WHAT THE FUNCTOR?: CATEGORY THEORY AND THE CONCEPT OF ADJOINTNESS

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ABSTRACT. Category theory provides a more abstract and thus more general setting for considering the structure of mathematical objects. In this paper we will define basic concepts related to category theory and discuss examples, such as groups and sets as categories and forgetful and free functors, following Eugenia Cheng's notes on category theory [1]. Our ultimate goal will be to examine the concept of adjointness through the example of free and forgetful functors.

CONTENTS

1. SOME DEFINITIONS AND EXAMPLES

First we define some basic concepts related to categories and see some helpful examples.

Definition 1.1. A *category* $\mathcal{C}$ consists of:

- A collection of objects, $\operatorname{ob}\mathcal{C}$
- For every pair of objects $X, Y \in \operatorname{ob}\mathcal{C}$, a collection $\mathcal{C}(X, Y)$ of morphisms (also called maps) $f : X \to Y$.

These morphisms are equipped with:

- For each object $X \in \operatorname{ob}\mathcal{C}$ an identity map $1_X \in \operatorname{ob}\mathcal{C}$
- For each $X, Y, Z \in \operatorname{ob}\mathcal{C}$ a composition map $m_{XYZ} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ that sends (g, f) to $g \circ f = gf$

Composition of morphisms satisfies the following:

- Unit laws: if $f : X \to Y$ then $1_Y \circ f = f = f \circ 1_X$.
- Associativity: if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$, then $h(gf) = (hg)f$.

Examples 1.2. Categories:

(1) The category **Set**, where objects are sets and morphisms are functions.
(2) The category **Grp**, where objects are groups and morphisms are group homomorphisms.

(3) The category **Top**, where the objects are topological spaces and the morphisms are continuous maps.
(4) The category $\mathbf{Set}_*$, where the objects are pointed sets and the morphisms are base point preserving functions.
(5) The category **Mon**, where the objects are monoids and the morphisms are monoid homomorphisms.

We see from these examples the wide range of mathematical structures described by category theory. We also have a relation between two categories through what is called a functor, which we now define.

Definition 1.3. A *functor* $F : \mathcal{C} \to \mathcal{D}$ relates two categories $\mathcal{C}$ and $\mathcal{D}$ in the following way:

- To each object $X \in \operatorname{ob} \mathcal{C}$ it associates an object $FX \in \operatorname{ob} \mathcal{D}$
- To each map $f \in \mathcal{C}(X, Y)$ it associates a map $Ff \in \mathcal{D}(FX, FY)$

such that the following properties hold:

- For each object $X \in \operatorname{ob} \mathcal{C}$, $F1_X = 1_{FX}$
- For a map $g \in \mathcal{C}(X, Y)$, and a map $f \in \mathcal{C}(Z, Y)$, we have $F(f \circ g) = Ff \circ Fg$.

Now we look at examples of functors of two sorts: free functors and forgetful functors. These two concepts will be useful later as examples for the concept of adjointness.

Examples 1.4. Forgetful functors:

(1) The forgetful functor: $\mathbf{Grp} \to \mathbf{Set}$ sends each group $G \in \mathbf{Grp}$ to its underlying set $U(G) \in \mathbf{Set}$ and each group homomorphism $f \in \mathbf{Grp}(G_1, G_2)$ to the corresponding set function $Uf \in \mathbf{Set}(U(G_1), U(G_2))$.
(2) The forgetful functor: $\mathbf{Grp} \to \mathbf{Set}_*$ sends a group G to its underlying set $U(G)$, and the identity in G to the basepoint in $U(G)$. It sends each group homomorphism $f \in \mathbf{Grp}(G_1, G_2)$ to the corresponding set function $Uf \in \mathbf{Set}_*(U(G_1), U(G_2))$.
(3) The forgetful functor: $\mathbf{Grp} \to \mathbf{Mon}$ sends a group G to itself as a monoid and each group homomorphism to the corresponding monoid homomorphism.
(4) The forgetful functors: $\mathbf{Set}_* \to \mathbf{Set}$ sends each pointed set S_* to itself as a set and each base point preserving function to the corresponding set function.

