BRAUER ALGEBRAS AND THE BRAUER GROUP

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ABSTRACT. An algebra is a vector space V over a field k together with a k-bilinear product of vectors under which V is a ring. A certain class of algebras, called Brauer algebras - algebras which split over a finite Galois extension - appear in many subfields of abstract algebra, including K-theory and class field theory. Beginning with a definition of the the tensor product, we define and study Brauer algebras, and in particular their connections with Galois cohomology and algebraic geometry. Key results include the fact that Brauer algebras are precisely the finite dimensional central simple algebras; that the Brauer algebras under the right equivalence relation form an abelian group; that this group can be identified with a first (and second) cohomology set; and that to every Brauer algebra corresponds a smooth projective variety.

CONTENTS

1. INTRODUCTION

We assume knowledge of the basic concepts of abstract algebra - group, ring, ideal, field, vector space, algebra, etc. - pausing only to remind the reader that an algebra is a vector space which is also a ring and to mention that our rings will be possibly non-commutative rings with 1. Also, we never consider fields of characteristic 2, although an analog of all of what we present here holds in such fields, and we reiterate this where important.

We begin with a crash course in the tensor product. Here we define the tensor product of two algebras over a fixed field:

Definition 1.1. Let k be a field, and A and B be k-algebras. Then the *tensor product of A and B*, denoted $A \otimes_k B$, is the k-algebra generated by k-linear combinations of elements of the form $a \otimes b$, $a \in A$, $b \in B$, together with the relation $\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$, with addition induced by the equalities $(a \otimes b) + (d \otimes b) = (a + d) \otimes b$ and $(a \otimes b) + (a \otimes c) = a \otimes (b + c)$, and multiplication induced by the equality $(a \otimes b)(d \otimes c) = ad \otimes bc$, for all $a, d \in A$, $b, c \in B$, and $\lambda \in k$. These identities extend in the obvious way to all elements of the tensor product.

Lemma 1.2. *Let $M_i(k)$ be the k-algebra of i-by-i matrices with entries from k. Then $M_n(k) \otimes_k M_m(k) \cong M_{nm}(k)$ for all $m, n > 0$.*

Proof: Note that $M_i(k) \cong End_k(k^i)$. Now given two elements $\phi \in M_m(k)$ and $\psi \in M_n(k)$, note that the pair (ϕ, ψ) induces an element of $End_k(k^m \otimes_k k^n)$: in particular, (ϕ, ψ) induces the map $m : \sum \alpha \otimes \beta \mapsto \sum \phi(\alpha) \otimes \psi(\beta)$. This map is an endomorphism of $k^m \otimes_k k^n$ as a k-vector space. But $k^m \otimes_k k^n = k^{mn}$, as is trivially checked, and the endomorphism algebra of k^{mn} is just $M_{mn}(k)$. So we now have an map $h : M_m(k) \otimes_k M_n(k) \to M_{mn}(k)$, and we must show that h is an isomorphism. But h is clearly injective as defined, and is therefore surjective by dimension reasons. □

A much more revealing use of the tensor product is the following:

Construction 1.3. Let A be a k-algebra, let K be a field extension of k, and consider $A \otimes_k K$. On the one hand, this is a k-algebra, with multiplication, etc. defined according to Proposition 1.2. However, in a natural sense, $A \otimes_k K$ is also a K-algebra, where scalar multiplication is defined as $x(\sum a_i \otimes b_i) = \sum(a_i \otimes (xb_i))$, where $x, b_i \in K$ and $a_i \in A$. This construction gives us a natural way to turn k-algebras into K-algebras, for any field extension $K|k$, which will be used frequently in this paper. This process is called *extending by scalars*.

Example 1.4. Let A be a k-algebra which has dimension n as a vector space, and let $K|k$ be a field extension. Then $A \otimes_k K$ has dimension n as a K-vector space.

Example 1.5. Let $n > 0$, $K|k$ a field extension. Then $M_n(k) \otimes_k K \cong M_n(K)$ as K-algebras.

Proofs: Exercise. For Example 1.4, a basis $\{a_i\}$ of A over k gives us a basis $\{a_i \otimes 1\}$ of $A \otimes_k K$ over K.

In this paper, we will study algebras which are "nearly isomorphic" to matrix algebras. To make this more precise, consider the following:

Construction 1.6. Let k be a field of characteristic $\neq 2$, $k^\times$ its group of units, and $a, b \in k^\times$. Then we define the *generalized quaternion algebra* $(a, b)_k$ as the unique 4-dimensional k-algebra with basis $\{1, i, j, ij\}$ as a vector space and multiplication defined by the relations $ii = a$, $jj = b$, $ij = -ji$.

Example 1.7. : If we let $k = \mathbb{R}$ and set $a = b = 1$, then $(1, 1)_\mathbb{R} \cong M_2(\mathbb{R})$, via the isomorphism which sends

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, ij \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If on the other hand we let $k = \mathbb{R}$ and set $a = b = -1$, we get Hamilton's original quaternions, $\mathbb{H}$. $\mathbb{H}$ is a division algebra, but not commutative, as is easily checked. Moreover, it is clear that $(a, b)_k \cong (b, a)_k$ for any $a, b \in k^\times$.

Fact 1.8. *The algebra $\mathbb{H}$ is not isomorphic to a matrix algebra.*

There are many ways to prove this, the simplest being to note that $\mathbb{H}$ is four-dimensional as a vector space and a division algebra, whereas any matrix algebra of dimension > 1 is not a division algebra. $\square$

However, consider the algebra $A = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, which is an algebra over the complex numbers via Construction 4. It is easily checked that A is isomorphic to the matrix algebra $M_2(\mathbb{C})$ via the isomorphism which sends

$$1 \otimes 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \otimes 1 \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, j \otimes 1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, ij \otimes 1 \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Note that for obvious reasons we should not use i for $\sqrt{-1}$. This strange result holds, in fact, for all generalized quaternion algebras:

Proposition 1.9. *If k is a field of characteristic not 2, then either $(a,b)_k \cong M_2(k)$ or $(a,b)_k \otimes_k k(\sqrt{a}) \cong M_2(k(\sqrt{a}))$.*

Proof: First, note that for all $a, b, u \in k$, $(a,b)_k \cong (u^2 a, b)_k$ via the obvious isomorphism. Suppose that either a or b is a square in k. WLOG, assume $a = x^2$ for some $x \in k$. Then by the above observation, $(a,b)_k \cong (1,b)_k$. In this event, the following homomorphism $m : (a,b)_k (\cong (1,b)_k) \to M_2(k)$, defined by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, ij \mapsto \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix},$$

is an isomorphism.

