EVEN WALKS ON VERTEX-TRANSITIVE GRAPHS

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ABSTRACT. The purpose of this paper is to show that a random walk of even length on a finite, vertex-transitive graph returns to its origin with greatest probability. In the course of showing this, I will introduce several relevant concepts and build lemmas that will ultimately allow us to demonstrate the theorem in question.

CONTENTS

1. GRAPHS AND RELATED TERMS

Definition 1.1. A *graph* G is a set of vertices (collectively denoted $V(G)$) and ordered pairs of vertices (called *edges* and denoted $E(G)$). Two nodes $a, b \in G$ are *connected* or *adjacent* if $(a, b) \in E(G)$. In an edge (a, b), we call the vertex a the *origin* of that edge and the vertex b the *target* of that edge.

A graph is *simple* if all of its edges are unique (there is at most one edge between any two vertices a and b) and none of its edges connect a vertex to itself.

A graph is *undirected* if $(a, b) \in E(G) \iff (b, a) \in E(G)$.

In practice, we usually describe graphs by drawing them, denoting the vertices with circles and the edges with lines. If a graph is directed, we usually signify an edge (a, b) without a corresponding edge (b, a) with an arrow pointing from a to b.

Definition 1.2. An undirected graph G is *k-regular* if every vertex in G is the source and target of k edges.

Definition 1.3. A *walk* is a sequence of edges $x_1, x_2, \ldots, x_n$, where the target of x_i is the origin of x_{i+1}. Each edge x_i is called a *step*. The length of the walk is equal to the number of steps it contains. A *random walk* is a walk that is constructed by:

(1) taking an initial vertex a
(2) choosing, with equal probability for all edges originating at a, a random next step in the walk.
(3) repeating this process with a as the target of the previously chosen edge.

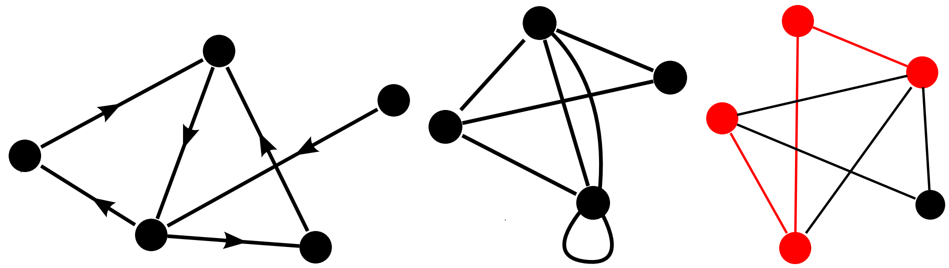


FIGURE 1. A directed graph; an undirected, non-simple graph; a walk of length 3

Definition 1.4. A *homomorphism* from a graph G to a graph H is a map $\sigma : V(G) \to V(H)$ that preserves adjacency. In other words, if $a, b \in V(G)$, $x, y \in V(H)$, $\sigma(a) = x$, and $\sigma(b) = y$, then $(a, b) \in E(G) \iff (x, y) \in E(H)$. A bijective homomorphism is called an *isomorphism*.

Definition 1.5. An *automorphism* is an isomorphism from a graph to itself.

Definition 1.6. Finally, a graph G is *vertex transitive* if $\forall a, b \in V(G)$ there is an automorphism σ on G where $\sigma(a) = b$. Informally, a vertex transitive graph is one that "looks the same" at every vertex, such as a cube or a lattice.

Note that all undirected, vertex-transitive graphs are k-regular.

For the purposes of this paper, we will be assuming that all of our graphs are simple, undirected, and finite, though this proof extends easily to non-simple graphs. We simply make this assumption for the sake of clarity. The other two conditions are necessary for our proof.

