THE GRIGORCHUK GROUP

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Abstract. In this survey we will define the Grigorchuk group and prove some of its properties. We will show that the Grigorchuk group is finitely generated but infinite. We will also show that the Grigorchuk group is a 2-group, meaning that every element has finite order a power of two. This, along with Burnside's Theorem, gives that the Grigorchuk group is not linear.

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Introduction

In a 1980 paper [2] Rostislav Grigorchuk constructed the Grigorchuk group, also known as the first Grigorchuk group. In 1984 [3] the group was proved by Grigorchuk to have intermediate word growth. This was the first finitely generated group proven to show such growth, answering the question posed by John Milnor [5] of whether such a group existed. The Grigorchuk group is one of the most important examples in geometric group theory as it exhibits a number of other interesting properties as well, including amenability and the characteristic of being just-infinite. In this paper I will prove some basic facts about the Grigorchuk group.

The Grigorchuk group is a subgroup of the automorphism group of the binary tree T, which we will call $\mathrm{Aut}(T)$. Section 1 will explore $\mathrm{Aut}(T)$. The group $\mathrm{Aut}(T)$ does not share all of its properties with the Grigorchuk group. For example we will prove:

Proposition 0.1. $\mathrm{Aut}(T)$ *is uncountable.*

The Grigorchuk group, being finitely generated, cannot be uncountable. The Grigorchuk group does inherit some properties from $\mathrm{Aut}(T)$, however. We will prove:

Proposition 0.2. $\mathrm{Aut}(T)$ *is residually finite.*

The Grigorchuk group, as a subgroup of a residually finite group, inherits this property. Finitely generated residually finite groups are Hopfian, thus we will also prove:

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Corollary 0.3. *The Grigorchuk group is Hopfian.*

We will give generators for the Grigorchuk group in Section 2 and exhibit some of the group's basic properties. We will prove that the generators all have order two (Proposition 2.3) and that some of the relations amongst the generators allow us to write down elements of the group in a useful way (Proposition 2.4). These facts will lead to a proof of the following:

Theorem 0.4. *The Grigorchuk group is infinite.*

Finally, in Section 3 we will prove the major result of this paper:

Theorem 0.5. *The Grigorchuk group is a 2-group.*

This shows that the Grigorchuk group is a finitely generated infinite group in which every element has finite order. A problem posed by William Burnside in 1902 stumped group theorists for a long time. The question was the following:

Question 0.6. *Must a finitely generated group where every element has finite order be finite?*

The Grigorchuk group, by definition and Proposition 2.3 and Theorem 0.4 answers this question negatively, though it is not the only such group. The first example of an infinite group with Burnside type was given by Golod and Shafarevich in 1964 [4]. The modified question to ask then, is what conditions must be imposed to get a positive answer. It turns out that it is sufficient for finite-ness to show that the group is linear, that is, homomorphic to a subgroup of $GL(n, \mathbb{K})$ for some integer n and some field $\mathbb{K}$. This result is beyond the scope of this paper, but taken for granted, it allows us to say that Γ is not a linear group.

1. Preliminaries

Definition 1.1. Let a *rooted binary tree* $T = (V, E)$ be a tree with vertex set V all finite sequences of elements of $\{0, 1\}$. We will write a vertex as $(j_1, j_2, \ldots, j_k)$. Two vertices are connected by an edge in E if their lengths as sequences differ by one, and the shorter sentence is obtained from the longer one by deleting its last term. Note that the empty set is a vertex, and it is considered the root of the tree.

We will refer to all seqences with length k as $L(k)$, and call this a level of T. Note that

$$V = \bigcup_{k=0}^{\infty} L(k).$$

Automorphisms of the binary tree fix the root $\emptyset$, and permute subtrees that begin on the same level. All adjacent vertices must remain adjacent, so we can think of an automorphism as a series of "twists" of branches of the tree. We will call the automorphism group $\mathrm{Aut}(T)$.

Consider an automorphism $g \in \mathrm{Aut}(T)$. It permutes the vertices at each level, but the allowed permutations at a given level depend on the permutations of the previous levels (since adjacent vertices must remain adjacent). We can write g as a sequence of permutations $\{x_i\}$, where x_1 is the permutation of $L(1)$, x_2 is the permutation of the vertices of $L(2)$ after x_1 is performed, and so on. This way of representing automorphisms can be used to prove that the cardinality of $\mathrm{Aut}(T)$ is uncountable.

