EQUITABLE PARTITIONS IN GRAPH THEORY

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Abstract. I will define the distance partition of a finite graph and show how this partition being equitable follows from and/or implies properties of the graph. In the process I will connect this partition to a number of fundamental ideas in graph theory and confirm an elementary identity of strongly regular graphs.

Contents

1. Introduction

Partitions of the vertices of graphs are a fundamental concept in graph theory, forming the basis for such fundamental graph properties as chromatic number. This paper will focus on one partition in particular: the distance-partition. The basic form of the distance-partition assigns a starting vertex to the zeroth cell, then groups the other vertices into cells according to their distance from the starting vertex. We introduce a generalized distance-partition in which vertices are partitioned by their distance from an initial set of vertices, rather than a single vertex.

Throughout this paper, graphs are nonempty, finite, simple, and undirected. Nearly all of the graphs we consider will be connected as well–this ensures that the distance-partition is well-defined and meaningful.

In order to determine graph properties from these distance-partitions, it is helpful to consider equitable partitions, a special class of partitions satisfying the following property: given two cells C_i and C_j of the partition, there is a constant $b_{i,j}$ such that a vertex $v_i \in C_i$ has exactly $b_{i,j}$ neighbors in C_j, regardless of the choice of v_i. Equitable partitions are related to many basic properties of graphs. For example, that the trivial partition (that is, the partition that puts every vertex in a single cell) is equitable only if the graph is regular. Equitable partitions have a number of applications in graph theory. Partitioning the vertices of a graph into their orbits under a group of automorphisms is always an equitable partition, and this fact has been exploited in the development of practical graph isomorphism algorithms ([4]).

In this paper we will derive properties of graphs that follow from the distance-partition being equitable, and discuss graph properties that imply that the distance-partition is equitable. In doing so, we connect the distance-partition to the well-known properties of strong regularity, distance-regularity, and diameter.

We end by introducing the notion of distance-equitable induced subgraphs: the set of induced subgraphs H of a graph G such that, for any $S \subseteq V(G)$ that induces H, the generalized distance partition beginning at S is equitable. K_1 being a distance-equitable induced subgraph is equivalent to the (usual) distance partition being equitable at any vertex, which is in turn equivalent to the graph being distance-regular. In the final section, we will show that $\overline{K_2}$ being distance-equitable implies diameter 2, and that K_1, K_2, and $\overline{K_2}$ being distance-equitable induced implies that every odd cycle induces a complete graph. Much of this paper has been informed by the classic texts of Biggs [1] and Godsil & Royle [3]. However, we believe that the generalized distance partition, the notion of distance-equitable induced subgraph, and the corresponding results are original.

The paper is organized as follows. In Section 2 we define the distance partition of a graph and discuss its basic properties. Section 3 will describe how the property of strong regularity relates to the distance partition.In Section 4 we discuss a generalized distance partition and the consequences of certain conditions. Finally, in Section 5 we mention some questions we have not had time to attempt.

2. Equitable Partitions and the Distance Partition

In this section, we give the definitions of equitable partition, the distance partition, and distance-regularity, culminating with a proof that a graph is distance-regular if any only if every distance partition is equitable.

Definition 2.1. A partition $\pi = \{C_0, C_1, \ldots, C_{m-1}\}$ of the n vertices of a graph G is *equitable* if for every pair of (not necessarily distinct) indices $i, j \in \{0, 1, \ldots, m-1\}$ there is a nonnegative integer $b_{i,j}$ such that each vertex v in C_i has exactly $b_{i,j}$ neighbors in C_j, regardless of the choice of v. The *partition matrix* is $B_\pi = (b_{i,j})$.

Note that the partition matrix B_π is well-defined if and only if the partition π is equitable.

Definition 2.2. Let G be a nonempty finite connected graph, and v a vertex of G. The *distance partition* $\pi_d(v)$ of G relative to v consists of the cells:

- $C_0 = \{v\}$
- $C_j = \{u \in G | d(u, v) = j\}$ for each $j = 1, 2, 3, \ldots$

That is to say, each cell consists of vertices that are a fixed distance from v_0.

Note that the number m of nonempty cells is bounded above by $1 + \mathrm{diam}(G)$, which is finite for any finite connected graph.