If, on the other hand, neither a nor b is a square in k, consider the homomorphism from $(a,b)_k \otimes_k k(\sqrt{a})$ to $M_2(k(\sqrt{a}))$ generated by its values on the standard basis elements:

$$1 \otimes 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \otimes 1 \mapsto \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{pmatrix}, j \otimes 1 \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, ij \otimes 1 \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & b\sqrt{a} \end{pmatrix}.$$

Since neither a nor b are squares in k, the image of the standard basis elements of $(a,b)_k \otimes_k k(\sqrt{a})$ is a linearly independent set with four elements. Thus, by dimension, the homomorphism thus induced is an isomorphism.

Thus, no matter what quaternion algebra we are given, either it is already isomorphic to a matrix algebra, or it becomes isomorphic to a matrix algebra via a very simple extension of scalars. In particular, the case of Hamilton's original quaternions reduces to the fact that $\mathbb{R}(\sqrt{-1}) \cong \mathbb{C}$. $\square$

We can make this phenomenon more precise through the language of Galois theory. Recall that, if $K|k$ is a field extension, $K|k$ is said to be *algebraic* if every element of K is a zero of some non-zero polynomial with coefficients in k. Thus, for instance, $\mathbb{C}|\mathbb{R}$ is algebraic, but $\mathbb{R}|\mathbb{Q}$ is not, since e.g. π is transcendental. An algebraic extension is said to be *Galois* if the field fixed by all automorphisms of K which fix k is precisely k. This condition, though it may seem trivial, does not hold of all algebraic extensions; for example, consider $\mathbb{Q}(\sqrt[3]{2})$, which is an algebraic field extension of $\mathbb{Q}$, but is not Galois, since the only possible automorphic image of $\sqrt[3]{2}$ is $\sqrt[3]{2}$. However, the extension $\mathbb{C}|\mathbb{R}$ is Galois, since the only elements of $\mathbb{C}$ fixed by both the identity automorphism and by conjugation are the real numbers. Given a Galois extension $K|k$, we can define the *Galois group* $G(K|k)$ as the group of all automorphisms of K which fix k pointwise (that is, $G(K|k) = \{\sigma \in Aut(K) : \forall\lambda \in k, \sigma(\lambda) = \lambda\}$. $K|k$ is said to be a *finite Galois extension* if K is finite dimensional as a k-vector space; this is equivalent to saying that the group $G(K|k)$ is of finite order, and in fact the order of $G(K|k)$ will be precisely the dimension of K as a k-vector space. Note that this equivalence need not hold if $K|k$ is not Galois. This gives us the language necessary for the following definition:

Definition 1.10. Let k be a field, A, B k-algebras. Then A is a *twisted form of* B if there is some finite Galois extension $K|k$ such that $A \otimes_k K$ and $B \otimes_k K$ are isomorphic as K-algebras. We denote this relation by $A \equiv B$. If e.g. $B = M_n(k)$, we say that K is a *splitting field* for A, and that A *splits over* K. If A is isomorphic to $M_n(k)$ as a k-algebra, then we say that A is split, or that k itself is a splitting field for A.

Example 1.11. We showed that the generalized quaternion algebra $(1,1)_{\mathbb{R}}$ is split, while Hamilton's original quaternions are not split but are a twisted form of the matrix algebra $M_2(\mathbb{R})$ and split over $\mathbb{C}$, since $M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$. Also, any two isomorphic algebras are clearly twisted forms of each other. From the concept of twistedness, we automatically get an equivalence relation, with k-algebras A and B being equivalent if and only if they are twisted forms of each other. This equivalence class is one of the fundamental concepts of this paper.

We now define the subject of this paper:

Definition 1.12. An algebra A over a field k is said to be *Brauer* if and only if A is a twisted form of $M_n(k)$ for some n. Alternatively, by Example 1.5, A is Brauer if and only if $A \otimes_k K \cong M_n(K)$ for some finite Galois extension $K|k$.

Lemma 1.13. *A Brauer k-algebra A is finite dimensional.*

Proof: By "finite dimensional," we mean finite dimensional as a k-vector space; another term used in the literature for this is "finite degree." Suppose A is Brauer. Then for some finite Galois extension $K|k$ and some $n > 0$, $A \otimes_k K \cong M_n(K)$. By Example 1.4, the dimension of A as a k-vector space is the same as the dimension of $A \otimes_k K$ as a K-vector space, which in turn is the same as the dimension of $M_n(K)$ since the two algebras are isomorphic, which is just n^2. Thus, A is finite dimensional. In fact, we have also proven that the dimension of any Brauer algebra is a perfect square, which is why Hamilton needed i, j and k to build a useful number system. $\square$

There are three main ways to study Brauer algebras. The first is to derive an alternative definition of Brauer algebras in elementary terms - that is, without the use of tensor product, Galois extension, etc. - and then prove things about Brauer algebras using that definition. The second is to use exploit a connection between Brauer algebras and group cohomology in order to use theorems about the one to prove theorems about the other, and vice versa. The third is to bring in some algebraic geometry. In this paper, we introduce all three methods, and as best as possible describe their uses and some of their basic results.

The reader should note that there are facets of the study of Brauer algebras the mere description of which is far beyond the scope of this paper to handle seriously. For example, two of the main applications of Brauer algebras are to K-theory and class field theory. However, these build on the three main approaches mentioned above, so by restricting our attention to these more basic directions, we can still develop a feel for the subject while not unduly sacrificing rigor or comprehension.

2. Properties of Central Simple Algebras

In this section we derive an alternate definition of Brauer algebras which uses only terms and ideas of a first course in modern algebra. We use this definition to prove facts about Brauer algebras which would be much more difficult to prove with Definition 1.12 alone.

Definition 2.1. Let A be a k-algebra. Then A is *simple* if A has no nontrivial proper two-sided ideals.

Definition 2.2. Let A be a k-algebra. Then a copy of k lives inside A in a canonical way: the set $k_A = \{\lambda \mathbf{1} : \lambda \in \mathrm{k}\}$, where $\mathbf{1}$ is the multiplicative identity of A, is a subalgebra of A isomorphic to k. Moreover, every element of k_A commutes with every element of A: for all $a \in A$, $(\lambda\mathbf{1})\mathrm{a} = \lambda(\mathbf{1}\mathrm{a}) = \lambda(\mathrm{a}\mathbf{1}) = \mathrm{a}(\lambda\mathbf{1})$. However, it is possible that k_A contains *all* of the elements of A which always commute, in which case A is in some sense as non-commutative as possible. In such a case, we say that A is central; more precisely, a k-algebra A is central if and only if the *center* of A, defined to be the set $Z(A) = \{x \in A : \forall y \in A, xy = yx\}$, is exactly k_A.

Definition 2.3. A k-algebra A is *central simple* if and only if it is both central and simple.

Example 2.4. A basic result of the study of matrices is that any matrix algebra with entries in a division algebra is central simple. In addition, the generalized quaternion algebras discussed in the introduction can be shown to be central simple without too much difficulty. However, the $\mathbb{R}$-algebra of ordered pairs of real numbers with multiplication being given by $(a, b)(c, d) = (ac, bd)$ is neither central nor simple.