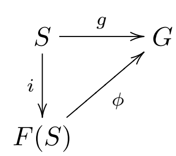
Definition 1.5. Some properties of functors:

- A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if for all X, $Y \in \operatorname{ob} \mathcal{C}$, the set function $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is injective.
- A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* if for all X, $Y \in \operatorname{ob} \mathcal{C}$, the set function $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is surjective.
- A functor $F : \mathcal{C} \to \mathcal{D}$ is *full and faithful* if for all X, $Y \in \operatorname{ob} \mathcal{C}$, the set function $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is bijective.

From our above examples, we see that the forgetful functor: $\mathbf{Grp} \to \mathbf{Mon}$ is both full and faithful since every group homomorphism is a monoid homomorphism and every monoid homomorphism between groups is a group homomorphism. The forgetful functor $\mathbf{Mon} \to \mathbf{Set}_*$ is faithful but not full because there are maps between monoids that are not monoid homomorphisms but which preserve the identity. The forgetful functor $\mathbf{Set}_* \to \mathbf{Set}$ is also faithful but not full since there are functions between pointed sets that do not preserve the basepoint.

Before we look at the examples of free functors we define the free group $F(S)$ on a set S.

Definition 1.6. Let S be a set. Suppose we are given a group $F(S)$ and a function $i : S \to F(S)$. The group $F(S)$ is a *free group* on S if for all groups G and for all functions $g : S \to G$, there exists a unique homomorphism $\phi : F(S) \to G$ such that

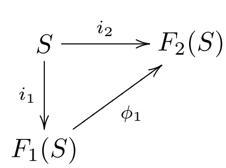


commutes, that is, such that $g = \phi \circ i$.

First we show such a group exists. Let S be a set, and suppose that for each element $s \in S$ we introduce a corresponding element s^{-1} in another set S^{-1} and call this element the inverse of s. We then form a group from S and S^{-1} by forming words from the elements of S and their inverses. We call the empty word the identity, and when we have an element adjacent to its inverse, we reduce the pair to the identity. For example, a word $s_1 s_2 s_2^{-1} s_3$ would reduce to $s_1 s_3$. Call this group $W(S)$. Now we show this group satisfies the universal property.

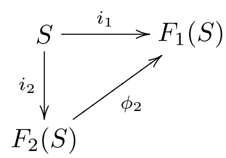
Let S be a set, and let $W(S)$ be the set of words in S described above. Let $i : S \to W(S)$ be the function sending each element in S to the corresponding one-letter word in $W(S)$. Then for any group G, and any function $h : S \to G$, we show that there exists a unique homomorphism $\phi : F(S) \to G$ such that the above diagram commutes. For an element $w \in W(S)$, we have by our construction $w = i(s_1)i(s_2)\cdots = s_1 s_2 \ldots$, where $s_i \in S$ and no s_i, s_{i+1} are inverses. Then we construct a function ϕ that sends each $w = s_1 s_2 \cdots \in W(S)$ to $h = g(s_1)g(s_2)\cdots \in G$. It is clear now that the diagram commutes since, by our construction $\phi \circ i = g$. Thus $W(S)$ is a candidate for the free group on S.

Now we show that the free group on S is unique, that is, that given two free groups on S, there exists an isomorphism between them that preserves S. Suppose $F_1(S)$ and $F_2(S)$ are free groups on S, with inclusions $i_1 : S \to F_1(S)$, $i_2 : S \to F_2(S)$. Then by the definition of $F_1(S)$ as a free group on S, since we have a group $F_2(S)$, and a set map $i_2 : S \to F_2(S)$, there exists a unique homomorphism $\phi_1 : F_1(S) \to F_2(S)$ such that

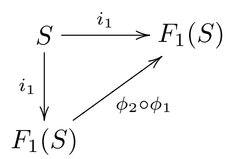


commutes, that is, such that $\phi_1 \circ i_1 = i_2$.