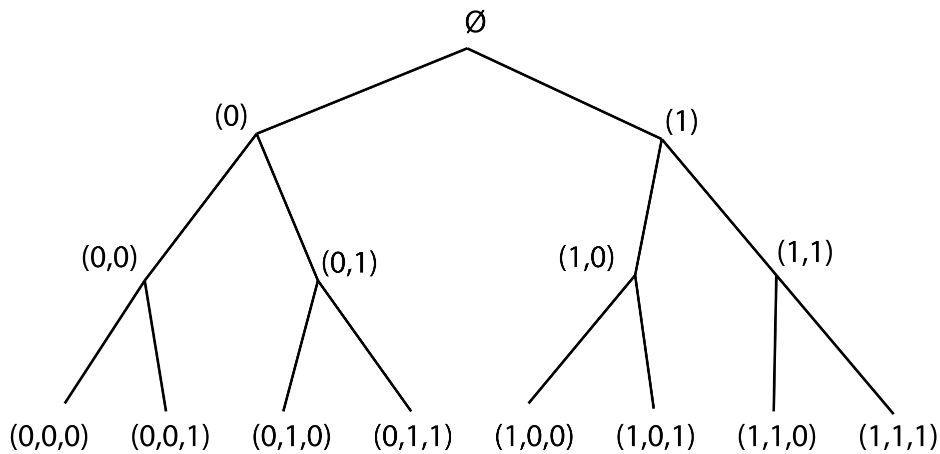

FIGURE 1. Here is part of a binary tree.

Proof of 0.1. [Aut(T) is Uncountable] At each level, let 0 correspond to the constant permutation. Given the lexicographic ordering of the level's vertices, let 1 be the transposition of the first two vertices. Consider an infinite sequence of 0's and 1's. This is a well-defined element of Aut(T), keeping in mind that the ith permutation permutes the vertices of $L(i)$ after all the permutations of lower levels have been performed. But this gives a bijection between a subset of Aut(T) and the set of infinite sequences of 0's and 1's, which is uncountable. □

Let $T(k)$ be the subtree of T spanned by all vertices with length at most k. Define $\mathrm{Aut}(T(k))$ analogously to $\mathrm{Aut}(T)$. Note that $\mathrm{Aut}(T(k))$ is finite for any k. Then let

$$St(k) = \{g \in \mathrm{Aut}(T) | \forall\ x \in T(k), g(x) = x\}.$$

be the set of automorphisms in $\mathrm{Aut}(T)$ that fix $T(k)$.

Proposition 1.2. *$St(k)$ is a normal subgroup of finite index in* $\mathrm{Aut}(T)$.

Proof. Let $\phi : \mathrm{Aut}(T) \to \mathrm{Aut}(T(k))$ be the function taking an automorphism $g \in \mathrm{Aut}(T)$ to the automorphism it performs on $T(k)$. This is a homomorphism, with kernel $St(k)$, thus $St(k)$ is normal in $\mathrm{Aut}(T)$. Then by the first isomorphism theorem, $St(k)$ has finite index in $\mathrm{Aut}(T)$. □

With further calculations, the following can be used to show facts about semi-direct products of subgroups of $\mathrm{Aut}(T)$.

Definition 1.3. A short exact sequence

$$1 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 1$$

is said to *split* if there is a homomorphism $h : C \to B$ such that $g \circ h$ is the identity on C.

Proposition 1.4. *The following is a short exact sequence that splits:*

$$1 \longrightarrow St(k) \xrightarrow{f} \mathrm{Aut}(T) \xrightarrow{g} \mathrm{Aut}(T(k)) \longrightarrow 1$$

where f is inclusion and g takes $a \in \mathrm{Aut}(T)$ to the induced automorphism on $T(k)$.

Proof. We know that $Im(f) = St(k) = Ker(g)$, as the automorphisms that fix $T(k)$ are exactly those that fix the first k levels of $\mathrm{Aut}(T)$, so the sequence is exact. Let $h : \mathrm{Aut}(T(k)) \to \mathrm{Aut}(T)$ map $b \in \mathrm{Aut}(T(k))$ to the automorphism of $\mathrm{Aut}(T)$ that performs b on the first k levels allowing levels $\geq k$ to rearrange only as needed. Then $f \circ h$ is the identity on $\mathrm{Aut}(T(k))$, and the sequence splits. $\square$

Now let us consider some characteristics that the Grigorchuk group will inherit.