A somewhat more useful result is that if A and B are central simple k-algebras, then $A \otimes_k B$ is also a central simple k-algebra. To show this, note that the center of $A \otimes_k B$ is the set of elements of the form $\sum x_i \otimes y_i$, where $x_i \in Z(A)$ and $y_i \in Z(B)$. So $Z(A \otimes_k B) \cong Z(A) \otimes_k Z(B)$. But $Z(A) \cong Z(B) \cong k$, so $Z(A \otimes_k B) \cong Z(A) \otimes_k Z(B) \cong k \otimes_k k \cong k$, so $A \otimes_k B$ is central. We leave to the reader the task of showing that $A \otimes_k B$ is a simple k-algebra if A and B are both simple. $\square$

Theorem 2.5. (Wedderburn) *Let A be a finite dimensional simple k-algebra. Then there is some finite-dimensional (as a k-vector space) division algebra $D \supset k$ and some $n > 0$ such that $A \cong M_n(D)$, and D, n are unique.*

Proof: The idea of the proof is as follows: Since A is finite dimensional, a descending sequence of left ideals terminates. Letting L be a minimal left ideal of L, we can show that $D = End_A(L)$ is a division ring, and that $A \cong End_D(L)$, where $End_X Y$ is the ring of endomorphisms of Y as an X-module. We then derive the natural isomorphism $End_D(L) \cong M_n(D)$. This shows the existence of such n and D. For uniqueness, we first show that any two simple left ideals of $M_n(D)$, for any $n > 0$ and division ring D, are isomorphic. This tells us that, if D and D' are division algebras and $m, n > 0$ such that $A \cong M_n(D) \cong M_m(D')$, then our minimal left ideal L satisfies $D^n \cong L \cong D^m$, since for any ring R and any $i > 0$ $M_i(R)$ has a simple left ideal isomorphic to R. But then we have that $D \cong End_A(D^n) \cong End_A(L) \cong End_A(D'^m) \cong D'$, so $D \cong D'$. From here, we have that $m = n$, by dimension. This completes the uniqueness statement. For a complete proof of Wedderburn's Theorem, see e.g. [2] or [3]. □

Theorem 2.6. *Let A be a finite-dimensional k-algebra. Then A is central simple if and only if there exists some finite (but not necessarily Galois) extension $K|k$ such that for some $n > 0$, $A \otimes_k K \cong M_n(K)$.*

To prove this, we make use of the following lemma:

Lemma 2.7. *Let A be a finite dimensional k-algebra, and $K|k$ a finite (but not necessarily Galois) field extension. Then A is central simple if and only if $A \otimes_k K$ is a central simple K-algebra.*

Proof: One direction is easy: if A is not simple, then letting I be a nontrivial two-sided ideal of A, we have that $I \otimes_k K$ must be a nontrivial two-sided ideal of $A \otimes_k K$ for dimension reasons. Moreover, if x is an element of $Z(A)$ not contained in k_A, then $x \otimes 1$ is an element of $Z(A \otimes_k K)$ not contained in $K_{A \otimes_k K}$. Thus, if A is *not* central simple, neither is $A \otimes_k K$ for any finite extension K.

The other direction is trickier: suppose A is central simple. By Wedderburn's theorem, we need only consider the case when A is a division algebra (exercise). We will show that in this case, $A \otimes_k K$ is both central and simple as a K-algebra.

Central: Let $\{w_1, w_2, ..., w_n\}$ be a basis for K as a k-vector space. Note that we can do this since $K|k$ is finite dimensional by assumption. Then looking at $A \otimes_k K$ as a A-module, we have that $1 \otimes w_1, 1 \otimes w_2, \ldots, 1 \otimes w_n$ form an A-basis for $A \otimes_k K$ since A is a division algebra. Note that $K_{A \otimes_k K} = \{\sum_{i=1}^n \alpha_i(1 \otimes w_i) : \alpha_i \in k_A\}$. Let $x \in Z(A \otimes_k K)$. Write $x = \sum_{i=1}^n \alpha_i(1 \otimes w_i)$. Since x is in $Z(A \otimes_k K)$, we have that for every $d \in A$, $dx = xd$ - remember, we are looking at $A \otimes_k K$ as a A-module, so it makes since to multiply x and d. But this means that $\sum_{i=1}^n d\alpha_i(1 \otimes w_i) = \sum_{i=1}^n \alpha_i d(1 \otimes w_i)$, which by the linear independence of the $1 \otimes w_i$, implies that $\alpha_i d = d\alpha_i$ for all i. But this holds for all $d \in A$! So $\alpha_i \in Z(A)$ for every i, which means since A is central that $\alpha_i \in k_A$. But this implies that $x \in K_{A \otimes_k K}$. So $A \otimes_k K$ is central.

Simple: Let J be a nonzero ideal in $A \otimes_k K$. We will show $J = A \otimes_k K$.

WLOG, let J be a principal ideal generated by α. Then we can extend $\{\alpha\}$ to a A-basis of $A \otimes_k K$ by adjoining some of the $1 \otimes w_i$. WLOG, $\{\alpha, 1\otimes w_2, 1\otimes w_3, ..., 1\otimes w_n\}$ is a basis for $A \otimes_k K$ as an A-module. Then we may write

$$1 \otimes w_1 = (\sum_{j=2}^{n} x_j(1 \otimes w_j)) + k\alpha,$$

where $k \in A$. Since J is a two-sided ideal, we must have $d^{-1}yd \in J$ for every $d \in A$ (why?), so there exists some $\beta \in A$ such that $d^{-1}yd = \beta y$. From this it follows, by left-multiplying by d^{-1} and right-multiplying by d and moving things around, that

$$1 \otimes w_1 - (\sum_{j=2}^{n}(d^{-1}\alpha_j d)(1 \otimes w_j)) = (\beta(1 \otimes w_1)) - (\beta \sum_{j=2}^{n} \alpha_j(1 \otimes w_j)).$$

From this, because the $1 \otimes w_m$ are linearly independent, we can conclude that $\beta = 1$ and $d^{-1}\alpha_j d = \alpha_j$ for all j. This last equality holds for all d, so we have that $\alpha_j \in Z(A)$. But this implies, since A is central, that $\alpha_j \in k$ [1], so we can move the α_j through the tensor signs and derive that y is of the form $1 \otimes z$ for some $z \in K$. But from this, because K is a field, we easily have that $J = A \otimes_k K$. So we are done. □