2. The Adjacency Matrix

One way of representing a graph is as an *adjacency matrix*. To create an adjacency matrix A for a graph G with vertices $\{v_1, ..., v_n\}$, one creates an $n \times n$ matrix, assigning the n^{th} row/column to the n^{th} distinct vertex, and determines the values with the formula:

$$A_{m,n} = \begin{cases} 1 & (v_m, v_n) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

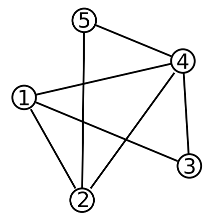
Consider the example in Figure 2. Notice, for example, because the first vertex is connected to the second, third, and fourth vertices, that there is a 1 in the second, third, and fourth columns of the first row.

You may also notice that in this particular undirected graph, the adjacency matrix is symmetric about its diagonal. This observation brings us to our first theorem.

Theorem 2.1. *In an undirected graph with adjacency matrix A, $A = A^T$.*

Proof. By definition, if a graph G containing vertices v_x, v_y is undirected, then $(v_x, v_y) \in E(G) \iff (v_y, v_x) \in E(G)$.
$\therefore A_{x,y} = A_{y,x}$, by the definition of an adjacency matrix.
$\therefore A = A^T$, by the definition of transpose.

□



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

FIGURE 2. An example of an adjacency matrix.

Corollary 2.2. $A^m = (A^m)^T$

Proof. We will prove this using induction. For the base case, take A^2. Because $(AB)^T = B^T A^T$, $(A \cdot A) = (A^T \cdot A^T) = (A \cdot A)^T$
Then, by the inductive hypothesis, assume $A^{m-1} = (A^{m-1})^T$.
$\therefore A^m = (A \cdot A^{m-1}) = \left(A^T \cdot (A^{m-1})^T\right) = (A^{m-1} \cdot A)^T = (A^m)^T$ □

However, the more important and surprising theorem is this one:

Theorem 2.3. *Given a graph with an $n \times n$ adjacency matrix A, ${A^m}_{i,j}$ is the number of walks of length m from v_i to v_j.*

Proof. We will prove this using induction. For the base case, take A^1.
Because a walk of length 1 consists of a single edge, the number of length-1 walks between v_i and v_j is equal to the number of edges between them. Because our graph is simple, there can be at most one edge between them.

(1) In the case that there is an edge between v_i and v_j, $A_{i,j} = 1$ by definition, so it correctly denotes the number of length-1 walks between them.
(2) In the case that there is not an edge between them, $A_{i,j} = 0$ by definition, so it still correctly denotes the number of length-1 walks between them.

Now, by the inductive hypothesis, assume $(A^{m-1})_{i,j}$ correctly denotes the number of walks of length $m-1$ between v_i and v_j. Then, by definition of matrix multiplication,

$$(A^{m-1} \cdot A)_{i,j} = \sum_{k=1}^{n} (A^{m-1})_{i,k} \cdot A_{k,j}$$

Because the penultimate edge in any walk terminating at v_j will necessarily terminate at a neighbor of v_j, it is sufficient to count the number of length-$(m-1)$ walks to a neighbor of v_j. Because $A_{k,j}$ will have a value of 0 if v_k is not a neighbor of v_j and 1 if it is, $(A^{m-1} \cdot A)_{i,j}$ is the sum of the number of length-$(m-1)$ paths to neighbors of v_j.
$\therefore (A^{m-1} \cdot A)_{i,j}$ is the number of length-m paths from v_i to v_j. □

3. Vertex Transitivity

There is one important property of walks on vertex-transitive graphs that will be used in our final solution, and so that will be discussed here. First, however, we will lay the foundation for proving this property with a lemma.

Lemma 3.1. *A graph isomorphism preserves the number of walks between two points*

