalence.

In this lemma, (1) is saying that $\mathcal{O}$ is a basis for a potentially smaller topology than B is given, and (2) is allowing for a "local" view of weak homotopy equivalence, rather than the more "global" idea of homotopy groups being isomorphic. Since we are dealing with relatively simple objects (the non-Hausdorff suspension's extra points are *only* contained in one open set — namely, the whole space), this is an extremely useful lemma to use. In fact, we will put it to use now in the following two lemmas.

Lemma 5.6. *Given a space X, the map $\gamma : SX \to \mathcal{S}X$ is a weak homotopy equivalence.*

Proof. For this and the next lemma, we closely follow the proof given in [4][6]. Take three subspaces X, $X \cup \{+\}$, and $X \cup \{-\}$ as our open cover of $\mathcal{S}X$. The open cover, then, consists of the space X, the space X and the added top point, $+$, and then the space X and the added bottom point, $-$. Notice that this open cover satisfies (i) in Lemma 5.5, since the pairwise intersection of these open sets is either itself (if we intersect the open set with itself) or X, which is included in the cover. If we consider the inverse images (under γ) of these open subsets in SX, we find that they are $X \times (0, 1)$, $X \times [0, 1)$, and $X \times (0, 1]$.

We claim that if we restrict γ to each of these subspaces, γ becomes a homotopy equivalence and, therefore, a weak homotopy equivalence[7]. The proof of this claim is clear: $\gamma(X \times (0, 1)) = X$, and it is trivial that X is homotopy equivalent to

[6] In [4], this lemma and the following two lemmas are condensed into a single theorem, labeled Theorem 3.4.

[7] The fact that a homotopy equivalence implies a weak homotopy equivalence is proven in [1] and [2]. Conversely, if X is a CW complex, we have that a weak homotopy equivalence *is* (!) a homotopy equivalence, but we will not need such a strong statement.

itself. The other two cases are similar. Since the two conditions in Lemma 5.5 are satisfied, we have that γ is a weak homotopy equivalence.

□

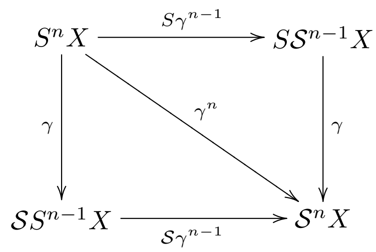
Lemma 5.7. *If $f : X \to Y$ is a weak homotopy equivalence, then the map $Sf : SX \to SY$ and the map $\mathcal{S}f : \mathcal{S}X \to \mathcal{S}Y$ are also weak homotopy equivalences.*

Proof. The proof of this is analogous to the proof of the preceding lemma and is left to the reader as an exercise.

□

Theorem 5.8. *For a space X, the map $\gamma^n : S^nX \to \mathcal{S}^nX$ is a weak homotopy equivalence.*

Proof. We appeal to the diagram



for the proof of this theorem. The commutativity of the diagram follows from how we've defined γ. We may assume inductively that γ^{n-1} is a homotopy equivalence. It follows that $S\gamma^{n-1}$ and $\mathcal{S}\gamma^{n-1}$ are also weak homotopy equivalences by the preceding lemma. By the commutativity of the diagram, we have that γ^n is also a weak homotopy equivalence. □

6. Applications to Familiar Spaces: S^nS^0 and $\mathcal{S}^nS^0$.

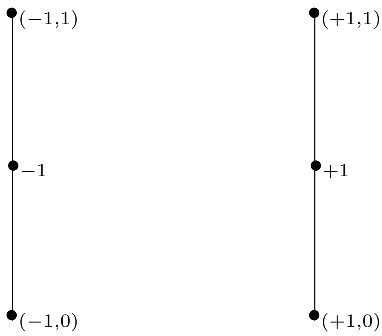
Now that we understand the connection between the suspension and the non-Hausdorff suspension of a space, let's investigate a relatively simple space.

Definition 6.1. The *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} | \, \|x\| = 1\}$ is the set of all points in $\mathbb{R}^{n+1}$ such that each point is distance exactly 1 from the origin[8].