Proof of Theorem 2.6: Sufficiency follows from the lemma and the fact that for any division ring D, $M_n(D)$ is central simple. For necessity, let $\overline{k}$ be a fixed algebraic closure of k; since algebraic closures are isomorphic, there is no ambiguity here. Note that $A \otimes_k \overline{k} \cong M_n(\overline{k})$ for some n, since every central simple algebra over an algebraically closed field is isomorphic to a matrix algebra; see Theorem 2.12. Now we can write $A \otimes_k \overline{k}$ as the union, over all finite field extensions $K|k$ such that $K \subset \overline{k}$, of $A \otimes_k K$. But this, together with the fact that A is finite-dimensional, implies that there is some finite extension $K|k$ contained in $\overline{k}$ such that each of the standard basis elements of $A \otimes_k \overline{k}$ - that is, the pre-images of the standard basis elements $e_1, ..., e_{n^2}$ of $M_n(\overline{k})$ - are contained in $A \otimes_k K$, and in addition all the pre-images of the elements $a_{ij} \in M_n(\overline{k})$ occurring in the relations $e_i e_j = \sum a_{ij} e_l$. This implies that $A \otimes_k K \cong M_n(K)$, and we are done. □

By a result of Noether and Koethe ([2], 2.2.5), we can let the extension $K|k$ in Theorem 2.6 be not just finite dimensional, but also Galois. This yields the following useful characterization of Brauer algebras:

Corollary 2.8. *Let A be a k-algebra. Then A is Brauer if and only if A is a finite-dimensional central simple algebra.*

Our next step is to define an equivalence relation on Brauer algebras:

Definition 2.9. Let A, B be Brauer k-algebras - or alternatively, finite-dimensional central simple k-algebras. Then A and B are *Brauer equivalent* , and we write $A \equiv_{Br} B$, if there exist $m, n > 0$ such that $A \otimes_k M_m(k) \cong B \otimes_k M_n(k)$. We denote the set of equivalence classes $Br(k)$. Note for instance that for all $m, n > 0$, $M_m(k) \equiv_{Br} M_n(k)$.

[1] Here we adopt a common practice and, in a central algebra, identify the center with the field.

We could have defined the set $Br(k)$ back in Section 1, in terms of twisted forms. However, the use of Corollary 2.8 allows us to derive simple algebraic properties of $Br(k)$ which would be nearly intractable to derive using only twisted forms:

Theorem 2.10. *The set $Br(k)$ forms an abelian group under the tensor product.*

Proof: Associativity, commutativity, closure, well-definedness, and the existence of an identity are trivial to check. In particular, note that the identity in the Brauer group is the equivalence class of $M_1(k)$. All that remains is to prove the existence of inverses. To do this, I need the following:

Construction 2.11. If R is a ring, we can define the *opposite ring of* R, written R^o, as the ring with the same elements and addition but with multiplication reversed, i.e., $x *_{R^o} y = y *_R x$. Moreover, if R is a k-algebra, then R^o is also a k-algebra in a natural way, and we call R^o the *opposite algebra* of R. For those interested in category theory, if we think of a ring as a category enriched over **Ab** (the category of abelian groups) with a single object, then the opposite ring is just the dual category. Note that it is trivial to show that if R is a central simple k-algebra, then R^o is also a central simple k-algebra.

Now the theorem can be proven. Let A be a k-algebra which splits over K, and consider the opposite algebra A^o. I claim that $A \otimes_k A^o$ is isomorphic to a matrix algebra. Consider the map m which sends an element $\sum \alpha_1 \otimes \alpha_2$ in $A \otimes_k A^o$ to the map $f(x) = \sum \alpha_1 x \alpha_2$, $f : A \to A$. Clearly, f is an endomorphism of A *as a k-vector space* , so I have a map $m : A \otimes_k A^o \to End_k(A)$. I want to show that m is an isomorphism. Linearity is trivial to check - the difficult step is bijectivity. But we can use a trick here: since A is central simple, A^o is central simple, so $A \otimes_k A^o$ is central simple. But this implies that m has either trivial kernel or kernel all of $A \otimes_k A^o$, because the kernel of m is a two-sided ideal of $A \otimes_k A^o$, and $A \otimes_k A^o$ is simple. But m is clearly nonzero (consider $m(1)$), so we have that $\mathrm{Ker}(m) = \{0\}$, so m is injective. But then by dimension reasons, m must be surjective. So m is an isomorphism. So A^o really is the inverse of A in the Brauer group, and we are done. [2] □

Because of this theorem, we call $Br(k)$ the *Brauer group of k.*

Theorem 2.12. *If k is an algebraically closed field, then $Br(k)$ is the trivial group.*

Proof: Let D be a finite-dimensional division algebra containing k. Suppose $D \neq k$. Let $d \in D \backslash k$. Then since D is finite dimensional as a k-vector space, the infinite set $\{1, d, d^2, ...\}$ is linearly dependent, so there must exist some polynomial $f \in k[x]$ such that $f(d) = 0$. Moreover, since D is a division algebra, we can choose some such f which is irreducible. This means that there is a natural homomorphism $\phi : k[x]/(f) \to D$ such that $d \in \mathrm{Im}(\phi)$; specifically, let ϕ be the map sending $k[x]/(f)$ to $k(d)$ lying inside D. But since k is algebraically closed, $k[x]/(f) \cong k$. Thus $d \in k$, yielding a contradiction. So there is no finite-dimensional division algebra strictly containing k. But then let C be some finite-dimensional central simple k-algebra. By Wedderburn's Theorem, there is some division algebra D containing k such that

[2] Note that by this construction, we ought to have that $M_n(k)^o \cong M_n(k)$ for all n, k. This is in fact the case, as is easily checked (the transposition map $Tr : A \mapsto A^T$ is an isomorphism).

$C \cong M_n(D)$ for some $n > 0$. Since C is finite-dimensional, D must also have finite dimension. But then $D = k$, by the result above. So $C \cong M_n(k)$ for some $n > 0$. So all finite dimensional central simple algebras over k are split, which together with Corollary 2.5 implies that $Br(k)$ is the trivial group. □

Theorem 2.13. *$Br(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, and is generated by the equivalence class of $(-1,-1)_{\mathbb{R}}$.*

In order to prove this theorem, we need the following Lemma:

Lemma 2.14. *Let k be a field. Then each Brauer equivalence class contains exactly one division algebra (up to isomorphism).*

Proof of Lemma: First, we show that for any k-algebra R, $R \otimes_k M_n(k) \cong M_n(R)$, which is a generalization of Example 1.5 to a not-as-trivial result. To do this, simply note that the obvious map $m : R \otimes_k M_n(k) \to M_n(R)$ given by $m(\sum r_i \otimes A_i) = \sum r_i A_i$ can be verified to be an isomorphism, by looking at what it does to the basis elements of $R \otimes_k M_n(k)$ as an R-module.

This lets us prove that each Brauer equivalence class contains at most one division algebra: suppose D_1, D_2 are Brauer division algebras which are Brauer equivalent. Then $D_1 \otimes_k M_m(k) \cong D_2 \otimes_k M_n(k)$ for some m, n. But this implies that $M_m(D_1) \cong M_n(D_2)$. But by Wedderburn's Theorem, together with the fact that D_1 and D_2 (being Brauer) are finite dimensional, implies that $m = n$ and $D_1 \cong D_2$.