At this point, it is natural to ask when $\pi_d(v)$ is equitable. We will see in Theorem 2.5 that $\pi_d(v)$ being equitable for every $v \in V(G)$ is equivalent to the well-studied property of distance-regularity (For an in-depth look at distance-regular graphs, see e.g. the work of Brouwer, *et al.* [2].)

Definition 2.3. A graph G is *distance-regular* if, for any pair of vertices v and u with $d(v, u) = i$,

(1) there is a number p_i such that there are p_i neighbors of u that are of distance $i - 1$ from v

(2) there is a number q_i such that there are q_i neighbors of u of distance $i+1$ from v,

where p_i and q_i depend only on i (and not on v nor u.)

The following lemma will be useful at several points in the remainder of this paper.

Lemma 2.4. *Let G be a graph, v_0 a vertex of G. If $\pi_d(v_0) = \{C_0, C_1, \dots\}$ and $|i-j| > 1$ then no vertex in C_i is adjacent to any vertex in C_j. In particular, if $\pi_d(v_0)$ is equitable and $B_\pi = (b_{i,j})$ is the associated partition matrix, then $b_{i,j} = 0$ whenever $|i-j| > 1$.*

Proof. Suppose for the purposes of contradiction that, for some pair of indices i, j with $|j-i| > 1$, there exist a vertex $v_i \in C_i$ that is adjacent to some vertex $v_j \in C_j$. We may assume without loss of generality that $j > i$. By definition of C_i, $d(v_0, v_i) = i$. Let P be a path of length i between v_0 and v_i, and let $P' = (P, v_j)$. P' is a path of length $i+1$ from v_0 to v_j, so $j = d(v_0, v_j) \leq i+1$. On the other hand, $|j-i| > 1$ and so $j > i+1$. This is a contradiction, proving the claim. □

Note that, by taking $i = 0$ in the definition of distance-regular, G is regular of degree q_0. This fact in addition to the preceding lemma will help us in the following proof.

Theorem 2.5. *The distance partition $\pi_d(v)$ of a graph G is equitable for every $v \in V(G)$ if and only if G is distance-regular.*

Proof. ($\Rightarrow$) Suppose $\pi_d(v)$ is equitable for any vertex v of a graph G. Let v, u be a pair of vertices of distance i apart. Consider $\pi_d(v)$ with cells $C_0, \dots, C_m$ as in Definition 2.2. Clearly $u \in C_i$. Note that any neighbors of u that are of distance $i-1$ from v are in C_{i-1}. The number of neighbors of u that are in C_{i-1} is given by the entry $b_{i,i-1}$ of the partition matrix of $\pi_d(v)$. Furthermore, as this same value holds for any vertex of C_i and therefore any vertex of distance i from v, it follows that $b_{i,i-1}$ satisfies the properties of p_i in the definition of distance-regular.

Similarly, $b_{i,i+1}$ satisfies the properties of q_i. Since both p_i and q_i are determined by values of the partition matrix, it follows that any graph where π_d is equitable for every vertex must be distance-regular.

($\Leftarrow$) Conversely, suppose G is distance-regular with p_i, q_i as in Definition 2.3. Let $v_0 \in V(G)$, and consider $\pi_d(v_0)$. We want to show that this partition is equitable, or equivalently that the values of the partition matrix B_π are well-defined. We will show that

$$b_{i,j} = \begin{cases} q_i & \text{If } i = j-1 \\ q_0 - p_i - q_i & \text{If } i = j \\ p_i & \text{If } i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

Consider the ith row of the partition matrix. Let $v_i \in C_i$, which is to say that $d(v_i, v_0) = i$. By definition of the distance-regular parameters, v_i has p_i neighbors that are of distance $i-1$ from v_0, independent of the choice of $v_i \in C_i$. Because these neighbors must by definition be in C_{i-1}, $b_{i,i-1}$ is well-defined and equal to p_i for all i. Similarly, v_i has q_i neighbors of distance $i+1$ from v_0 regardless the choice of $v_i \in C_i$, so $b_{i,i+1}$ is well defined and equal to q_i for all i.

