THE FIRST FUNDAMENTAL THEOREM OF WELFARE ECONOMICS

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Abstract. A brief proof of the First Welfare Theorem aimed particularly at providing an accessible way to understand the theorem. Some effort has been made to translate the mathematics into more economic terms and a number of economic terms are used without proper definition as we assume the reader has some background knowledge in economics. The paper will end by discussing the implications of the theorem and further applications.

1. Introduction

The First Welfare Theorem is really a mathematical restatement of Adam Smith's famous "invisible hand" result. It relates competitive equilibria and Pareto optimality, which we will define more precisely further on. But before that, some preliminary terms are introduced.

We let S be a normed vector space with norm $\|\cdot\|$. We interpret S to be a commodity space, each vector representing a basket of commodities. There are I consumers, $1, 2, \ldots, I$. Each consumer i chooses from a set of commodity points in the set $X_i \subseteq S$, giving each commodity point (that is, basket of commodities) a value via the utility function $u_i : X_i \to \mathbb{R}$. There are J firms, $1, 2, \ldots, J$. Each firm j chooses among points in the set $Y_j \subseteq S$, each point being a vector describing production of the firms in the various commodities under technological limitations.

The members of the economy, both firms and consumers, interact with each other in the demand and supply of factors of production and final goods. Firms demand factors of production and supply final goods (negative and positive components of the y_j's, respectively) while consumers supply factors of production (for example, in the form of labour) and demand final goods (negative and positive components of the x_i's, respectively). The signs of the components simply represent the effect of demanding or supplying factors of production and final goods on the members of the economy, with positive being an increase in utility or profit and negative being a decrease in utility or profit. From a consumer's point of view, providing factors of production (that is, working) would be negative since it robs him of leisure, and consuming final products is positive since it provides him with pleasure. There is a similar explanation for the firms.

The choices made by firms and consumers are constrained across the whole economy by $\sum_i x_i - \sum_j y_j = 0$. This is the mathematical statement that all markets must *clear*: all produced goods are consumed, and there is no unemployment. The $(I+J)$-tuple $[(x_i), (y_j)]$ describes the consumption x_i of each consumer and the

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production y_j of each firm and is called an *allocation* for this particular economy. If $x_i \in X_i$ for all i and $y_j \in Y_j$ for all j, and $\sum_i x_i - \sum_j y_j = 0$ then the allocation is *feasible*. Finally, we take the *price system* to be a continuous linear functional $\phi : S \to \mathbb{R}$. This means that a vector in S representing a basket of commodities has an associated price for each commodity and can be assigned a single number which is the expenditure of purchasing that basket of commodities for consumers, or the profit of producing that basket of commodities for firms. We are now ready to define Pareto optimality and competitive equilibrium.

Definition 1.1. An allocation $[(x_i), (y_j)]$ is *Pareto Optimal* if it is feasible and if there is no other feasible allocation $[(x'_i), (y'_j)]$ such that $u_i(x'_i) \geq u_i(x_i)$ for all i and $u_i(x'_i) > u_i(x_i)$ for some i.

Pareto optimality, then, is the allocation at which no individual can increase his utility without reducing the utility of others.

Definition 1.2. An allocation $[(x_i^0), (y_j^0)]$ together with a price system $\phi : S \to \mathbb{R}$ is a *competitive equilibrium* if the following three conditions are satisfied.

(C1) The allocation $[(x_i^0), (y_j^0)]$ is feasible.
(C2) For each i, $x \in X_i$ and $\phi(x) \leq \phi(x_i^0)$ implies $u_i(x) \leq u_i(x_i^0)$ (equivalently, $u_i(x) > u_i(x_i^0)$ implies $\phi(x) > \phi(x_i^0)$).
(C3) For each j, $y \in Y_j$ implies $\phi(y) \leq \phi(y_j^0)$.

By condition (C3), the allocation is profit maximising for each firm at the given price system. By condition (C2), the allocation is clearly utility maximising for individual consumers at the given price system. We make one final definition before launching into the welfare theorem.

Definition 1.3. The *local nonsatiation condition for consumers* is satisfied if for each consumer i and each $x \in X_i$, and for every $\varepsilon > 0$, there exists $x' \in X_i$ such that $||x' - x|| \leq \varepsilon$ and $u_i(x') > u_i(x)$. That is to say, for any basket of commodities there is another basket of commodities arbitrarily nearby that is strictly preferred to it.