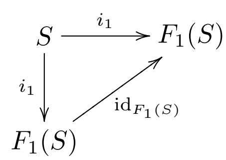
Now by the definition of $F_2(S)$ as a free group on S, since we have a group $F_1(S)$ and a set map $i_1 : S \to F_1(S)$, there exists a unique ϕ_2 such that



commutes, that is, such that $\phi_2 \circ i_2 = i_1$. Now substituting $i_2 = \phi_1 \circ i_1$ from the first equation into this we obtain $\phi_2 \circ \phi_1 \circ i_1 = i_1$. This tells us the following diagram commutes:



But we know there is another map: $F_1(S) \to F_1(S)$ such that the diagram commutes:

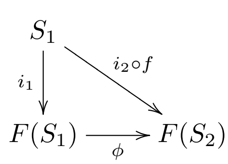


By the definition of the free group $F_1(S)$ on S, this homomorphism is unique, so we must have $\phi_2 \circ \phi_1 = \mathrm{id}_{F_1(S)}$. Similarly $\phi_1 \circ \phi_2 = \mathrm{id}_{F_2(S)}$. Thus $F_1(S)$ and $F_2(S)$ are isomorphic, and the free group on S is unique up to isomorphism preserving S.

Here are some examples of functors involving free groups:

Examples 1.7. Free functors:

(1) The free functor: **Set** $\to$ **Grp** sends each set S to the free group $F(S)$ on S. It sends each set function to a group homomorphism in the following way. Say we have $i_1 : S_1 \to F(S_1)$, $i_2 : S_2 \to F(S_2)$. Given $f : S_1 \to S_2$, we have a map $i_2 \circ f$ from S_1 to $F(S_2)$. Then by the universal property, there exists a unique homomorphism $\phi : F(S_1) \to F(S_2)$ such that the following diagram commutes:



We define $Ff = \phi$. Then the diagram commutes by construction. So the free functor sends each set function f to the group homomorphism Ff.

(2) The free functor: $\mathbf{Set}_*$ $\to$ **Grp** sends each pointed set S_* to the free group $F(S_*)$ that assigns the basepoint to the identity on the free group. It sends each set function to the corresponding group homomorphism as described above.

There are similar free functors from $\mathbf{Set}_*$ to **Mon** and from **Set** to $\mathbf{Set}_*$, but we will only be using the example from **Set** to **Grp** for the rest of the paper.

2. Natural Transformations and Adjoint Functors

Here are a few more concepts we need to understand the definition of adjointness.

Definition 2.1. Let $\mathcal{C}$ be a category. The *dual* or *opposite* category $\mathcal{C}^{op}$ consists of the following:

- $\operatorname{ob}\mathcal{C}^{op} = \operatorname{ob}\mathcal{C}$
- $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$
- the identity maps are the same
- $f^{op} \circ g^{op} = (g \circ f)^{op}$

Definition 2.2. A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{op}$ to $\mathcal{D}$. Equivalently, F is a contravariant functor if:

- F sends objects $X \in \operatorname{ob}\mathcal{C}$ to objects $FX \in \operatorname{ob} D$
- F sends morphisms $f \in \mathcal{C}(X,Y)$ to morphisms $Ff \in \mathcal{D}(FY, FX)$
- identities are preserved
- $F(f \circ g) = Fg \circ Ff$

What we have called "functor" so far is also called a *covariant* functor.

Examples 2.3. Here are examples of a contravariant functor and a covariant functor.