Next, note that the sum of any row of the partition matrix must add up to q_0. This property follows from G being q_0-regular, as remarked above. By Lemma 2.4, all entries on the ith row of B_π other than $b_{i,i-1}$, $b_{i,i}$, and $b_{i,i+1}$ are well-defined and equal to zero. Thus $b_{i,i} = q_0 - p_i - q_i$ is well-defined, so $\pi_d(v)$ is equitable for every $v \in G$. □

3. Strongly Regular Graphs

Definition 3.1. A graph G is *strongly regular* with parameters (n, k, a, c) if it is not K_n nor $\overline{K_n}$ and:

1. G has n vertices,
2. G is k-regular,
3. any pair of adjacent vertices has exactly a common neighbors,
4. any pair of non-adjacent vertices has exactly c common neighbors.

Lemma 3.2. *Let G be a strongly regular graph. Then diam$(G) = 2$.*

Proof. Let G be a strongly regular graph with parameters (n, k, a, c), and suppose there are two vertices u, x of G that are distance 3 apart. Let (u, v, w, x) be a path of length 3. The pair of vertices u and x are non-adjacent, therefore they must have exactly c neighbors in common. It is clear in this case that c must equal zero: if u and x shared some neighbor y, then (u, y, x) would be a path of length 2 from u to x, a contradiction.

The pair of vertices u and w must also be non-adjacent, or else (u, w, x) would be a path of length 2 from u to x. As u and w are non-adjacent, they have $c = 0$ common neighbors. But, v is a common neighbor of u and w, a contradiction. Thus G must have a diameter of at most 2.

Suppose $\text{diam}(G) = 1$. Then there is a path of length 1 from any vertex v to any other vertex u, which is to say that there is an edge between them. Thus, G is complete, and is therefore excluded from being strongly regular by definition. Likewise, if $\text{diam}(G) = 0$, it is clear that G consists only of a single point and is therefore K_1.

□

Strongly regular graphs are distance-regular (as we shall see in the next result), so it follows from Theorem 2.5 that π_d of a strongly regular graph relative to any vertex v is equitable. In order to understand how π_d partitions a strongly regular graph, we will examine the properties of the corresponding cells. We will first find the partition matrix B_π, with $b_{i,j}$ in terms of the parameters of any strongly regular graph.

Theorem 3.3. *The distance-partition $\pi_d(v_0)$ of a strongly regular graph G with parameters (n, k, a, c) is equitable with partition matrix*

$$B_\pi = \begin{pmatrix} 0 & k & 0 \\ 1 & a & k-1-a \\ 0 & c & k-c \end{pmatrix}$$

In particular, every strongly regular graph is distance regular.

Proof. We proceed term by term, indexing rows and columns starting at zero.

$b_{0,0}$ is the number of neighbors of v_0 within C_0. As C_0 consists only of v_0 and G has no loops, $b_{0,0} = 0$.

$b_{0,1}$ is the number of neighbors of v_0 within C_1. As G is k-regular, v_0 has k neighbors, and the neighbors comprise C_1 as they are all the points of distance 1 from v_0. Therefore, it follows that $b_{0,1} = k$.

$b_{0,2}$ is zero by Lemma 2.4.

$b_{1,0}$ is 1, as each member of C_1 has precisely one neighbor in C_0 – namely, v_0.

$b_{1,1}$ can be computed as follows: Let $w \in C_1$. Then v_0 and w are adjacent, so by the definition of strongly regular v_0 and w have a common neighbors. As all of these a neighbors are neighbors of v_0, they are in C_1. Moreover, any neighbor of w in C_1 is necessarily a neighbor of v_0. Thus, any vertex in C_1 has a neighbors in C_1, and so $b_{1,1} = a$.

$b_{1,2}$ is $k - 1 - a$ since each row must sum to k.

$b_{2,0}$ is zero by Lemma 2.4.

$b_{2,1}$ is c. Let $w \in C_2$; then v_0 and w are non-adjacent, and so by the definition of strongly regular they must have c common neighbors. These c neighbors are neighbors of v_0, so they must be in C_1. Conversely, any common neighbor of v_0 and w is, in particular, adjacent to v_0 and thus in C_1, so this set contains all the neighbors of w in C_1. Therefore any $w \in C_2$ has exactly c neighbors in C_1.