2. The First Welfare Theorem

The First Welfare Theorem draws a relationship between the two concepts explored above, Pareto optimality and competitive equilibria. We begin with a competitive equilibrium and show that it is surely Pareto optimal. Armed with the preliminary definitions, we can finally state and prove the theorem proper.

Theorem 2.1 (First Welfare Theorem)**.** *If the local nonsatiation condition for consumers is satisfied and* $[(x_i^0), (y_j^0), (\phi)]$ *is a competitive equilibrium, then the allocation* $[(x_i^0), (y_j^0)]$ *is Pareto optimal.*

Proof. Let $[(x_i^0), (y_j^0), (\phi)]$ be a competitive equilibrium. We first show that under the condition of nonsatiation,

$$\text{for each } i,\ u_i(x) = u_i(x_i^0) \text{ implies } \phi(x) \geq \phi(x_i^0). \tag{2.2}$$

Suppose instead that

$$u_i(x) = u_i(x_i^0) \text{ and } \phi(x) < \phi(x_i^0).$$

Now by the local nonsatiation condition, we can let $\{x_n\}$ be a sequence in X_i converging to x such that

$$u_i(x_n) > u_i(x) = u_i(x_i^0), i = 1, 2, \dots.$$

Since ϕ is continuous, this implies, by our supposition, that for all sufficiently large n,

$$\phi(x_n) < \phi(x_i^0).$$

However, by condition (C2) in the definition of a competitive equilibrium,

$$u_i(x_n) > u_i(x_i^0) \text{ implies } \phi(x_n) > \phi(x_i^0)$$

Hence by contradiction, (2.2) is true.

For the sake of contradiction, we suppose that the initial allocation $[(x_i^0), (y_j^0), (\phi)]$ is not Pareto optimal. Then there is another feasible allocation $[(x_i'), (y_j')]$ such that $u_i(x_i') \geq u_i(x_i^0)$ for all i, with strict inequality for some i. Employing condition (C2), this strict inequality gives that

$$\text{for some } i,\ u_i(x_i') > u_i(x_i^0) \text{ implies } \phi(x_i') > \phi(x_i^0). \tag{2.3}$$

From equation (2.2) and the fact that ϕ is linear, we see that for $k \in i$ where $u_k(x_k') = u_k(x_k^0)$, $\sum_k \phi(x_k') \geq \sum_k \phi(x_i^0)$. For $l \neq k$ where $u_l(x_l') > u_l(x_l^0)$, $\sum_l \phi(x) > \sum_l \phi(x_l^0)$ Summing across all i, this gives

$$\phi\left(\sum_i x_i'\right) = \sum_i \phi(x_i') > \sum_i \phi(x_i^0) = \phi\left(\sum_i x_i^0\right).$$

Since both allocations are feasible, we find that

(2.4)

$$\sum_j \phi(y_j') = \phi\left(\sum_j y_j'\right) = \phi\left(\sum_i x_i'\right) > \phi\left(\sum_i x_i^0\right) = \phi\left(\sum_j y_j^0\right) = \sum_j \phi(y_j^0),$$

and this contradicts (C3). □

2.1. **Some Notes about the Proof.** The crux of the proof lies in considering an alternate feasible allocation that improves the utility of at least one consumer without decreasing the utility of any other consumer, while retaining the conditions of competitive equilibrium. Let us consider the scenario where just one consumer increases his utility and everyone else remains the same (this is sufficient to prevent the competitive equilibrium from being Pareto optimal). The reason for showing equation (2.2) is to reveal the effect on expenditure if one's utility stays the same (it either increases or stays the same). This is the case for all the consumers except one. This inequality is not directly obvious from the conditions of competitive equilibrium. The next step, equation (2.3), shows us the effect on expenditure for that one consumer whose utility has increased (hence the strict inequality between the utilities). From these we see quite clearly now that even if only one consumer increases his utility with the new allocation (and everyone else maintains status quo), the total expenditure of all the consumers combined will surely increase. As a result, the feasibility condition and condition (C3) come together in equation (2.4) to yield the contradiction needed to prove the theorem.