The other part of the lemma is proved similarly. Let A be a Brauer algebra. Then $A \cong M_n(D)$ for some finite-dimensional division algebra D, by Wedderburn's Theorem. But $M_n(D) \cong D \otimes_k M_n(k)$, which since $M_n(D)$, being isomorphic to A, is Brauer, implies that D is Brauer, since the Brauer algebras form a group. But then D and A are Brauer equivalent. So each Brauer equivalence class contains at least one division algebra. This, together with the previous paragraph, proves the lemma. □

Proof of Theorem 2.13 : We first recall Frobenius' Theorem: that if A is a division algebra over $\mathbb{R}$, then A is isomorphic to either $\mathbb{R}, \mathbb{C}$, or $\mathbb{H} = (-1,-1)_{\mathbb{R}}$. This theorem, and its proof, can be found in [4]. Note that $\mathbb{C}$ is not central, and $\mathbb{R}$ as an algebra over itself corresponds to the trivial element of $Br(\mathbb{R})$, so the only Brauer division algebra over $\mathbb{R}$ which might correspond to a non-trivial element of the Brauer group is $(-1,-1)_{\mathbb{R}}$. This, together with Lemma 2.14, quickly proves the theorem: since $(-1,-1)_{\mathbb{R}}$ is a division algebra, it cannot lie in the same Brauer equivalence class as $\mathbb{R}$, so the equivalence class of $(-1,-1)_{\mathbb{R}}$ is non-trivial. And since $\mathbb{R}$ and $(-1,-1)_{\mathbb{R}}$ are the only central simple $\mathbb{R}$-division algebras, there are only two equivalence classes of central simple $\mathbb{R}$-algebras. Thus, $Br(\mathbb{R})$ has precisely two elements, which means that $Br(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. And the generator is clearly (the equivalence class of) $(-1,-1)_{\mathbb{R}}$. □

Other specific Brauer groups which are known are: $Br(F) = \{0\}$ for any finite field F or algebraic extension thereof, and $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ for any p (where $\mathbb{Q}_p$ denotes the field of p-adic numbers). The first result follows from Wedderburn's Theorem, and the second from local class field theory.

Corollary 2.8 also lets us prove results about individual Brauer algebras:

Theorem 2.15. *Any automorphism of a Brauer algebra A is inner, i.e., is of the form $f(x) = c^{-1}xc$ for some $c \in A$.*

Proof: This follows from the well-known theorem of Noether and Skolem (see [2]) that the automorphisms of a finite-dimensional central simple algebra are inner. $\square$

We end this section by mentioning one of the more powerful theorems about Brauer algebras:

Theorem 2.16. (Merkurjev) *Let k be a field of characteristic not 2, and let A be a Brauer division algebra over k such that $A \otimes_k A$ is split (this latter condition is sometimes abbreviated by saying that A is of period two). Then there exist positive integers $m_1, m_2, n > 0$ and quaternion algebras $Q_1, Q_2, ..., Q_n$ such that $A \otimes_k M_{m_1}(k) \cong Q_1 \otimes_k Q_2 \otimes_k ... \otimes_k Q_n \otimes_k M_{m_2}(k)$.*

The proof of this theorem is unfortunately beyond the scope of this paper, but see [2]. Besides being interesting in and of itself, it vindicates the use of generalized quaternion algebras as representatives and test cases of central simple algebras. There is a much more powerful version of Merkurjev's Theorem, dealing with algebras which split over a field extension with a cyclic Galois group, which is called the Merkurjev-Suslin Theorem, but it is even more beyond the scope of this paper. Merkurjev's Theorem, and the Merkurjev-Suslin Theorem, play a major role in the study of central simple algebras, and so we mention them in passing.

3. Galois Theory and Group Cohomology

Here we use techniques of Galois theory and group cohomology to study the relative Brauer group $Br(K|k)$. In particular, we show that this group can be thought of as a first cohomology set. We begin with a few important concepts. We begin by restricting the Brauer group to equivalence classes which split over a fixed field:

Definition 3.1. $Br(K|k)$, called the *Brauer group of k relative to K*, is the abelian group of Brauer equivalence classes of central simple algebras which split over K. The Brauer group $Br(k)$ is then called the *absolute Brauer group.*

Checking that $Br(K|k)$ is in fact an abelian group is completely analogous to the case of $Br(k)$ and is left to the reader.

Definition 3.2. Let G and A be groups. Then a *left G-action on A* is a map $m : G \times A \to A$ satisfying

(i) $m(g, m(h, a)) = m(gh, a)$, and
(ii) $m(g, ab) = m(g, a)m(g, b)$

for all $a, b \in A$ and $g, h \in G$. We usually denote $m(g, a)$ by $g(a)$. Given groups G and A and a left G-action, we say that *G acts on A on the left.* There is of course an analogous notion of right action; however, in this paper, we will only be concerned with left actions, and by "group action" we shall always mean "left group action."

Example 3.3. Classic examples of group actions include S_n acting on $\mathbb{R}^n$ by sending $\sum_{i\leq n} \alpha_i e_i$ to $\sum_{i\leq n} \alpha_i e_{\pi(i)}$, where e_i are the standard basis vectors; $\mathbb{Z}$ acting on any group G via $z(g) = g^z$; and any of the groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n,\mathbb{R})$, or $\mathrm{SO}(n,\mathbb{R})$ acting on $\mathbb{R}^n$.

Definition 3.4. Let G be a group and A a group on which G acts on the left. A *1-cocycle of G with values in A* is a map, denoted a_-, from G to A, such that $a_{\sigma\tau} = a_\sigma\sigma(a_\tau)$ for all $\sigma,\tau \in G$. This equality is called the *cocycle relation*. We say that two 1-cocycles a_- and b_- are *cohomologous* if and only if for some $c \in A$, the relation $a_\sigma = c^{-1}b_\sigma c$ holds for all $\sigma \in G$. It is easily checked that "are cohomologous" is an equivalence relation. We denote the set of equivalence classes of 1-cocycles by $H^1(G,A)$, and call this the *first cohomology set of G with values in A*. Note that $H^1(G,A)$ is a *pointed* set, the basepoint being given by (the equivalence class of) the trivial cocycle $triv_- : \sigma \mapsto 1$.

Example 3.5. If a_- is a cocycle of G with values in A, then for any $b \in A$, the map $a^*_- : G \to A$ defined by $a^*_\sigma = b^{-1}a_\sigma\sigma(b)$ is a cocycle as well.

Example 3.6. One of the fundamental concepts of Galois theory is the norm map: given a finite Galois extension $K|k$ and an element $\alpha \in K$, the *norm of α* , written $N_{K|k}(\alpha)$, is defined to be $\prod_{\sigma\in G(K|k)} \sigma(\alpha)$, i.e., the product of the automorphic images of α. Since $K|k$ is finite, this is always defined. As this example shows, there is a relationship between the norm of an element and 1-cocycles of the Galois group.

