A BRIEF INTRODUCTION TO RAMSEY THEORY OF GRAPHS

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Abstract. This paper provides proof of Ramsey's Theorem as it applies to (undirected) graphs, along with a short discussion of the thereom and the difficulty of finding Ramsey numbers.

1. Introduction: Background and Definitions

The first results in what is today considered Ramsey Theory were actually proved earlier by Issai Schur and Bartel L. Van der Waerden, and pertained to arithmetic progressions and regular equations, respectively. The field, which is named after Frank P. Ramsey, deals with the study of sets and under what conditions order arises in subsets. Ramsey's only contribution to this field was the theorem bearing his name, which guarantees that a complete subgraph of a given size with a monochromatic edge coloring can be found in any complete graph with r-color edge coloring, provided the graph is of large enough size.

1.1. Definitions.

Definition 1.1. A *graph* G is a pair of sets (V, E), denoted $G(V, E)$, where V is a nonempty set of elements called *vertices*, and E is a set of unordered pairs of distinct vertices called *edges*.

The *order* of a graph is the number of vertices, which in the following will usually be denoted by n. If there exists an edge between two vertices, v_1 and v_2, then they are said to be *adjacent*.

An *isolated vertex* is a vertex that has no other vertices adjacent to it.

An *m-independent* set is a graph of m isolated vertices.

Example 1.2. Figure 1.1 shows G, a graph of order 6, consisting of the set of vertices

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and the set of edges

$$E = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_4v_5, v_5v_6\}.$$

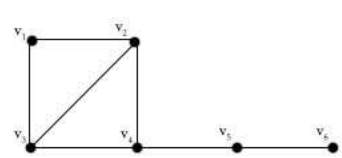

Figure 1.1.

Date: July 31, 2008.

Definition 1.3. A graph $G'(V', E')$ is said to be a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

Example 1.4. The graph H in figure 1.2 is a subgraph of G from the previous example.

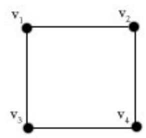

FIGURE 1.2.

Definition 1.5. A *complete graph* on $n \in \mathbb{N}$ vertices, denoted by K_n, is a graph such that each vertex of K_n is adjacent to all other vertices.

Example 1.6. Figure 1.3 shows K_4, K_3, and K_2 graphs, respectively.

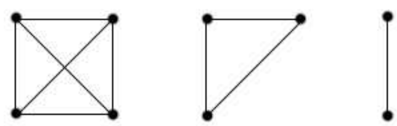

FIGURE 1.3.

Definition 1.7. An *r-coloring* of a set S is a function $\chi : S \to C$, where $|C| = r$.

Let χ be an r-coloring of the edges of a graph K_k^i. We say that (K_k^i, χ) (or simply K_k^i) is a *monochromatic complete graph* on k vertices if K_k^i is a complete graph on k vertices and $\chi(e) = i$ for each edge e of K_k^i.

1.2. **Motivation.** With these concepts defined, let us now provide some motivation for Ramsey's Theorem. The first example is known as the party problem.

Example 1.8. At a party there are a number of people who either know each other or don't know each other. Now, what is the smallest party (number of people) that is necessary such that there exist three people who mutually know each other or three people where no two know each other? We define the concept of "knowing each other" as follows. If A *knows* B, then B *knows* A. If A, B, C, and D "mutually know each other" then each member *knows* every other member. So, this problem corresponds to finding a least positive integer n such that a 2-coloring of K_n, let's say with colors red and blue, admits either a red or blue triangle, i.e. a monochromatic K_3 subgraph. We now show that $n = 6$.

By the pigeonhole principle, there exists a person, say A, who either knows or doesn't know at least 3 of the others. So, suppose that A knows B, C, and D. Now, if any two B, C, and D know each other, then we are done, since they also know A. If no two of B, C, or D know each other then we are done because we found 3 people who mutually don't know each other. If A doesn't know B, C, or D, then a similar argument yields our conclusion.

We have shown that $n \leq 6$; now we show that $n > 5$, which implies that $n = 6$. More specifically, we will prove that there exists a 2-coloring of the edges of K_5 which does not contain a monochromatic triangle. We use the colors red and blue. Label the vertices of the K_5 with $(A, \ldots, E)$. Color the edges AB, BC, CD, DE, and EA red. Color the remaining edges blue. Now there exist no red or blue triangles.

2. Ramsey's Theorem

2.1. Ramsey's Theorem for Complete Graphs.

Theorem 2.1 (General Ramsey's Theorem). *Let $k_i \geq 1$ for all $1 \leq i \leq r$. Then there exists a least positive integer $R = R(k_1, \ldots, k_r)$ such that every r-coloring of the edges of K_R forces K_R to have a monochromatic complete subgraph K_{k_i} of color i, for some $1 \leq i \leq r$.*

Proof. First note that for any $k_i = 1$, $R(k_1, \ldots, k_r) = 1$: the set of edges for a complete graph on 1 vertex is empty and thus monochromatic of any color, in particular color i. Also note that $R(n) = n$ since K_n with a 1-coloring is by definition monochromatic, and $R(n, 2) = n$ since for any 2-coloring of K_n, either one of the edges has the second color or all of the edges have the first color. Given that $R(k_1, \ldots, k_r)$ exists, then by the same reasoning, $R(k_1, \ldots, k_r, 2) = R(k_1, \ldots, k_r)$.

We claim that if $k_i = 2$ for all i, then $R(k_1, \ldots, k_r)$ exists and equals 2. To see this, note that in K_2 there exist two vertices with one edge between them of some color i. Taking the subgraph consisting of the entire graph, we have a monochromatic $K_2 = K_{k_i}$.