$b_{2,2}$ may be computed in a similar manner to $b_{1,2}$: the rows must add up to k, so $b_{2,2} = k$ minus the sum of the other elements of the row: $k - c$.

Since these values are well-defined, $\pi_d(v_0)$ is equitable. Since the values are independent of the choice of v_0, $\pi_d(v_0$ is equitable for any $v_0 \in V(G)$. Thus, by Theorem 2.5, G is distance-regular. □

In the next result, we calculate the sizes of the cells of the distance partition of a strongly regular graph explicitly, and in doing so re-derive a well-known formula relating the parameters of a strongly regular graph.

Theorem 3.4. *Let G be a strongly regular graph with parameters (n, k, a, c) and let $\pi_d(v_0) = \{C_0, C_1, C_2\}$ be its distance partition. Then $|C_0| = 1$, $|C_1| = k$, $|C_2| = n - k - 1 = \frac{(k-1-a)(k)}{c}$. In particular,*

$$c(n - k - 1) = k(k - 1 - a)$$

Proof. The first part follows trivially from the definition of C_0. The second part follows from G being k-regular: v_0 has k neighbors, and these neighbors comprise C_1.

As G is strongly regular and therefore must have diameter of at most 2, there are no points of distance 3 or greater from v_0, and so C_3 and all higher cells are empty. As all the vertices must be covered by C_0, C_1, and C_2, their sizes must add up to the total number of vertices n, which is to say $|C_2| = n - k - 1$. Furthermore, this result shows that C_2 is empty $\iff k = n - 1$, i.e., $G = K_n$.

The following combinatorial argument shows that $|C_2| = \frac{(k-1-a)(k)}{c}$. Consider the number of edges going from C_1 to C_2. There are k vertices in C_1 and we see from entry $b_{1,2}$ of B_π above that each vertex in C_1 has $(k - 1 - a)$ neighbors in C_2. Thus the number of edges (u, v) with $u \in C_1$ and $v \in C_2$ is $(k - 1 - a)(k)$. From entry $b_{2,1}$ we see that every vertex in C_2 has c neighbors in C_1, so the size of C_2 is the number of edges from C_1 to C_2 divided by c. Thus, the actual number of vertices in C_2 is $\frac{(k-1-a)(k)}{c}$. As $|C_2| = \frac{(k-1-a)(k)}{c} = n - k - 1$ from the result above, it follows that $c(n - k - 1) = k(k - 1 - a)$. □

So far, we have shown that a strongly regular graph is distance-regular of diameter 2. We now prove the converse of this result, which will be useful in Section 4.

Theorem 3.5. *Let G be a nonempty finite connected graph with n vertices. Suppose G is distance-regular and diam$(G) = 2$. Then G is strongly regular with parameters $(n, q_0, q_0 - q_1, p_2)$, where p_i, q_i are the distance-regular parameters, as in Definition 2.3.*

Proof. G is q_0-regular, so q_0 fulfills the criteria for the parameter k of a strongly regular graph.

Let u and v be adjacent vertices of G. G is distance regular, so there are q_1 neighbors of u that are of distance 2 from v. In particular, there are q_1 neighbors of u that are not neighbors of v (including v itself). As G is q_0-regular, u has q_0 neighbors in total. By subtraction, the number of neighbors of u that are also neighbors of v is given by $q_0 - q_1$, regardless of the choice of u and v among the adjacent vertices of G. Consequently, $q_0 - q_1$ fulfills the criteria for the parameter a of a strongly regular graph.

As diam$(G) = 2$, there are vertices u, w such that $d(u, w) = 2$, i.e., u and w are non-adjacent. As G is distance-regular, there are p_2 neighbors of u that are of distance 1 from w, regardless of the choice of u and w. This is precisely to say that there are exactly p_2 common neighbors of any two non-adjacent points u and w in G, and so p_2 fulfills the criteria for the parameter c of a strongly regular graph. □

Corollary 3.6. *A graph G is strongly regular if and only if it is distance-regular of diameter 2.*

Proof. This simply combines the results of Theorems 3.3 and 3.5. □

4. A Generalized Distance Partition

Previous sections discussed the distance-partition relative to a single vertex. To further explore the properties of the distance-partition, we introduce the distance-partition relative to any subset S of $V(G)$. In essence, the vertices of G are sorted into cells according to their distance from (the closest vertex of) S.