- Fix $X \in \operatorname{ob} C$. The contravariant hom functor, $\mathcal{C}(-,X) : \mathcal{C}^{op} \to \mathbf{Set}$ sends any $Y \in \operatorname{ob}\mathcal{C}^{op}$ to the set of morphisms from Y to X. For any $g : Y \to Y'$, $\mathcal{C}(-,X)$ gives us a map $\mathcal{C}(Y',X) \to \mathcal{C}(Y,X)$ by sending $f \in \mathcal{C}(Y',X)$ to $f \circ g$. This is an example of a contravariant functor.
- Fix $X \in \operatorname{ob} C$. The covariant hom functor $\mathcal{C}(X,-) : \mathcal{C} \to \mathbf{Set}$ sends $Y \in \operatorname{ob}\mathcal{C}$ to the set of morphisms from X to Y. For any morphism $g : Y \to Y'$, $\mathcal{C}(X,-)$ gives us a map $\mathcal{C}(X,Y) \to \mathcal{C}(X,Y')$ by sending $f \in \mathcal{C}(X,Y)$ to $g \circ f$. This is an example of a covariant functor.

Here is a definition that will be important in understanding adjointness.

Definition 2.4. Let F, G be functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A transformation $\alpha : F \to G$, consisting of component maps $\alpha_X : FX \to GX$, for all $X \in \operatorname{ob}\mathcal{C}$, is a *natural transformation* if for all maps $f : X \to X'$ in $\mathcal{C}$,

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FX' & \xrightarrow{\alpha_{X'}} & GX' \end{array}$$

commutes, that is, $\alpha_{X'} \circ Ff = Gf \circ \alpha_X$.

This also gives a definition of natural transformations of contravariant functors.

Examples 2.5. The contravariant and covariant functors in Examples 2.3 also provide examples of natural transformations:

(1) Let X, $Y \in \operatorname{ob}\mathcal{C}$, $h : X \to Y$, and let α be the transformation between the two contravariant functors $\mathcal{C}(-,X)$ and $\mathcal{C}(-,Y)$ consisting of the component transformations: $\{\alpha_Z : \mathcal{C}(Z,X) \to \mathcal{C}(Z,Y)\}$, where $Z \in \mathcal{C}$. These component transformations are given by composing $g \in \mathcal{C}(Z,X)$ with h.

For this to be a natural transformation, we need to show that given any $f : Z \to Z'$, where $Z,\ Z' \in \mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(Z', X) & \xrightarrow{\alpha_{Z'}} & \mathcal{C}(Z', Y) \\ {\scriptstyle \mathcal{C}(f,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(f,\mathrm{id})} \\ \mathcal{C}(Z, X) & \xrightarrow[\alpha_Z]{} & \mathcal{C}(Z, Y) \end{array}$$

To see if this commutes, take $g : Z' \to X$. Apply $\alpha_{Z'}$ to get $h \circ g \in \mathcal{C}(Z', Y)$. Then apply $\mathcal{C}(f, \mathrm{id})$ to get $h \circ g \circ f \in \mathcal{C}(Z, Y)$. Now take this same g but apply $\mathcal{C}(f, \mathrm{id})$ first to get $g \circ f \in \mathcal{C}(Z, X)$. Then apply α_Z to get $h \circ g \circ f \in \mathcal{C}(Z, Y)$. This tells us $\mathcal{C}(f, \mathrm{id}) \circ \alpha_{Z'} = \alpha_Z \circ \mathcal{C}(f, \mathrm{id})$, or that the diagram commutes and α is a natural transformation.