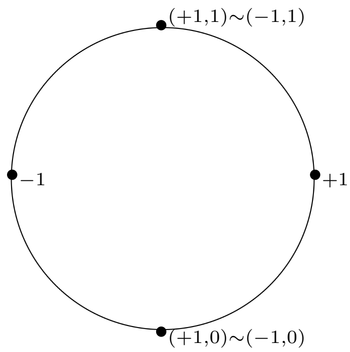
Example 6.2. S^1 is the standard unit circle in the Euclidean plane. S^2 is the standard unit sphere in Euclidean 3-space. An easy, but perhaps less intuitively obvious example is S^0 which is just the the points $\{-1\}$ and $\{1\}$ on the real line.

Let's consider S^0, which is just two points. Let's build SS^0, which is the suspension on S^0. Well, what is this? We cross the two points with the interval, and then identify the "bottom two points" and the "top two points" but none of the points in between. The process looks like this:

[8]Rarely is there ever a topology on the n-sphere which differs from the topology induced by the one-point compactification of $\mathbb{R}^n$, so for the rest of this paper, we will assume S^n has this topology.



First we cross the two points with the interval $[0, 1]$, and then we notice that we just need to identify $(+1, 1) \sim (-1, 1)$ and $(+1, 0) \sim (-1, 0)$. Therefore, we obtain something that looks like:

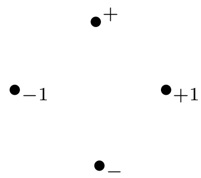


But now the observant reader should be awestruck — "But...this is a circle!" And, indeed, it is a circle. We jot this down: $SS^0 \cong S^1$. The interested reader should now note that we can do the same thing for the circle, S^1. By gluing two hollow cones onto the circle (one upwards, one downwards) we can imagine "blowing it up like a balloon" until it becomes a sphere. We jot this down, too: $SS^1 \cong S^2$, which means that $SSS^0 = S^2S^0 \cong S^2$. By exploring, you can find that this is true for all n, and the proof is just looking at what the suspension does to S^{n-1}. Let's record this as a theorem so that we may refer to it later.

Theorem 6.3. $S^nS^0 \cong S^n$.

In other words, the suspension of S^0 iterated n times is homeomorphic, and, therefore, homotopy equivalent, to the the n-sphere, S^n.

Now let's note that the non-Hausdorff suspension of S^0 is sort of funny: it's simply 4 points!



and if we iterate this once more, we simply get two more points: $\{+'\}$ and $\{-'\}$. Similarly, starting with S^0 and iterating this process n times, we obtain $2n$ new points in addition to the 2 we started with. We write this down as a theorem so that we can easily refer to it.

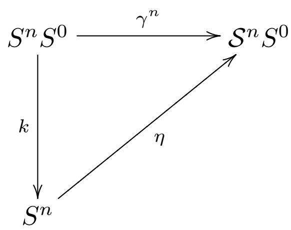
Theorem 6.4. *$\mathcal{S}^n S^0$ is a finite minimal space with $2n+2$ points.*

Note that we have made this space minimal by the arguments in the first part of this paper. It is T_0 trivially, since we consider S^0 with the standard topology as a T_0 space, and by a previous comment we have that $\mathcal{S}^n S^0$ will give us a T_0 space.

Now, let us sit back and piece some of these theorems together to make a simple, but satisfying, corollary. With all of the machinery we've built up, the proof is nothing but citing a few theorems at this point. The reader is encouraged to look back and try to see how each step fits together since this is a really wild idea.

Corollary 6.5. *The n-sphere S^n is weak homotopy equivalent to a finite minimal T_0-space with $2n+2$ points.*

Proof. By theorem 5.8, we have that $\gamma^n : S^n S^0 \to \mathcal{S}^n S^0$ is a weak homotopy equivalence. By Theorem 6.3, we have that the map $k : S^n S^0 \cong S^n$ is a weak homotopy equivalence. By Theorem 6.4, we have that $\mathcal{S}^n S^0$ is a finite minimal T_0-space with $2n+2$ points. We now consider the diagram



and since k and γ^n are weak homotopy equivalences, their composition $\eta = \gamma^n \circ k^{-1}$ gives a weak homotopy equivalence from S^n to the finite minimal T_0-space with $2n+2$ points. □

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