3. Beyond the First Welfare Theorem

3.1. **The Fundamental Theorems of Welfare Economics.** The First Welfare Theorem is often coupled with the Second Welfare Theorem which is converse to it (together they are referred to as the Fundamental Theorems of Welfare Economics). As the First Welfare Theorem states that a competitive equilibrium is Pareto optimal, the Second begins with a Pareto optimal allocation and concludes that there will be a suitable price system such that an equilibrium not unlike the competitive one above (albeit a little weaker) is reached. The proof for the Second Welfare Theorem is considerably more involved and even the assumptions that precede it require significant effort to explain. In the interest of simplicity and brevity, we will be satisfied with briefly stating the Second Welfare Theorem and the additional conditions needed to strengthen its result to a proper competitive equilibrium.

Theorem 3.1 (Second Welfare Theorem)**.** *Under the appropriate conditions, given a Pareto optimal allocation* $[(x_i^0), (y_j^0)]$, *there exists a nonzero price system* $\phi : S \to \mathbb{R}$ *such that*
(S1) for each i, $x \in X_i$ *and* $u_i(x) \geq u_i(x_i^0)$ *implies* $\phi(x) \geq \phi(x_i^0)$*; and*
(S2) for each j, $y \in Y_j$ *implies* $\phi(y) \leq \phi(y_j^0)$.

Note here that while condition (S2) is identical to (C3) under the definition of competitive equilibrium (thus fulfilling the profit maximizing condition for firms), condition (S1) is quite a bit weaker than (C2). This condition only minimises the expenditure for consumers, rather than maximising utility as in condition (C2). It is not too difficult, however, to improve (S1). It is sufficient to impose the conditions that the consumption set be convex, the utility function $u_i : X_i \to \mathbb{R}$ be continuous, and for there always to exist a "cheaper" alternative basket of commodities for each consumer. We will not, in this paper, go further into what exactly these conditions correspond to in an economy or why they would allow us to reach a competitive equilibrium from the "weak" equilibrium of the Second Welfare Theorem. The important point is that it can be strengthened.

3.2. **Applications of the Fundamental Theorems of Welfare.** The First Welfare Theorem is of particular interest in the debate on whether or not we should intervene in markets. It serves as a double edged sword, either for arguing in favour of market intervention or against it. As we mentioned in the opening lines of this paper, the First Welfare Theorem is the "invisible hand" result presented and proved mathematically. The "invisible hand" result states that as long as each member of the economy maximises utility or profit for himself, the best possible outcome for all members as a whole will emerge. This then seems to be in favour of market non-intervention when under ideal conditions (perfect competition in economic terms), since economies should ideally settle into a competitive equilibrium leading to a Pareto optimal result. These ideal conditions for a perfectly competitive market correspond with the conditions and definitions we have set up for the preparation of the proof of the First Welfare Theorem. For example, the linearity of the given price function, $\phi(\lambda x) = \lambda\phi(x)$ for $\lambda \in \mathbb{R}$, assures us that the price for any good in the commodity basket stays constant over all units of the good, which is similar to price being constant in a perfectly competitive market. The problem here is the "ideal conditions" which are needed for the theorem to apply.

Some argue that since those very conditions are never met in the real world, the First Welfare Theorem never comes into play and we never reach a Pareto optimal result from a competitive equilibrium unless we intervene in markets. We will not comment on this issue further but leave it as a point of interest.

The two theorems together are also useful in dealing with dynamic programming aimed at achieving Pareto optimal outcomes, either through deterministic models or their stochastic counterparts. The two Welfare Theorems assure us that under the conditions of the theorems, the set of competitive equilibrium allocations and the set of Pareto optimal allocations coincide exactly, so that finding competitive equilibria gives us the set of solutions of optimising problems that employ dynamic programming. The Second Welfare Theorem is also encouraging since it indicates that under a fixed set of conditions, solutions to dynamic programming models are supportable (that is stable with both consumers and producers satisfied) as competitive equilibria. Therefore, under these conditions, *all* optimisations can be found by finding competitive equilibria.

Further research and understanding in the two theorems is most definitely fruitful for anyone with a serious interest in economics. Hopefully this paper has managed to clearly enunciate and draw out the economic ideas which correspond to the mathematical proof of the First Welfare Theorem, an exercise which is particularly useful for those who have thus far only had a rather less than rigorous approach to ideas in economics.

References

[1] Nancy L. Stokey and Robert E. Lucas, Jr. with Edward C. Prescott. Recursive Methods in Economic Dynamics. Harvard University Press. 1989