(2) Let $X,\ Y \in \mathrm{ob}\, \mathcal{C}^{op}$, $h : Y \to X$, and let α be the transformation between the two covariant functors $\mathcal{C}(X, -)$ and $\mathcal{C}(Y, -)$ consisting of the component transformations $\{\alpha_Z : \mathcal{C}(X, Z) \to \mathcal{C}(Y, Z)\}$, for $Z \in \mathcal{C}$. These component transformations are given by composing $g \in \mathcal{C}(X, Z)$ with h. For this to be a natural transformation, we need to show that given any $f : Z \to Z'$, where $Z,\ Z' \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(X, Z) & \xrightarrow{\alpha_Z} & \mathcal{C}(Y, Z) \\ {\scriptstyle \mathcal{C}(\mathrm{id},f)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\mathrm{id},f)} \\ \mathcal{C}(X, Z') & \xrightarrow[\alpha_{Z'}]{} & \mathcal{C}(Y, Z') \end{array}$$

To check if this commutes, take $g : Z \to Z'$. Apply α_Z to get $g \circ h \in (Y, Z)$. Then apply $\mathcal{C}(\mathrm{id}, f)$ to get $f \circ g \circ h \in \mathcal{C}(Y, Z')$. Now take g again and first apply $\mathcal{C}(\mathrm{id}, f)$ to get $f \circ g \in \mathcal{C}(X, Z')$. Then apply $\alpha_{Z'}$ to get $f \circ g \circ h \in \mathcal{C}(Y, Z')$. So we see the diagram does commute and α is a natural transformation.

Now we define adjointness.

Definition 2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are *adjoint* if there exists an isomorphism

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

that is natural in X and Y. We then say F is *left adjoint* to G, and G is *right adjoint* to F, and denote this by $F \dashv G$. We will denote the component transformations of such an isomorphism as $\{\eta_{X,Y} : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)\}$

Now we explain what naturality in X and Y means. First notice that if we fix $X \in \mathrm{ob}\, \mathcal{C}$, then $\mathcal{D}(FX, -)$ and $\mathcal{C}(X, G-)$ are both covariant functors $\mathcal{D} \to \mathbf{Set}$. Similarly, if we fix $Y \in \mathrm{ob}\, \mathcal{D}$, then $\mathcal{D}(F-, Y)$ and $\mathcal{C}(-, GY)$ are both contravariant functors $\mathcal{C} \to \mathbf{Set}$.

Thus, by naturality in Y, we mean that for our fixed $X \in \mathrm{ob}\, \mathcal{C}$, then $\eta_{X,-} : \mathcal{D}(FX, -) \to \mathcal{C}(X, G-)$ is a natural transformation. That is, for any $g : Y \to Y'$, where Y, $Y' \in \operatorname{ob}\mathcal{D}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(FX,Y) & \xrightarrow{\eta_{X,Y}} & \mathcal{C}(X',GY) \\ \Big\downarrow{\scriptstyle \mathcal{D}(\mathrm{id},g)} & & \Big\downarrow{\scriptstyle \mathcal{C}(\mathrm{id},Gg)} \\ \mathcal{D}(FX,Y') & \xrightarrow[\eta_{X,Y'}]{} & \mathcal{C}(X,GY') \end{array}$$

Similarly, by naturality in X, we mean that if we fix $Y \in \operatorname{ob}\mathcal{D}$, then $\eta_{-,Y} : \mathcal{D}(F-,Y) \to \mathcal{C}(-,GY)$ is a natural transformation. That is, for any $f : X \to X'$, where X, $X' \in \operatorname{ob}\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(FX',Y) & \xrightarrow{\eta_{X',Y}} & \mathcal{C}(X',GY) \\ \Big\downarrow{\scriptstyle \mathcal{D}(Ff,\mathrm{id})} & & \Big\downarrow{\scriptstyle \mathcal{C}(f,\mathrm{id})} \\ \mathcal{D}(FX,Y) & \xrightarrow[\eta_{X,Y}]{} & \mathcal{C}(X,GY) \end{array}$$

3. The Free and Forgetful Functors as Adjoints

Now that we defined the concept of adjointness, here is the example we have been working up to.