Definition 1.5. A group G is *residually finite* if for every nontrivial element $g \in G$, there exists a homomorphism from G to a finite group that maps g to a nontrivial element.

Lemma 1.6. *A group G is residually finite if*

$$\bigcap_{\substack{H \leq G \\ finite\ index}} H = \{1\}.$$

Proof. Let $g \in G,\ g \neq 1$. Because the intersection of the finite index subgroups is trivial, there exists a finite index subgroup H that doesn't contain g. Let

$$N = \bigcap_{h \in G} h^{-1} H h.$$

Note that $N \trianglelefteq H$ is also of finite index, as there are finitely many conjugates each of finite index, and the intersection of these must be of finite index. Since $g \notin N$, the quotient map $f : G \to G \setminus N$ takes G to a finite set $G \setminus N$ and maps g to a nontrivial element. Thus G is residually finite. $\square$

Proof of 0.2. [$\mathrm{Aut}(T)$ is residually finite] I will present two proofs.

First proof: Let $g \in \mathrm{Aut}(T),\ g \neq 1$. Then there is some $x \in L(k)$ for some k such that $g(x) \neq x$. Define $\phi : \mathrm{Aut}(T) \to Sym(L(k))$ such that $\phi(h)$ is the permutation h performs on $L(k)$. Note that $Sym(L(k))$ is finite. By construction of ϕ we know that $\phi(g) \neq 1$, and $\mathrm{Aut}(T)$ is residually finite.

Second proof: Note that $\{St(k) | k \in \mathbb{N}\} \subseteq \{H \leq \mathrm{Aut}(T) | H\ of\ finite\ index\}$ by Proposition 1.2 so by Lemma 1.6 is sufficient to prove the following:

$$\bigcap_{k=1}^{\infty} St(k) = \{1\}.$$

Let $g \in St(j),\ g \neq 1$. Then $\exists\ x \in L(i)$ for some i such that $g(x) \neq x$. But then $g \notin St(i)$, so $g \notin \bigcap_{k=1}^{\infty} St(k)$. $\square$

Given a subgroup of a residually finite group, restriction of homomorphisms to the subgroup gives the desired property. Thus subgroups of residually finite groups are also residually finite.

Corollary 1.7. *The Grigorchuk group, being a subgroup of a residually finite group, is residually finite.*

Definition 1.8. A group G is *Hopfian* if any surjective homomorphism $G \to G$ is also injective.

Proposition 1.9. *Any finitely generated residually finite group is Hopfian.*

Proof. We will prove the contrapositive. Let G be a non-Hopfian group. Then there exists a surjective homomorphism $\phi : G \to G$ that is not injective. Let $g_0 \in Ker(\phi)$ such that $g_0 \neq 1$. Assume G is residually finite, so there is a homomorphism $\pi : G \to A$ with A a finite group, such that $\pi(g_0) \neq 1$.

Using the surjectivity of ϕ, for each $n \geq 1$, choose some $g_n \in G$ such that $\phi^n(g_n) = g_0$. Let $\pi_n : G \to A$ such that for all $g \in G$, $\pi_n(g) = \pi(\phi^n(g))$. A little calculation shows that $\pi_n(g_n) = \pi(\phi^n(g_n)) = \pi(g_0) \neq 1$, similarly, $\pi_m(g_m) \neq 1$. If $m \neq n$, we can assume without loss of generality that $m > n$, since otherwise we could make the following argument about g_m. Let $m > n$. Then $\pi_m(g_n) = \pi(\phi^{m-n-1}(\phi(\phi^n(g_n)))) = \pi(\phi^{m-n-1}(\phi(g))) = \pi(\phi^{m-n-1}(1)) = \pi(1) = 1$. Since $\pi_n(g_n) \neq 1$ this shows all the π_i are distinct. But G is finitely generated, and A is finite, so there are only finitely many possible homomorphisms $G \to A$, contradiction. So G is not residually finite, and our claim holds. $\square$

Proof of 0.3. The Grigorchuk group, being finitely generated and residually finite, is Hopfian. $\square$

2. The Definition of the Grigorchuk Group

Here we will define the Grigorchuk group. To do so we must define four automorphisms on the binary tree that will be its generators.