Definition 4.1. Let G be a nonempty finite connected graph, and S a nonempty subset of $V(G)$. The *(generalized) distance partition* $\pi_d(S)$ of G consists of the cells:

- $C_0 = S$
- $C_j = \{v \in G | \min_{u \in S} d(v, u) = j\}$ for each $j = 1, 2, 3, \ldots$

(We may abuse notation by writing $\pi_d(v_0, v_1, \ldots, v_k)$ instead of $\pi_d(\{v_0, v_1, \ldots, v_k\})$.)

Definition 4.2. Let G be a nonempty finite connected graph. An isomorphism type H is a *distance-equitable induced subgraph* of G if G contains an induced subgraph isomorphic to H, and for every subset $S \subseteq V(G)$ that induces a copy of H, $\pi_d(S)$ is equitable. The set of distance-equitable induced subgraphs of G is denoted $\mathcal{DE}(G)$.

We can now restate Theorem 2.5 as: $K_1 \in \mathcal{DE}(G)$ if and only if G is distance-regular. In the remainder of this section, we will discuss the implications of K_2 and $\overline{K_2}$ being in $\mathcal{DE}(G)$.

Lemma 4.3. *Let G be a nonempty finite connected graph. If $\overline{K_2} \in \mathcal{DE}(G)$, then diam$(G) = 2$.*

Proof. As $\overline{K_2}$ is an induced subgraph of G, it is clear that G is not complete and thus $\text{diam}(G) > 1$. Suppose for the sake of contradiction that $\text{diam}(G) \geq 3$. Then there is a path $P = (u, v, w, x)$ such that u is not adjacent to w and x is not adjacent to v. Thus, $\overline{K_2}$ is induced by $\{u, w\}$. Consider $\pi = \pi_d(u, w)$. The cell C_0 consists of vertices u and w. The cell C_1 contains vertices v and x. By the definition of P, v has two neighbors in C_0, namely u and w, whereas x has exactly one neighbor in C_0, w. This implies that π is not equitable, a contradiction. Thus, $\text{diam}(G) = 2$. □

Corollary 4.4. *Let G be a nonempty finite connected graph. If $\{K_1, \overline{K_2}\} \subseteq \mathcal{DE}(G)$, then G is strongly regular.*

Proof. Suppose $\{K_1, \overline{K_2}\} \subseteq \mathcal{DE}(G)$. Then by Theorem 2.5, G is distance-regular, and by Lemma 4.3, $\text{diam}(G) = 2$. Thus by Theorem 3.5, G is strongly regular. □

The converse of this corollary is false, however: both the five-cycle and the octahedron graph are strongly regular, yet neither has $\overline{K_2}$ as a distance-equitable induced subgraph. Furthermore, the $n \times n$ grid graphs (in which two vertices are connected exactly when they share a row or column) also have this property. It is interesting to note that all of these graphs are line graphs (the $m \times m$ grid is the line graph of $K_{m,m}$).

From the last corollary, K_1 and $\overline{K_2}$ being in $\mathcal{DE}(G)$ is already enough to imply that G is strongly regular. We will now see that K_1, K_2 and $\overline{K_2}$ being in $\mathcal{DE}(G)$ is an exceptionally strong restriction on G.

Theorem 4.5. *Let G be a nonempty finite connected graph. If $\{K_1, K_2, \overline{K_2}\} \subseteq \mathcal{DE}(G)$, then every odd cycle of length ℓ in G induces the complete graph K_ℓ.*

Proof. Let $C = (v_0, v_1, \ldots, v_{\ell-1})$ denote the vertices of a cycle of odd length ℓ in G. As G is distance-regular, it is enough to show that v_0 is connected to every other vertex of C, since the same argument may be used for any vertex of the cycle.