Example 3.1. Let F be the free functor from **Set** to **Grp** and let U be the forgetful functor from **Grp** to **Set**. Then $F \dashv U$. That is, there exists a natural isomorphism

$$\mathbf{Grp}(F(S),G) \cong \mathbf{Set}(S,U(G))$$

where here, and throughout the following explanation, U represents the forgetful functor: $\mathbf{Grp} \to \mathbf{Set}$, F the free functor: $\mathbf{Set} \to \mathbf{Grp}$, S a set, and G a group.

To see why this is true, we first need to show that there exists a bijection between the two sets. This means we need to find a map : $\mathbf{Grp}(F(S),G) \to \mathbf{Set}(S,U(G))$ and a map : $\mathbf{Set}(S,U(G)) \to \mathbf{Grp}(F(S),G)$ such that their composition gives the identity.

A map from $\mathbf{Set}(S,U(G)) \to \mathbf{Grp}(F(S),G)$ is given us by the definition of a free group. If $i : S \to U(F(S))$ is the inclusion given in the definition, then given a morphism $f \in \mathbf{Set}(S,U(G))$ we have a unique morphism $\psi \in \mathbf{Grp}(F(S),G)$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{f} & U(G) \\ \Big\downarrow{\scriptstyle i} & \nearrow{\scriptstyle U\psi} & \\ U(F(S)) & & \end{array}$$

This also gives us a map from $\mathbf{Grp}(F(S),G) \to \mathbf{Set}(S,U(G))$. If $i : S \to U(F(S))$, then given a group homomorphism $\psi \in \mathbf{Grp}(F(S),G)$, take $U\psi \circ i$ which we can see from the above diagram is in $\mathbf{Set}(S,U(G))$.

Now we show that the composition of these maps gives the identity. Start with $\psi : F(S) \to G$. Taking $U\psi \circ i$ $(i : S \to U(F(S)))$, we get a morphism in $\mathbf{Set}(S,U(G))$. Now given this morphism, the definition of the free group gives us a unique homomorphism $\phi : F(S) \to G$ such that

$$\begin{array}{ccc} S & \xrightarrow{U\psi \circ i} & U(G) \\ {\scriptstyle i}\downarrow & \nearrow_{\phi} & \\ U(F(S)) & & \end{array}$$

commutes, that is, such that $\phi \circ i = U\psi \circ i$. Since $U\psi$ makes the diagram commute, and the homomorphism for which the diagram commutes is unique, we must have $\phi = U\psi$, which gives us back our original $\psi \in \mathbf{Grp}(F(S), G)$. Starting with $f : S \to U(G)$, the universal property tell us there exists a unique homomorphism $\psi \in \mathbf{Grp}(F(S), G)$, such that the following diagram commutes. Then taking $U\psi \circ i = f$ we get back our original function in $\mathbf{Set}(S, U(G))$.

Now we show that this isomorphism is natural in S and in G. First notice, if we fix $S \in \mathbf{Set}$ we have two covariant functors $\mathbf{Grp}(F(S), -)$ and $\mathbf{Set}(S, U(-))$, and if we fix $G \in \mathbf{Grp}$ we have two contravariant functors $\mathbf{Grp}(F(-), G)$ and $\mathbf{Set}(-, U(G))$.

We now show naturality in G. Fix $S \in \mathbf{Set}$. Let $\alpha_{S,-} : \mathbf{Grp}(F(S), -) \to \mathbf{Set}(S, U(-))$ be as defined above. Let $i : S \to F(S)$. Given any $g : G \to G'$, where G, $G' \in \mathbf{Grp}$, we need to check that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Grp}(F(S), G) & \xrightarrow{\alpha_{S,G}} & \mathbf{Set}(S, U(G)) \\ {\scriptstyle \mathbf{Grp}(\mathrm{id}, g)}\downarrow & & \downarrow{\scriptstyle \mathbf{Set}(\mathrm{id}, Ug)} \\ \mathbf{Grp}(F(S), G') & \xrightarrow[\alpha_{S,G'}]{} & \mathbf{Set}(S, U(G')) \end{array}$$