Let $K|k$ be a finite Galois extension with Galois group G such that G is cyclic of order n. Let A be the multiplicative group of K, and let σ be a generator of G. Then any 1-cocycle a_- is determined by its value on σ, a_σ. Using the cocycle relation over and over again, we get $a_{\sigma^i} = a_\sigma\sigma(a_\sigma)\sigma^2(a_\sigma)...\sigma^{i-1}(a_\sigma)$ for $1 \leq i \leq n$. Setting $i = n$ gives $a_{\sigma^n} = a_\sigma\sigma(a_\sigma)\sigma^2(a_\sigma)...\sigma^{n-1}(a_\sigma)$. This latter equals a_1, since $\sigma^n = 1$ for all $\sigma \in G$. But $a_1 = 1$, again by the cocycle relation. Recall, however, that $a_\sigma\sigma(a_\sigma)\sigma^2(a_\sigma)...\sigma^{n-1}(a_\sigma)$ is just the norm of a_σ, since G is a cyclic group generated by σ. Thus, we have that an element of K has norm 1 if it is in the image of some cocycle.

Moreover, it is clear that if $\alpha \in K$ has norm 1, then α is in the image of some cocycle. In particular, consider the cocycle a_- defined by $a_{\sigma^i} = \prod_{1\leq j\leq i} \sigma^j(\alpha)$. It is trivial to check that this is indeed a cocycle, and clearly $\alpha = a_\sigma$.

Thus, if $K|k$ is a cyclic Galois extension of degree n, an element of K has norm 1 if and only if it is in the image of some cocycle of $G(K|k)$ with values in $K/\{0\}$. $\square$

There is a subtle connection between cocycles and twisted forms. Let k be a field, A a central simple k-algebra not isomorphic to a matrix algebra, and $K|k$

a finite Galois extension with Galois group G over which A splits. Let $n > 0$ be such that $A \otimes_k K \cong M_n(K)$. Let i be an isomorphism which takes $A \otimes_k K$ to $M_n(K)$. I want to associate a cocycle a^i_- of G with values in $Aut_K(A \otimes_k K)$, where $Aut_K(A \otimes_k K)$ is the group of automorphisms of $A \otimes_k K$ as a K-algebra, to i in a natural way..

Let $\sigma \in G$. Then $\sigma : K \to K$. Tensoring on both sides by A gives a map $\sigma^* : A \otimes_k K \to A \otimes_k K$, defined by $\sigma(\sum \alpha_i \otimes \lambda_i) = \sum(\alpha_i \otimes \sigma(\lambda_i))$. Similarly, tensoring by $M_n(k)$ gives a map $\sigma_* : M_n(K) \to M_n(K)$ via Lemma 1.5. Now any K-linear map $f : A \otimes_k K \to M_n(K)$ induces a map $\#_\sigma(f) : A \otimes_k K \to M_n(K)$ defined by $\#_\sigma(f) = \sigma_*(f(\sigma^{*-1}))$. This in turn induces the following map associated to the isomorphism i, $a^i_- : G \to Aut_K(A \otimes_k K)$, given by $a^i_\sigma = i^{-1}(\#_\sigma(i))$. One can check that the map a^i_- satisfies the cocycle relation, and so is a 1-cocycle.

The key here is the following observation: let j be any *other* isomorphism from $A \otimes_k K$ to $M_n(K)$. Then the cocycles a^i_- and a^j_- are cohomologous (exercise; use Theorem 2.15), and so lie in the same equivalence class of $H^1(G, Aut(A))$. This gives us a map Θ from $CSA_n(K)$ (where $CSA_n(K)$ is the set of n-dimensional central simple algebras which split over K) to $H^1(G, Aut_K(A \otimes_k K))$. Our instinct is to assert that this map is in fact a bijection. In order to do this, we will exhibit a map $\Omega : H^1(G, Aut_K(A \otimes_k K)) \to CSA_n(K)$ such that Ω is the inverse of Θ. This will show that Θ is indeed a bijection.

The intuitive idea for coming up with Ω is based on the following construction:

Construction 3.7. Let G and A be groups, with a group action of G on A. Also, suppose X is a set on which G and A act in a compatible way, i.e., $\sigma(a(x)) = \sigma(a)\sigma(x)$ for all $\sigma \in G$, $a \in A$, $x \in X$. Finally, let a_- be a 1-cocycle of G with values in A. Then we define the *twisted action of G on X by a_-* to be the map $(\sigma, x) \mapsto a_\sigma(\sigma(x))$. This is, possibly despite appearances, a G-action: by the cocycle relation, $(\sigma\tau, x) = a_{\sigma\tau}((\sigma\tau)(x)) = a_\sigma\sigma(a_\tau)((\sigma\tau)(x)) = a_\sigma\sigma(a_\tau\tau(x)) = (\sigma, (\tau, x))$. Note also that if X has some algebraic structure - for instance, if X is an algebra - and G and A act on X by automorphisms, then the twisted action will also be an automorphism. We write ${}_aX$ to denote that X is equipped with the twisted action given by the cocycle a_-, and we will write ${}_aX^G$ to denote the subset of X fixed by the twisted G-action.

Now we can define Ω: given a cocycle of G with values in A, apply the above construction with $G = G(K|k)$, $A = Aut_K(A \otimes_k K)$, $X = A \otimes_k K$, and define $\Omega(a_-)$ to be the k-algebra ${}_aA^G$. From here, all we must show is that $\Omega(a_-)$ is actually a twisted form of A, and that if a_- and b_- are cohomologous, then $\Omega(a_-) = \Omega(b_-)$. The proof of these facts is beyond the scope of this paper (see [2]), but we mention the key idea:

Definition 3.8. Let $K|k$ be a finite Galois extension, G its group, V a K-vector space. Then a *semi-linear G-action* is a left G-action on the group V of vectors such that, for all $\lambda \in K$, $g \in G$, $v \in V$, $g(\lambda v) = g(\lambda)g(v)$, where $g(\lambda)$ is the image of λ under the K-automorphism represented by g. We define $V^G = \{x \in V : (\forall g \in G)(g(x) = x)\}$.

Lemma 3.9. (Speiser) *Let $K|k$ be a finite Galois extension with Galois group G, and let V be a K-vector space given a* semi-linear G-action. *Then the map $\theta : V^G \otimes_k K \to V$, given by $\theta(\sum v_i \otimes \lambda_i) = \sum(\lambda_i v_i)$, is an isomorphism.*

This lemma essentially tells us that if we take a vector space and use Construction 1.3 to extend scalars, we can recover the original vector space from the extended one, up to twistedness. The result we are trying to prove - that Ω is just the inverse of Θ, and thus that Θ is a bijection - is just a generalization of this lemma to include algebras. Speiser's Lemma is the beginning of an area of algebra known as *descent theory*; for a relatively survivable introduction to this field, the reader is referred to Keith Conrad's paper [0].