Let T_j be the subtree of all vertices whose sequence begins with j. Define the homomorphism $\delta_j : T \to T_j$ that takes a vertex x to the vertex jx by concatenation of the sequences j and x. Given two automorphisms g_0 and g_1, we can define $g = (g_0, g_1)$ which acts on T_i as $\delta_i g_i \delta_i^{-1}$.

Definition 2.1. If $j \in \{0, 1\}$, define $\bar{j}$ by $\bar{0} = 1$ and $\bar{1} = 0$. Then let $a : V \to V$ be defined by $a(j_1, j_2, \ldots, j_k) = (\bar{j}_1, j_2, \ldots, j_k)$. Let e be the identity automorphism. Define b, c, and d recursively. All fix the root and $L(1)$ and then using the notation defined above,

$$b = (a, c) \qquad c = (a, d) \qquad d = (e, b).$$

An explicit example gives

$$\begin{aligned} b(1,0,1,1) &= \delta_1(c(\delta_1^{-1}(1,0,1,1))) = \delta_1(c(0,1,1)) = \delta_1(\delta_0(a(\delta_0^{-1}(0,1,1))) \\ &= \delta_1(\delta_0(a(1,1))) = \delta_1(\delta_0(0,1)) = \delta_1(0,0,1) = (1,0,0,1) \end{aligned}$$

In shorthand, we have

$$\begin{aligned} b(0, j_2, j_3, \ldots, j_k) &= (0, \bar{j}_2, j_3, \ldots, j_k) \\ b(1, j_2, j_3, \ldots, j_k) &= (1, c(j_2, j_3, \ldots, j_k)) \\ c(0, j_2, j_3, \ldots, j_k) &= (0, \bar{j}_2, j_3, \ldots, j_k) \\ c(1, j_2, j_3, \ldots, j_k) &= (1, d(j_2, j_3, \ldots, j_k)) \\ d(0, j_2, j_3, \ldots, j_k) &= (0, j_2, j_3, \ldots, j_k) \\ d(1, j_2, j_3, \ldots, j_k) &= (1, b(j_2, j_3, \ldots, j_k)) \end{aligned}$$

Here are some more examples:

$$b(1,1,0,1,1,0,1) = (1, c(1,0,1,1,0,1)) = (1,1,d(0,1,1,0,1)) = (1,1,0,1,1,0,1)$$

$$c(1,1,0,1,1,0,1) = (1, d(1,0,1,1,0,1)) = (1,1,b(0,1,1,0,1)) = (1,1,0,0,1,0,1)$$
$$d(1,1,0,1,1,0,1) = (1, b(1,0,1,1,0,1)) = (1,1,c(0,1,1,0,1)) = (1,1,0,0,1,0,1)$$

Definition 2.2. Define the *Grigorchuk group* Γ as the group of automorphisms generated by a, b, c, and d:

$$\Gamma = \langle a, b, c, d \rangle$$

Let's prove some facts about these generators and a few of the relations amongst them.

Proposition 2.3. *We have*

$$a^2 = b^2 = c^2 = d^2 = e$$

Proof. For a:

$$a(a(j_1, j_2, \dots, j_k)) = a(\bar{j}_1, j_2, \dots, j_k) = (j_1, j_2, \dots, j_k)$$

so $a^2 = e$.

Let's consider b, c, and d. I will prove by induction on n that all the vertices in a level $L(n)$ are held constant by b^2, c^2, or d^2. By definition of each, the claim holds for $L(1)$. Now assume $L(k)$ is held constant for all $k < n$ when b^2, c^2, or d^2 is applied. We know $(x, y)^2 = (x^2, y^2)$ since performing automorphisms on different subtrees vertices is commutative. This means that

$$b^2 = (a^2, c^2) = (e, c^2)$$
$$c^2 = (a^2, d^2) = (e, (e^2, d^2)) = (e, (e, d^2))$$
$$d^2 = (e^2, b^2) = (e, b^2)$$

Consider a vertex of $L(n)$, $v = (j_1, \dots, j_n)$. Either $v = (0, j_2, \dots, j_n)$ or $v = (1, j_2, \dots, j_n)$. If we take $b^2(v)$ we get $(0, e(j_2, \dots j_n)$ or $(1, c^2(j_2, \dots, j_n))$ respectively. The first is trivially v, the second is v by the inductive hypothesis, since $(j_2, \dots, j_n)$ is a sequence of length $n-1$. Similar arguments can be made when applying c^2 and d^2. $\square$