We will show by induction that v_0 is connected to every vertex of index $2k+1$ (all indices are taken modulo ℓ) for all k, and since C has odd length, this implies v_0 is connected to every vertex in the cycle. By our choice of numbering, v_0 is connected to v_1. Suppose v_0 is connected to a vertex v_{2k+1}. Consider $\pi_d(v_0, v_{2k+2}) = \{C_0, C_1, ...\}$, which is equitable by assumption[1]. As v_{2k+1} and v_{2k+3} are adjacent to v_{k+1}, both are in C_1. v_{2k+1} is connected to v_0 by assumption, and is also connected to v_{2k+2}. It therefore has two neighbors in C_0, and since $\pi_d(v_0, v_{2k+2})$ is equitable, all other members of C_1 must also have two neighbors in C_0. In particular, v_{2k+3} has two neighbors in C_0, i. e., $v_{2(k+1)+1}$ is adjacent to v_0. Thus v_0 is connected to every vertex of index $2k+1$, and since the cycle has odd length, this includes every vertex in the cycle. □

Again, the converse of this theorem is false: obviously C_5 is a strongly regular graph in which a 5-cycle doesn't induce K_5. The remainder of this section will discuss the family of graphs satisfying the *consequence* of Theorem 4.5.

[1]Since both K_2 and $\overline{K_2}$ are distance-equitable induced subgraphs, $\pi_d(v_0, v_{2k+2})$ is equitable whether or not v_0 and v_{2k+2} are adjacent. This is the key that allows this argument to be applied starting from any vertex of C.

4.1. **Graphs in which every odd cycle induces a complete graph.** We begin by proving that this family is infinite by showing that any complete bipartite graph of the form $K_{m,m}$ for $m > 3$ has this property. These graphs have no odd cycles, so the strong consequence ofTheorem 4.5 is vacuously satisfied.

Definition 4.6. A graph G is *bipartite* if there exist two disjoint sets $V_0, V_1 \subseteq V(G)$ whose union is $V(G)$ and that have the property that any vertex in V_i has no neighbors in V_i.A bipartite graph is *complete bipartite* if any vertex v in V_0 is connected to every member of V_1 and vice versa. These graphs are uniquely determined by the sizes of V_0 and V_1 up to isomorphism and are denoted $K_{a,b}$, where $a = |V_0|$ and $b = |V_1|$.

Lemma 4.7. *The complete bipartite graph $K_{a,b}$ is strongly regular if and only if $a = b$.*

Proof. Suppose $K_{a,b}$ is strongly regular. Let V_0 and V_1 be the bipartition of $K_{a,b}$ and let $v_0 \in V_0$ and $v_1 \in V_1$. As $K_{a,b}$ is complete bipartite, v_0 is connected to every member of V_1 and therefore has degree $|V_1| = b$. Likewise v_1 has degree $|V_0| = a$. As $K_{a,b}$ is strongly regular, $b = k = a$ and so $a = b$.

Conversely, suppose $a = b = m$. $K_{a,b}$ then has $2m$ vertices. Any vertex has m neighbors, namely all those vertices that are in the opposite V_i. Any pair of adjacent vertices has no neighbors in common, as each has neighbors exclusively within the other's V_i. Any pair of non-adjacent vertices has m neighbors in common, as they are both connected to each member of the opposite V_i. Thus $K_{a,b} = K_{m,m}$ is strongly regular with parameters $(2m, m, 0, m)$. □

Theorem 4.8. $\{K_1, K_2, \overline{K_2}\} \subseteq \mathcal{DE}(K_{m,m})$.

Proof. By the preceding lemma, $K_{m,m}$ is strongly regular with parameters $(2m, m, 0, m)$. In particular, $K_{m,m}$ is distance regular, and so by Theorem 2.5 $K_1 \in \mathcal{DE}(K_{m,m})$.

Let V_0, V_1 be the bipartition of $K_{m,m}$. Let $v_0 \in V_0, v_1 \in V_1$ be a pair of adjacent vertices of $K_{m,m}$. Now consider $\pi_d(v_0, v_1) = \{C_0, C_1, \dots\}$. Since v_1 is connected to every member of V_0 and v_0 is connected to every member of V_1, C_1 contains all vertices of $K_{m,m}$ except for v_0 and v_1. Now, we will show that $\pi_d(v_0, v_1)$ is equitable with partition matrix

$$B_\pi = \begin{pmatrix} 1 & m-1 \\ 1 & m-1 \end{pmatrix}$$

- $b_{0,0}$ is 1 as each vertex in C_0 has one neighbor in C_0, namely the other vertex.
- $b_{0,1}$ is $m-1$ as G is m-regular and so the rows of the partition matrix must add up to m.
- $b_{1,0}$ is 1 as each vertex in C_1 is connected to the single vertex in C_0 that is in the opposite part V_i.
- $b_{1,1}$ is $m-1$, by similar logic to $b_{0,1}$.