Thus we have the following result:

Theorem 3.10. *Let n be a fixed natural number. Let $CSA_K(n)$ denote the set of central simple algebras which split over K and which, when so split, become isomorphic to $M_n(K)$, let G be the Galois group of $K|k$, and let A be the group of automorphisms of $M_n(K)$. Then the map Θ defined above is an isomorphism.* [3]

We can in fact state this theorem in a more concise way via the following lemma:

Lemma 3.11. *The automorphism group of the matrix algebra $M_n(K)$, $Aut_K(M_n(k))$, is the group $PGL_n(K)$.*

Proof: $PGL_n(K)$, the projective general linear group, is the group $GL_n(K)/(\{\lambda I_n : \lambda \in K\})$, where I_n is the identity matrix; $PGL_n(K)$ can be thought of as $GL_n(K)$ quotiented in such a way as to ignore multiplication by scalars in K. There is a natural homomorphism $h : GL_n(K) \to Aut(M_n(K))$ sending $C \in GL_n(K)$ to the automorphism $m : M \mapsto C^{-1}MC$. Since matrix algebras are central simple, we have that all automorphisms of matrix algebras are inner (see Theorem 2.13), and thus h is surjective; and the kernel of h is readily seen to be the center of $M_n(K)$, which is just $\{\lambda I_n : \lambda \in K\}$, since $M_n(K)$ is central simple. Thus, the first isomorphism theorem yields an isomorphism $GL_n(K)/Z(M_n(K)) \cong Aut(M_n(K))$. But $GL_n(K)/Z(M_n(K))$ is just $PGL_n(K)$. So we are done. $\square$

This allows us to restate Theorem 3.10 as "$CSA_n(K) \cong H^1(G, PGL_n(K))$," since $A \otimes_k K \cong M_n(K)$ by assumption.

We now use Theorem 3.10 to give a characterization of the relative Brauer group $Br(K|k)$ in terms of cohomology sets using one final algebraic construction:

Construction 3.12. (Direct Limits) Let $\mathcal{C}$ be a category. Let I be a set partially ordered by $\leq_D$, and suppose we are given a collection of objects $\{X_i : i \in I\} \subset \mathcal{C}$ together with morphisms $f_{ij} : X_i \to X_j$ such that for all $i \leq_D j \leq_D k$, $f_{ij} \circ f_{jk} = f_{ik}$. If $A \in \mathcal{C}$ and $\{\phi_i : i \in I\}$ is a set of morphisms with $\phi_i : X_i \to A$ such that for all $i \leq_D j$, $\phi_j \circ f_{ij} = \phi_i$, then we say that $(A, (\phi_i))$ is a *cone*. The *direct limit of*

[3] The reader should know that there is a much more general form of this theorem, which deals not just with algebras but with twisted forms of ANY type of k-vector space equipped with extra structure. This more general version of the theorem is proved in the same way but to describe formally what it says would take us too far afield. Essentially, it says that the bijection between twisted forms and cocycles generalizes to almost any context we would want it to.

$\{X_i\}$ *via the* f_{ij} is the unique (up to isomorphism) cone $(X, (\phi_i))$ (if such a cone exists) such that for any cone $(Y, (\psi_i))$, there is exactly one morphism $u : X \to Y$ such that for all $i, j \in I$, with $i \leq_D j$, the following diagram commutes:

$$\begin{tikzcd}
X_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & X_j \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\
& X \arrow[d, "u"] & \\
& Y &
\end{tikzcd}$$

In the case where the maps f_{ij} are injections, the direct limit is intuitively the infinite union of the X_i, where the injections f_{ij} are treated as actual inclusions. So, for instance, this allows us to take the "union" of the sets $\{(x_1, ..., x_n) : x_i \in \mathbb{R}\}$ over all $n \in \mathbb{N}$ by identifying $(x_1, ..., x_n)$ with $(x_1, ..., x_n, 0)$ for all n; the result is then isomorphic to the set of sequences of real numbers all but finitely many of which are zero (exercise; note that this also allows us to construct infinite direct sums). In the following theorem, we use the direct limit to extend Theorem 3.10 to the whole relative Brauer group $Br(K|k)$; we do not do so rigorously, due to constraints of space, however, and thus the task of making the arguments complete is unfortunately left to the reader.

Theorem 3.13. *The group $Br(K|k)$ can be realized as a first cohomology set.*

Proof: Again, let $K|k$ be a finite Galois extension with group G. For any $m, n > 0$, consider the natural injective map $\psi_{m,n} : GL_m(K) \hookrightarrow GL_{mn}(K)$ given by sending an m-by-m matrix M to the block matrix given by placing n copies of M along the diagonal and zeroes everywhere else. Quotienting by scalar multiples of the identity matrix, we get injections $\psi_{m,n} : PGL_m(K) \hookrightarrow PGL_{mn}(K)$ for every $m, n < 0$. These induce injective maps $\lambda_{m,n} : H^1(G, PGL_m(K)) \hookrightarrow H^1(G, PGL_{mn}(K))$. We now take the direct limit of the $H^1(G, PGL_i(K))$ via the inclusion maps $\lambda_{m,n}$. Call this direct limit $H^1(G, PGL_\infty(K))$. This is not yet a very meaningful name, as we have no idea what $PGL_\infty(K)$ actually is. If, however, we let $PGL_\infty(K)$ be the direct limit of the $PGL_i(K)$ via the maps $\psi_{m,n}$, we then get an actual group. It can then be proven that the first cohomology set of G with values in $PGL_\infty(K)$ is precisely the direct limit of the $H^1(G, PGL_i(K))$, thus justifying our choice of notation. Correspondingly, we can write $Br(K|k)$ as the direct limit of the $CSA_n(K)$ (why?). Thus, the relative Brauer group can be identified with a first cohomology set. Moreover, it should be noted that a natural product operation can be placed on this cohomology set, which turns the identification mentioned above into an isomorphism of (abelian) groups. $\square$

In a similar vein, we can identify the *entire* Brauer group $Br(k)$ with a first cohomology set, but the specifics would take us too far afield.

In actual applications, we generally find it much more useful to identify the relative (and absolute) Brauer group with a *second* cohomology set of G. However, this result is beyond the scope of this paper to prove or study in detail. For completeness' sake, however, we state it.

Definition 3.14. Let G be a group and A a group on which G acts from the left. Then a *2-cocycle of G with values in A* is a map from $G \times G$ to A, denoted $a_{-,-}$, satisfying

$$\sigma_1 a_{\sigma_2,\sigma_3} - a_{\sigma_1\sigma_2,\sigma_3} + a_{\sigma_1,\sigma_2\sigma_3} - a_{\sigma_1,\sigma_2} = 0$$

for all $\sigma_1, \sigma_2, \sigma_3 \in G$. This equality is called the *2-cocycle relation*. As before, there exists a "nice" equivalence relation on 2-cocycles, and the set of equivalence classes of 2-cocycles of G with values in A is denoted $H^2(G, A)$ and called the *second cohomology set*.