Proposition 2.4. *The following are some of the relations in* Γ*:*

$$bc = cb = d \qquad cd = dc = b \qquad db = bd = c$$

Proof. I will prove by induction on n that the stated relations hold on vertices of $L(n)$. First note that the relations hold on $L(1)$ trivially since b, c, and d are all constant on $L(1)$. Assume the relations hold on $L(k)$ for all $k < n$. Because automorphisms on different subtrees are commutative, we know the following:

$$bc = (a, c)(a, d) = (a^2, cd) = (e, cd)$$
$$cd = (a, d)(e, b) = (a, db)$$
$$db = (1, b)(a, c) = (a, bc)$$

Now consider a vertex $v = (j_1, \dots, j_n)$ of $L(n)$. If $v = (0, j_2, \dots, j_n)$ then

$$bc(v) = (0, e(j_2, \dots, j_n)) = d(v),$$

and if $v = (1, j_2, \dots, j_n)$ then

$$bc(v) = (1, cd(j_2, \dots, j_n)) = (1, b(j_2, \dots, j_n)) = d(v)$$

where the second equality holds by the inductive hypothesis since $(j_2, \dots, j_n)$ is a sequence of length $n-1$. Similar arguments can be made for the other stated relations. $\square$

Remark 2.5. An element of Γ can be written as a reduced word in a, b, c, and d. As all the generators are their own inverses, we need only positive letters, and given the above relations, we can collapse any repeated letters to one or the empty letter, and any combinations of b, c, and d to a single letter. Thus any word can be written as

$$u_0 a u_1 a u_2 a \dots u_{l-1} a u_l$$

where $u_1, \dots, u_{l-1} \in \{b, c, d\}$ and $u_0, u_l \in \{\emptyset, b, c, d\}$.

We will now show that Γ is infinite. To do so, recall $St(1)$, the subgroup of $\mathrm{Aut}(T)$ from Section 1 that holds $L(1)$ constant. Here we will examine $St_\Gamma(1)$, which we will take to be the intersection of $St(1)$ with Γ.

Proposition 2.6. *A word in $\{a, b, c, d\}$ is in $St_\Gamma(1)$ if and only if it has an even number of occurrences of a.*

Proof. First consider a word g with an even number of occurences of a. Since b, c, and d all hold $L(1)$ constant, performing g on $L(1)$ merely requires flipping 0 and 1 as many times as a occurs. Since this is an even number, $L(1)$ will return to its starting position, thus making $g \in St_\Gamma(1)$.

If the word has an odd number of occurences of a, 0 and 1 in $L(1)$ will be exchanged an odd number of times, thus $L(1)$ will not be kept constant, and the claim holds. $\square$

Definition 2.7. Define

$$\psi = (\phi_0, \phi_1) : St_\Gamma(1) \to \Gamma \times \Gamma$$

as we defined b, c, and d above, i.e. for $g \in St_\Gamma(1)$, for $x \in T_i$ $g(x) = \delta_i \phi_i \delta_i^{-1}(x)$. So $\psi(b) = (a, c)$, etc. In particular, $\phi_0(b) = a$.

Let us calculate ψ for some useful elements of $St_\Gamma(1)$. First the easy ones.

$$\psi(b) = (a, c) \qquad \psi(c) = (a, d) \qquad \psi(d) = (e, b)$$

Going beyond those requires a little hard work.

Lemma 2.8. *The following hold:*

$$\psi(aba) = (c, a) \qquad \psi(aca) = (d, a) \qquad \psi(ada) = (b, 1)$$

Proof. To calculate $\psi(aba)$ consider $(0, j_2, \dots, j_n),\ (1, j_2, \dots, j_n) \in T$:

$$\begin{aligned} aba(0, j_2, \dots, j_n) &= ab(1, j_2, \dots, j_n) \\ &= a(1, c(j_2, \dots, j_n) \\ &= (0, c(j_2, \dots, j_n)) \end{aligned}$$

$$\begin{aligned} aba(1, j_2, \dots, j_n) &= ab(0, j_2, \dots, j_n) \\ &= a(0, a(j_2, \dots, j_n) \\ &= (1, a(j_2, \dots, j_n)) \end{aligned}$$

So $\phi_0(aba) = c$, and $\phi_1(aba) = a$, thus giving $\psi(aba) = (c, a)$. Similar calculations give the other two results. $\square$