The fact that these values are well-defined independent of v_0 and v_1 proves that the partition is equitable for any pair of adjacent vertices $\{v_0, v_1\} \subseteq V(K_{m,m})$. That is to say $K_2 \in \mathcal{DE}(K_{m,m})$

Now, let v_0, v_1 be a pair of nonadjacent vertices of $K_{m,m}$. As $K_{m,m}$ is complete bipartite, they must both be in either V_0 or V_1. Without loss of generality, suppose they are in V_0. Then consider $\pi_d(v_0, v_1) = \{C_0, C_1, \dots\}$. Since the neighbors of v_0 and v_1 are precisely the members of V_1, it follows that $V_1 = C_1$. The remaining

$m-2$ vertices comprise C_2, as they are neighbors of every point in V_1 but not of v_0 or v_1. The following argument shows that $\pi_d(v_0, v_1)$ is equitable with partition matrix

$$B_\pi = \begin{pmatrix} 0 & m & 0 \\ 2 & 0 & m-2 \\ 0 & m & 0 \end{pmatrix}$$

$b_{0,0}$ is 0 as v_0 is not connected to v_1 by assumption.
$b_{0,1}$ is m as each of v_0 and v_1 is connected precisely to the m members of V_1.
$b_{0,2}$ is 0 as vertices in C_0 and C_2 are all in V_0 and therefore have no connecting edges.
$b_{1,0}$ is 2 as each vertex in C_1 is connected to both v_0 and v_1.
$b_{1,1}$ is 0 as each vertex in C_1 is a member of V_1 and therefore has no neighbors within V_1.
$b_{1,2}$ is $m-2$, as G is regular of degree m and thus the rows must add to m.
$b_{2,0}$ is 0 as members of V_0 cannot be connected.
$b_{2,1}$ is m as each member of C_2 is connected to all of the m members of $V_1 = C_1$.
$b_{2,2}$ is 0 by the same logic as the $b_{2,0}$ case.

These values are well-defined independent of v_0 and v_1, and so $\pi_d(v_0, v_1)$ is equitable for any pair of non-adjacent vertices $\{v_0, v_1\} \subseteq V(K_{m,m})$. That is to say that $\overline{K_2} \in \mathcal{DE}(K_{m,m})$. □

5. Open Questions

The work we have done so far suggests sevral open questions, which we have not yet had the time to attempt. In no particular order:

- Classify the (strongly regular) graphs G satisfying $\{K_1, \overline{K_2}\} \subseteq \mathcal{DE}(G)$.
- Classify the graphs G satisfying $\{K_1, K_2, \overline{K_2}\} \subseteq \mathcal{DE}(G)$. Are there any besides the complete bipartite graphs $K_{m,m}$? Are there infinitely many more? We note that any such graphs would have girth 3, as by Corollary 4.4 they are strongly regular and thus by Lemma 4.7 they are either $K_{m,m}$ or they are non-bipartite, which is to say they have odd cycles and therefore triangles by Theorem 4.5. They must also have a degree of at least $\ell - 1$, where ℓ is the largest odd cycle; this fact follows trivially from Theorem 4.5.
- What does $\overline{K_2} \in \mathcal{DE}(L(G))$ imply about G (where $L(G)$ is the line graph of G)? What about $\{K_1, K_2, \overline{K_2}\} \in \mathcal{DE}(G)$? More generally, clarify the relationship between $\mathcal{DE}(G)$ and $\mathcal{DE}(L(G))$. This is motivated by the observation that the only examples we found of strongly regular graphs with $K_2 \nsubseteq \mathcal{DE}(G)$ are the five-cycle, the octahedron graph, and the grid graphs, and these are all line graphs.
- What can be said about graphs in which every odd cycle induces a complete subgraph?
- Are there conditions on distance-equitable induced subgraphs that imply that every cycle in a graph induces a complete subgraph? What can be said about graphs with this property?

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