That is, we need to check $\mathbf{Set}(\mathrm{id}, Ug) \circ \alpha_{S,G} = \alpha_{S,G'} \circ \mathbf{Grp}(\mathrm{id}, g)$. We show that chasing the diagram both ways gives the same result. Take $\psi \in \mathbf{Grp}(F(S), G)$. Then apply $\alpha_{S,G}$ to get $U\psi \circ i \in \mathbf{Set}(S, U(G))$. Then apply $\mathbf{Set}(\mathrm{id}, Ug)$ to get $U(g \circ \psi) \circ i = Ug \circ U\psi \circ i \in \mathbf{Set}(S, U(G'))$. Now take the same $\psi \in \mathbf{Grp}(F(S), G)$ and apply $\mathbf{Grp}(\mathrm{id}, g)$ to get $g \circ \psi \in \mathbf{Grp}(F(S), G')$. Then apply $\alpha_{S,G'}$ to get $U(g \circ \psi) \circ i = Ug \circ U\psi \circ i \in \mathbf{Set}(S, U(G'))$. We have shown $\mathbf{Set}(\mathrm{id}, Ug) \circ \alpha_{S,G} = Ug \circ \psi \circ i$ and $\alpha_{S,G'} \circ \mathbf{Grp}(\mathrm{id}, g) = Ug \circ U\psi \circ i$, so the diagram commutes.

Now we show naturality in S. Fix $G \in \mathbf{Grp}$ and let $\eta_{-,G} : \mathbf{Grp}(F(-), G) \to \mathbf{Set}(-, U(G))$ be as defined above. Let S, $S' \in \mathbf{Set}$, $i : S \to U(F(S))$, $i' : S' \to U(F(S'))$. Given any $f : S \to S'$, we need to check that

$$\begin{array}{ccc} \mathbf{Grp}(F(S'), G) & \xrightarrow{\eta_{S',G}} & \mathbf{Set}(S', U(G)) \\ {\scriptstyle \mathbf{Grp}(Ff, \mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathbf{Set}(f, \mathrm{id})} \\ \mathbf{Grp}(F(S), G) & \xrightarrow[\eta_{S,G}]{} & \mathbf{Set}(S, U(G)) \end{array}$$

commutes, that is, that $\eta_{S',G} \circ \mathbf{Set}(f, \mathrm{id}) = \mathbf{Grp}(Ff, \mathrm{id}) \circ \eta_{S,G}$. We show that chasing the diagram both ways gives the same result. Take $\psi \in \mathbf{Grp}(F(S'), G)$, then apply $\eta_{S',G}$, to get $U\psi \circ i' \in \mathbf{Set}(S', U(G))$. Then applying $\mathbf{Set}(f, \mathrm{id})$ we get $U\psi \circ i' \circ f \in \mathbf{Set}(S, U(G))$. Now again take $\psi \in \mathbf{Grp}(F(S'), G)$ and apply $\mathbf{Grp}(Ff, \mathrm{id})$, to get $\psi \circ Ff \in \mathbf{Grp}(F(S), G)$. Then apply $\eta_{S,G}$ to get $U(\psi \circ Ff) \circ i = U\psi \circ UFf \circ i$. The diagram commutes if $\eta_{S',G} \circ \mathbf{Set}(f, \mathrm{id}) = \mathbf{Grp}(Ff, \mathrm{id}) \circ \eta_{S,G}$, or if $U\psi \circ i' \circ f = U\psi \circ UFf \circ i$. To show this we need only check that $i' \circ f = UFf \circ i$, or that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow[f]{} & S' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ U(F(S)) & \xrightarrow[UFf]{} & U(F(S')) \end{array}$$

But we showed in Examples 1.7 that this diagram does commute. Thus $i' \circ f = UFf \circ i$, and so U and F are adjoints.

References

[1] Eugenia Cheng Category Theory www.dpmms.cam.ac.uk/ elgc2