AN EXPLORATION OF COMPLEX JACOBIAN VARIETIES

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Abstract. In this paper, I will describe my thought process as I read in [1] about Abelian varieties in general, and the Jacobian variety associated to any compact Riemann surface in particular. I will also describe the way I currently think about the material, and any additional questions I have. I will not include material I personally knew before the beginning of the summer, which included the basics of algebraic and differential topology, real analysis, one complex variable, and some elementary material about complex algebraic curves.

Contents

1. Abelian Sums (I)

The first thing I read about were Abelian sums, which are sums of the form

$$\psi(L) = \sum_{i=1}^{3} \int_{p_0}^{p_i} \omega, \tag{1.1}$$

where ω is the meromorphic one-form dx/y, L is a line in $\mathbb{P}^2$ (the complex projective plane), p_0 is a fixed point of the cubic curve $C = (y^2 = x^3 + ax^2 + bx + c)$, and p_1, p_2, and p_3 are the three intersections of the line L with C. The fact that C and L intersect in three points counting multiplicity is just Bezout's theorem. Since C is not simply connected, the integrals in the Abelian sum depend on the path, but there's no natural choice of path from p_0 to p_i, so this function depends on an arbitrary choice of path, and will not necessarily be holomorphic, or even

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continuous. It's easy to see that if we have two such paths, γ and γ', then

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\gamma^{-1}\circ\gamma'} \omega, \tag{1.2}$$

so the integral of ω over any two such paths differ by the integral of ω over some loop, but since this integral only depends on the homology class of the loop, we get a better function if we define ψ to have domain $\mathbb{C}/\Lambda$, where Λ is the image of $H_1(C,\mathbb{Z})$ under the map $[\gamma] \mapsto \int_\gamma \omega$, which is obviously well-defined.

The genus formula says that the genus of C is 1, so $H_1(C,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. This suggests that Λ is a lattice in $\mathbb{C}$, but we have to prove this claim. To do this, we select two elements γ_1, γ_2 which generate $H_1(C,\mathbb{Z})$. If their images under the integration map, $\int_{\gamma_1} \omega$ and $\int_{\gamma_2} \omega$ were linearly dependent over $\mathbb{R}$, then there would be real numbers k_1, k_2 not both 0 such that $k_1 \int_{\gamma_1} \omega + k_2 \int_{\gamma_2} \omega = 0$. This would also show that $k_1 \int_{\gamma_1} \overline{\omega} + k_2 \int_{\gamma_2} \overline{\omega} = 0$. If we assume that ω and $\overline{\omega}$ generate $H^1_{dR}(C)$, then we would have a contradiction. This is because de Rham's theorem tells us that $(H^1_{dR}(C))^* \cong H_1(C,\mathbb{C})$. Since by assumption, the functional defined by $k_1\gamma_1 + k_2\gamma_2$ is zero on generators of $H^1_{dR}(C)$, it must be the zero functional, which means that $k_1\gamma_1 + k_2\gamma_2 = 0$, which is impossible.

In order to show that ω and $\overline{\omega}$ generate $H^1_{dR}(C)$, I needed to read a lot more, though.

2. Complex Manifolds

First of all, I needed some basic facts about complex manifolds.

Any complex n-manifold is a real differentiable $2n$-manifold, so it has a real tangent bundle. This can be complexified at any given point by taking the tensor product with $\mathbb{C}$ to get the complex tangent bundle. The complex tangent space at a point is $T_p(M) = \mathbb{C}\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$. Defining $\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i})$ and $\frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i})$, we see that the tangent space at a point is $\mathbb{C}\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}\}$.

Taking just $\mathbb{C}\{\frac{\partial}{\partial z_i}\}$, we get the holomorphic tangent bundle, $T'_p(M)$. Similarly, we get the antiholomorphic tangent bundle, $T''_p(M)$, by taking $\mathbb{C}\{\frac{\partial}{\partial \overline{z}_i}\}$. Considering the complex tangent bundle to be locally the space of $\mathbb{C}$-linear derivations, $T'_p(M)$ consists of those derivations which vanish on antiholomorphic functions, and we have an analogous interpretation for $T''_p(M)$. The complex tangent space at a point is the direct sum of the holomorphic and antiholomorphic tangent spaces.