Then we have the following result:

Theorem 3.15. *Let $K|k$ be a finite Galois extension with Galois group G. Then $Br(K|k) \cong H^2(G, K^\times)$ and $Br(k) \cong H^2(k, k_s^\times)$, where $k_s^\times$ is the multiplicative group of the separable closure of k.*

Proof: See [2]. The value of this theorem can actually be immediately seen, by noting the lack of direct limits or other complicated constructions. □

4. Algebraic Geometry and Severi-Brauer Varieties

In the last section of this paper, we use techniques of algebraic geometry to study Brauer algebras. This requires a significant amount of algebraic geometry, which we do not have the requisite space to develop in full, so we forego all pretension of rigor. We refer the reader to [1] for a treatment of algebraic geometry, and to [2] for a more complete, if more advanced, exposition of the applications to Brauer algebras.

Definition 4.1. Let K be a field. Then we define *n-dimensional K-projective space* $K\mathbb{P}^n$as $K\mathbb{P}^n = (K^n \backslash \{(0, 0, ..., 0)\}) / \approx$, where $\approx$ is the equivalence relation defined by $(x_1, ..., x_{n+1}) \approx (y_1, ..., y_{n+1})$ iff $(y_1, ..., y_{n+1}) = (\lambda x_1, ..., \lambda x_{n+1})$ for some $\lambda \in K \backslash \{0\}$. n-dimensional K-projective space can be thought of as the set of lines through the origin in $(n+1)$-dimensional K-Cartesian space, which yields a particularly useful geometric intuition.

Construction 4.2. Let k be a field of characteristic $\neq 2$, let $\overline{k}$ be an algebraic closure of k, and let $(a, b)_k$ be a generalized quaternion algebra. Then we define the *associated conic of $(a, b)_k$* as the set $C_k(a, b) = \{[(x, y, z)] \in \overline{k}\mathbb{P}^n : ax^2 + by^2 = z^2\}$, where [ ] denotes "the equivalence class of." Note that this is well-defined: if $ax^2 + by^2 = z^2$, then $a(\lambda x)^2 + b(\lambda y)^2 = (\lambda z)^2$ for any $\lambda \neq 0$.

Definition 4.3. $C_k(a,b)$ has a *k-rational point* iff $(\exists x,y,z \in k)(ax^2+by^2=z^2)$.

We then have the following useful result:

Theorem 4.4. *The properties of quaternion algebras are captured to a large extent by their associated conic. Specifically,*

(i) $C_k(a,b)$ has a k-rational point iff $(a,b)_k \cong M_2(k)$.

(ii) **(Witt's Theorem)** $(a,b)_k \cong (c,d)_k$ if and only if $C_k(a,b) \cong C_k(c,d)$, that is, if and only if there is a bijection $m : C_k(a,b) \to C_k(c,d)$ which is given by some rational function with coefficients from k.

Proof: To prove (i) one first defines the norm of an element x of a finite field extension K of k as the product of the automorphic images of x (see Example 3.6), and then shows *(a)* that $C_k(a,b)$ has a rational point if and only if $b = N_{k(\sqrt{a})}(\alpha)$ for some $\alpha \in k(\sqrt{a})$, and *(b)* that $b = N_{k(\sqrt{a})}(\alpha)$ for some $\alpha \in k(\sqrt{a})$ if and only if $(a,b)_k \cong M_2(k)$. This yields the desired result.

Part (ii) is unfortunately well beyond the scope of this paper. A proof can be found in [2], but this proof is quite difficult and uses a large amount of algebraic geometry, more even than is strictly necessary. □

A natural question to ask at this point is whether we can extend a version of this result to *all* Brauer algebras. The answer to this question is yes, and the method of extension is one of the more beautiful concepts in the study of central simple algebras.

Before we describe the more general version of the associated conic, we must define one of the main objects of study of algebraic geometry:

Definition 4.5. Let S be a set of homogeneous polynomials (that is, for every polynomial $p \in S$, the terms of p have the same total degree - so $x^2+xy+zy+zx$ is homogeneous, but x^2+x+1 is not; see [1]) in n variables. An *algebraic variety in* $\overline{k}\,\mathbb{P}^n$ is then the solution set $Z(S) = \{(x_1,...,x_n) \in \overline{k}\,\mathbb{P}^n : (\forall f \in S)(f(x_1,...,x_n) = 0)\}$.

Now, in a natural sense, $k\,\mathbb{P}^n$ lives inside $\overline{k}\,\mathbb{P}^n$: identify $k\,\mathbb{P}^n$ with the set of (equivalence classes of) points in $\overline{k}\,\mathbb{P}^n$ all of whose coordinates lie in k. Furthermore, in the same way, for any finite Galois extension $K|k$, or for that matter any field extension at all contained in $\overline{k}$, we have that $k\,\mathbb{P}^n \subset K\,\mathbb{P}^n \subset \overline{k}\,\mathbb{P}^n$. Now given an algebraic variety $\mathbb{V}$ and a finite Galois extension $K|k$, let $\mathbb{V}_K$ be the set of points in $\mathbb{V}$ lying in $K\,\mathbb{P}^n$. We then have the following result:

Theorem 4.6. *Let k be a field. Then for any Brauer k-algebra A, there is an algebraic variety $SB(A)$ of $\overline{k}$-projective space - called the* Severi-Brauer variety *of A - such that if $K|k$ is a finite Galois extension, then $SB(A)_K \neq \emptyset$ if and only if $A \otimes_k K \cong M_n(K)$ where $n = \dim(A)$.*

The proof of this thorem is far beyond the scope of this paper. However, this is possibly the most beautiful facet of Brauer algebras, as Severi-Brauer varieties arise in algebraic geometry independently of Brauer algebras, and thus yield a serious application of Brauer algebras to algebraic geometry.

5. Addenda

The main source for this paper was the book "Central Simple Algebras and Galois Cohomology" by Gille and Szamuely. This book contains all of the information presented here and much more, but assumes a fairly strong knowledge of Galois theory and commutative algebra, and a little algebraic geometry. There is unfortunately no elementary introduction to the subject at this time. However, there is a paper by Jorg Jahnel ("The Brauer-Severi Variety Associated With A Central Simple Algebra: A Survey"), which deals with group cohomological and algebro-geometric techniques in the study of central simple algebras, many parts of which are readable, and which covers descent in detail. In addition, the paper [5] contains a description of the basic algebraic properties of central simple algebras, including a proof of the Noether-Skolem Theorem and a generalization of the second half of Example 2.4. A source which the author has not had sufficient time to examine, but appears at first glance to be both readable and relevant, is the book "Noncommutative Algebra" by R. Keith Dennis and Benson Farb.

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