Now we will induct on the sum $\sum k_i$. Note that the proof is complete when $r = 1$ since $R(n) = n$, so we only consider $r \geq 2$. We use the previous claim as the base case for our induction. Now for the inductive step we assume that R exists when the sum of its r entries is $(\sum k_i) - 1$. Let

$$m = R(k_1, \ldots, k_r - 1) + \ldots + R(k_1 - 1, \ldots, k_r).$$

Pick a vertex from K_m and call it v, so now there are $m - 1$ edges from v to the other vertices. Let S_i be the set of edges of color i coming out of v. Since $\bigcup S_i$ is the set of all edges coming out of v, note that

$$\sum |S_i| = m - 1.$$

Now for some i it must be true that $|S_i| \geq R(k_1, \ldots, k_i - 1, \ldots, k_r)$, because if

$$|S_i| < R(k_1, \ldots, k_i - 1, \ldots, k_r)$$

for all i then

$$|S_i| \leq R(k_1, \ldots, k_i - 1, \ldots, k_r) - 1$$

for all i, so $\sum |S_i| \leq m - r < m - 1$, a contradiction. So, we may assume without loss of generality that $|S_1| \geq R(k_1 - 1, \ldots, k_r)$.

Let V be the set of vertices connected to v by an edge of color 1 so that $|V| \geq R(k_1 - 1, \ldots, k_r)$. We denote by K_V the complete subgraph with vertices V. By the inductive hypothesis, K_V contains either a monochromatic K_{k_i} subgraph of color i for some $2 \leq i \leq r$ or a monochromatic $K_{k_1 - 1}$ subgraph of color 1. If K_V contains a subgraph of the first kind then we are done. If K_V contains a subgraph of the second kind, then add v, which is connected to each vertex of K_{k_1-1} by an edge of color 1, giving us a monochromatic K_{k_1} subgraph of K_R of color 1. □

2.2. **Computation of Ramsey Numbers.** Despite this proof that the Ramsey numbers exist, the upper bound on these numbers grows very fast, as one can tell by the proof. On the other hand, for lower numbers it is relatively accurate. Take the case discussed in Example 1.8, which corresponds to $R(3,3) = 6$. The proof of Ramsey's Theorem gives an upper bound of $R(3,3) \leq 6$, and in the example we showed that indeed $R(3,3) = 6$. For $R(3,3,3)$, the proof gives an upper bound of 18, and it has been proven to be 17 (see [1]).

Ramsey numbers have been notoriously hard to compute. Using various methods, people have been able to narrow the bounds of some numbers, but for most cases the difference in the lower and upper bounds is still very large. For example, take $R(r,s)$ such that $r = s$:

n	Bounds on $R(n,n)$(see[4])
1	$R(1,1) = 1$
2	$R(2,2) = 2$
3	$R(3,3) = 6$
4	$R(4,4) = 18$
5	$43 \leq R(5,5) \leq 49$
6	$102 \leq R(6,6) \leq 165$
7	$205 \leq R(7,7) \leq 540$
8	$282 \leq R(8,8) \leq 1870$
9	$565 \leq R(9,9) \leq 6588$
10	$798 \leq R(10,10) \leq 23556$

An example of an improvement that can be made on the upper bounds of Ramsey's numbers comes from the case of two colors, which we call red and blue. From the proof, we get the bound

$$R(r,s) \leq R(r-1,s) + R(r,s-1). \tag{*}$$

By making the following observation we can improve upon this (see [2]). Let $n = R(r-1,s) + R(r,s-1) - 1$. If K_n is 2-colored with neither a red K_r subgraph nor a blue K_s subgraph, then each vertex v is connected to the remaining $n-1$ vertices by precisely $R(r-1,s) - 1$ red edges and $R(r,s-1) - 1$ blue edges. This is because if a vertex v has $R(r-1,s) - 1$ red edges, then the set of vertices that is connected to v by red edges, call it V, has $|V| = R(r-1,s) - 1$. Therefore, we can find a graph for which V has no K_{r-1} monochromatic red subgraph or K_s monochromatic blue subgraph, which means that connecting v to each vertex of V by a red edge cannot give a K_r monochromatic red subgraph. The same holds for the blue edges. So, the total number of red edges is exactly $n(R(r-1,s) - 1)/2$, where this number is the number of red edges per vertex multiplied by the number of vertices and divided by two for double counting. This number must be an integer, which is impossible if $R(r-1,s)$ and $R(r,s-1)$ are even. Since we must allow for the case when $R(r-1,s)$ and $R(r,s-1)$ are even, our supposition that there exists neither a red K_r subgraph or a blue K_s subgraph must be incorrect, and we can infer that the inequality (*) must be strict.

3. Conclusion

While Ramsey's theorem is relatively easy to prove, actually finding the Ramsey number for each case has proven to be extremely difficult. Even though many minor improvements have been made on the bounds on the $R(n, n)$ over the last quarter century, it may be the case that we'll never know what $R(n, n)$ is for $n \geq 5$.

References

[1] Beineke, L. W., Wilson, R. J., Cameron, P. J., *Topics in Algebraic Graph Theory.* 2004
[2] Graham, R. L., Rothchild, B. L., Spencer, J. H. *Ramsey Theory.* 1980
[3] Landman, B. M., Robertson, A. *Ramsey Theory on the Integers.* 2004
[4] Weisstein, Eric W. "Ramsey Number." . http://mathworld.wolfram.com/RamseyNumber.html