Proposition 2.9. *The homomorphism $\phi_1 : St_\Gamma(1) \to \Gamma$ is surjective.*

Proof. Note that the following hold:

$$\phi_1(b) = c \qquad \phi_1(c) = d \qquad \phi_1(d) = b$$

Also, above we showed that $\phi_1(aba) = a$, which means that ϕ_1 maps to all of the generators of Γ and, as it is a homomorphism, must be surjective. □

Proof of Theorem 0.4. [Γ is infinite] We know $St_\Gamma(1)$ is a strictly proper subgroup of Γ since a is an element of Γ but not of $St_\Gamma(1)$. $St_\Gamma(1)$ is mapped onto Γ by ϕ_1. This is only possible if Γ is infinite. □

3. The Grigorchuk Group is a 2-group

In order to prove Theorem 0.5 we will need to define the length of an automorphism in Γ. The proof will proceed by induction on the length of elements of Γ. We will also prove base cases for the inductive argument as Propositions 3.2 and 3.3.

Definition 3.1. For an automorphism γ in Γ define its *length* $l(\gamma)$ as the smallest integer n for which there exists a sequence $(s_i)_{i=1}^n$ with $s_i \in \{a,b,c,d\}$ such that $s_1 s_2 \cdots s_n = \gamma$. In other words, $l(\gamma)$ is the minimum number of letters required to make a word in $\{a,b,c,d\}$ that represents γ.

Proposition 3.2. *For $\gamma \in \Gamma$ such that $l(\gamma) = 2$, $\gamma^{16} = 1$.*

Proof. By Proposition 2.4 any pairing of elements of $\{b,c,d\}$ can be reduced to a single letter, and any repeated letter is the identity, so we must only show that the elements ab, ba, ac, ca, ad, and da have order dividing 16.

(1) Consider ad and da first. By Lemma 2.8, $ada = (b,1)$ (where the ada and $\psi(ada)$ are considered to be the same automorphism). So ad has order 4 by the following calculation:

$$(ad)^4 = (adad)^2 = ((b,1)(1,b)) = (b,b)^2 = (b^2,b^2) = (e,e)$$

Similarly for da:

$$(da)^4 = (dada)^2 = ((1,b)(b,1)) = (b,b)^2 = (b^2,b^2) = (e,e)$$

(2) Now consider ac and ca. By Lemma 2.8, $aca = (d,a)$. So we have:

$$\begin{aligned}(ac)^2 &= acac = (d,a)(a,d) = (da,ad)\\ (ca)^2 &= caca = (a,d)(d,a) = (ad,da)\end{aligned}$$

Since ad and da were shown in (1) to have order 4, the calculations above give that ac and ca have order 8.

(3) Lastly, let's look at ab and ba. Above we found that $aba = (c,a)$, giving:

$$\begin{aligned}(ab)^2 &= abab = (c,a)(a,c) = (ca,ac)\\ (ba)^2 &= baba = (a,c)(c,a) = (ac,ca)\end{aligned}$$

Since ac and ca were shown in (2) to have order 8, these calculations show that ab and ba have order 16.

□

Corollary 3.3. *All words in $\{a,b\}$ have order dividing 16.*

Proof. Let w be a reduced word in $\{a, b\}$. Assume w has an odd number of letters. Then w starts and ends with the same letter, so $w = uvuvu \cdots uvu$. It is clear that the following holds:

$$ww = (uvuvu \cdots uvu)(uvuvu \cdots uvu) = u(v(u \cdots u(v(uu)v)u \cdots u)v)u = e$$

So w has order 2, and the claim holds. Now assume w has an even number of letters, k. Then $w = x^{\frac{k}{2}}$ where x is ab or ba. Note then that $w^{16} = (x^{\frac{k}{2}})^{16} = (x^{16})^{\frac{k}{2}} = e$ since the order of ab or ba is 16. Thus the order of w divides 16, proving the claim. $\square$

Proof of Theorem 0.5. [Γ is a 2-group] Let $k = l(\gamma)$ be the length of γ and let w be a reduced word of length k representing γ. We will prove this theorem by induction on k.

We will treat the cases of k odd and even separately. We will rely on orders of some elements that we have already calculated, and on some of the relations we have proven between the generators a, b, c, and d. In both cases we will make counting arguments about the number of letters needed in words. In the even case we will use the homomorphism ψ to make such a counting argument.