Proof. Let G and H be two isomorphic graphs with an isomorphism $\sigma : V(G) \to V(H)$. Let $v_x, v_y \in G$. Because σ preserves adjacency, for any step s in any arbitrary path from v_x to v_y there must exist an equivalent step r under transformation by σ. Furthermore, because σ is bijective and therefore injective, both the source and the target of s map to unique vertices in $V(H)$, so no other edge in $E(G)$ maps to r. Because every step s in every path from v_x to v_y maps to a distinct edge in $E(H)$ under σ, every path from v_x to v_y maps to a distinct path. Therefore, the number of paths from $\sigma(v_x)$ to $\sigma(v_y)$ is at least the number of paths from v_x to v_y. Furthermore, because σ is bijective, every path from $\sigma(v_x)$ to $\sigma(v_y)$ will map to a distinct path from v_x to v_y under σ^{-1} by the same reasoning. Therefore, the number of paths from v_x to v_y is at least the number of paths from $\sigma(v_x)$ to $\sigma(v_y)$. Therefore there must be the same number of length-m paths from v_x to v_y as there are from $\sigma(v_x)$ to $\sigma(v_y)$. □

The major property of vertex transitive graphs, however, is this:

Theorem 3.2. *Given a vertex-transitive graph G with an adjacency matrix A, for any m, every row of A^m contains the same entries, though possibly in a different arrangement.*

Proof. Let A^m_f be an arbitrary row of A^m. Because G is vertex-transitive, there is an automorphism σ mapping v_f, the vertex in G corresponding to A^m_f, to v_1, the vertex in G corresponding to A^m_1. Then, for any vertex $v_j \in G$, there will be a corresponding $v_k = \sigma(v_j)$. Because σ is an isomorphism, and so preserves the number of walks between two points, the number of walks between v_f and v_j will equal the number of walks from v_1 to v_k Therefore, by Theorem 2.3, for $1 \leq f, j \leq n$ the j^{th} entry in A^m_f will be equal to a corresponding k^{th} entry in A^m_1. □

The significance of this is that if one views the rows of A^m as vectors, then each vector has the same magnitude. This will be useful in our final proof.

4. Tying it together

Here we will present our final proof. Note that the theorem in question only asserts that the probability of a random walk returning to its origin is never less than the probability of its arriving elsewhere[1].

Theorem 4.1. *A random walk of even length on a vertex-transitive graph returns to its origin with greatest probability.*

[1] As an example of the probability of a random walk returning to its origin equaling the probability of its arriving elsewhere, consider a square. It is a vertex transitive graph, but any even-length random walk will be as likely to return to the corner opposite its origin as to its origin.

Proof. To show this, we'll prove that the given statement is equivalent to "given an arbitrary even length, an arbitrary walk of that length returns to its origin with greatest probability" and then prove this new statement.

Consider all walks of arbitrary even length m on a vertex-transitive graph G with adjacency matrix A. Because G is vertex-transitive and therefore k-regular, the chance of a length-m random walk matching any given length-m walk is $\frac{1}{|V(G)|} \cdot \frac{1}{k^m}$. Because a random walk is equally likely to match any length-m walk, taking a length-m random walk on G is equivalent to arbitrarily choosing a length-m walk from the set of all possible length-m walks. So in order to show that a length-m random walk returns to its origin with greatest probability, we must show that an arbitrarily chosen length-m walk returns to its origin with greatest probability. By Theorem 2.3, the number of length-m walks that return to their origin are represented along the diagonal of A^m, so in order to show that an arbitrarily chosen length-m walk terminates at its origin with greatest probability, we can show that no non-diagonal entries of the matrix A^m are greater than any diagonal entries.

Because m is even, we can express A^m as $\left(A^{\frac{m}{2}}\right)^2$. By Corollary 2.2, $A^{\frac{m}{2}} = (A^{\frac{m}{2}})^T$, and we know, by the definition of dot product, that:

$$(A^m)_{i,j} = (A^{\frac{m}{2}})_i \bullet (A^{\frac{m}{2}})_j$$

Now, we can express the dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ as $\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$, and because all row vectors in $A^{\frac{m}{2}}$ have the same magnitude, by Theorem 3.2, the sole determinant of the size of their product is the cosine of the contained angle. Because the contained angle will be 0 when $i = j$, as the two row vectors are the same in this case, no entries in A^m can be greater than any entry along the diagonal. Therefore a random walk of arbitrary even length m originating from any vertex in G returns to its origin with greatest probability. □

References

[1] J. H. van Lint. A Course in Combinatorics Cambridge University Press. 2001.