If $k = 0$, then $\gamma = e$, and if $k = 1$ then $\gamma^2 = 1$ by Proposition 2.3. We showed above in Proposition 3.2 that if $k = 2$ the claim holds.

Assume $k \geq 3$ and that the claim holds for words with length up to $k - 1$.

First assuming k is odd, if the first letter of w is a, the last letter is also (see Remark 2.5), in which case $w = axa$ for some word x of length $k - 2$. By the inductive hypothesis, x has order 2^M for some $M \geq 0$. Since conjugation doesn't affect order, w has order 2^M and the claim holds.

If the first letter is not a, then the first and last letters are in $\{b, c, d\}$. So we have $w = uxv$ with $u, v \in \{b, c, d\}$, and $l(x) = k - 2$. Consider $uwu = uuxvu = xvu$. Since $v, u \in \{b, c, d\}$ they reduce to one letter by Proposition 2.4 so we have that uwu has length at most $l(x) + 1 = k - 1$. Therefore uwu has order 2^M for some $M \geq 0$. Again, since conjugation does not affect order, w has the same order as uwu.

Now let's consider the case where k is even. By replacing γ with $b\gamma b$, $c\gamma c$, or $d\gamma d$ as necessary, we can assume without loss of generality that w begins with a, so $w = au_1au_2 \cdots au_l$ where $l = \frac{k}{2}$ and $u_i \in \{b, c, d\}$.

Consider the subcase where l is even, $l = 2m$ for some natural number m. We have l a's in w, an even number, so $\gamma \in St_G(1)$ by Proposition 2.6, so we can take $\psi(\gamma)$:

$$\psi(\gamma) = \psi(au_1a)\psi(u_2) \ldots \psi(au_{2m-1}a)\psi(u_2m) = (\gamma_0, \gamma_1)$$

with the middle product having $2m$ terms. For all u_i, $\phi_j(au_ia) = v$ for some $v \in \{a, b, c, d, e\}$, so each term of the middle product contributes at most one letter to the words representing γ_0 and γ_1. By the inductive hypothesis, then, $\gamma_0^{2^M} = e = \gamma_1^{2^N}$ for some $M, N \geq 0$. The order of γ must divide the least common multiple of 2^M and 2^N, which must be a power of 2.

Now assume l is odd. Then $k = 4m - 2$ for some $m \geq 2$. This gives the following:

$$\begin{aligned}
\gamma^2 = ww &= (au_1au_2 \ldots u_{2m-2}au_{2m-1})(au_1au_2 \ldots u_{2m-2}au_{2m-1}) \\
&= (au_1a)u_2 \ldots u_{2m-2}(au_{2m-1}a)u_1(au_2a) \ldots (au_{2m-2}a)u_{2m-1}
\end{aligned}$$

Note the final product has $8m - 4$ terms. Again, $\gamma^2 \in St_G(1)$ so we can take $\psi(\gamma^2)$:

$$\psi(\gamma^2) = \psi(au_1a)\psi(u_2) \ldots \psi(u_{2m-2})\psi((au_{2m-1}a)\psi(u_1)\psi(au_2a) \ldots \psi(au_{2m-2}a)\psi(u_{2m-1})$$

$$= (\alpha, \beta)$$

Both α and β are of length less than or equal to $4m - 2$, but this isn't sufficient to use the inductive hypothesis, so we are going to split up the subcase a little further.

(1) Assume for some j, $u_j = d$. Then $\phi_1(au_ja) = e$ and $\phi_0(u_j) = e$, so α and β are each represented by a word of length at most $4m - 3 = k - 1$, so the inductive hypothesis applies, and the order of γ divides the least common multiple of the orders of α and β, each a power of two.

(2) Assume for some j, $u_j = c$. Then $\phi_0(au_ja) = d$ and $\phi_1(u_j) = d$. Either α and β are both words of length $4m - 2$ involving d, so by 1 each has order a power of 2, or both have length shorter than $4m - 2$ and the inductive hypothesis holds. Either way the order of γ divides the least common multiple of the orders of α and β, and both are powers of two.

(3) If there are no c's or b's in w then it is a word solely in $\{a, b\}$, and so we can apply Corollary 3.3.

□

The fact that Γ is a 2-group gives us the interesting, non-intuitive fact that Γ is finitely generated and every element has finite order, but the group itself is infinite. Thus according to the discussion in the introduction, we have the following:

Corollary 3.4. Γ *is not a linear group.*

References

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