The holomorphic and anti-holomorphic tangent spaces are in fact independent of the choice of coordinates, and the conjugation map on $\mathbb{C}$ in the tensor product which defines the complex tangent space sends the holomorphic tangent space to the antiholomorphic tangent space and vice versa. Also, the map on the tangent bundle induced by a holomorphic function between complex manifolds preserves the decomposition.

This decomposition of the complex tangent bundle also applies to the complex cotangent bundle, so

$$\bigwedge^n T^*_z(M) = \bigoplus_{p+q=n} (\bigwedge^p T'^*_z(M) \otimes \bigwedge^q T''^*_z(M)), \tag{2.1}$$

and we can apply this decomposition to complex-valued differential forms to get $A^{p,q}(M)$, the space of (p,q) forms.

The de Rham differential sends sections of $\bigwedge^p T'^*(M) \otimes \bigwedge^q T''^*(M)$ to sections of $(\bigwedge^p T'^*(M) \otimes \bigwedge^q T''^*(M)) \wedge T^*(M) = (\bigwedge^{p+1} T_z'^*(M) \otimes \bigwedge^q T_z''^*(M)) \oplus (\bigwedge^p T_z'^*(M) \otimes \bigwedge^{q+1} T_z''^*(M))$, applying the decomposition of the cotangent bundle, so it sends $A^{p,q}$ to $A^{p+1,q} \oplus A^{p,q+1}$.

By composing d with the natural projection maps to each coordinate, we get maps ∂ and $\overline{\partial}$. $\overline{\partial}^2 = 0$, so we can define the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(M)$. We can also define cohomology groups $H^{p,q}(M)$ as the quotient group of d-closed (p,q)-forms by d-exact (p,q)-forms. By definition, a holomorphic differential is a $(p,0)$ form ω with $\overline{\partial}\omega = 0$. The sheaf of holomorphic p-forms is denoted Ω^p. It is immediately apparent that the cocycle representatives in the Dolbeault cohomology groups $H^{p,0}_{\overline{\partial}}(M)$ are holomorphic. In fact, the relationship is a lot closer than this, as shown by

Theorem 2.2 (Dolbeault's Theorem). $H^q(M, \Omega^p) \cong H^{p,q}_{\overline{\partial}}(M)$.

A Hermitian metric on a complex manifold is a collection of positive definite Hermitian (sesquilinear and conjugate-symmetric) inner products on the holomorphic tangent space at each point z of the manifold which depends smoothly on z. More generally, we can define a Hermitian metric on any vector bundle over a complex manifold. Dualizing, the metric is given coordinate-wise by $ds^2 = \sum h_{ij}(z) dz_i \otimes d\overline{z}_j$, where

$$h_{ij}(z) = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle_z \tag{2.3}$$

A coframe is a collection of n $(1,0)$ forms φ_i which form an orthonormal basis for the holomorphic cotangent bundle with respect to the inner product induced by the Hermitian metric. The Gram-Schmidt process allows you to construct coframes locally.

A consequence of this definition is that

$$ds^2 = \sum \varphi_i \otimes \overline{\varphi}_i. \tag{2.4}$$

The real part of the Hermitian metric $\operatorname{Re} ds^2$ is a Riemannian metric, i.e. a smoothly varying inner product on the real tangent space at each point, since conjugate-symmetry and sesquilinearity become symmetry and bilinearity in this case.

On the other hand, taking $\operatorname{Im} ds^2$ is an alternating form for analogous reasons, so it represents a real 2-form. It is fairly obvious from the definition of a Hermitian metric that it is a $(1,1)$-form, and $-\frac{1}{2}\operatorname{Im} ds^2$ is called the associated $(1,1)$-form of the metric.

Conversely, any real (1,1)-form ω induces a quasi-Hermitian inner product on each holomorphic tangent space since for each point z of the manifold, $\omega(z) \in T_z'^*(M) \otimes T_z''^*(M) \cong (T_z'(M) \otimes \overline{T'}_z(M))^*$, so $\omega(z)$ is a sesquilinear functional on pairs of holomorphic tangent vectors. We multiply this functional by the appropriate scalar to make sure that the metric induced by the (1,1)-form associated with a given Hermitian metric is in fact the original metric. The (1,1) form ω is called positive if the pseudo-Hermitian inner product it induces is actually positive definite. It is immediate that the (1,1)-form associated with a prior Hermitian inner product is positive.

The Riemannian metric and associated (1,1)-form are closely related by

Theorem 2.5 (Wirtinger's Theorem). *If M is a complex manifold, then*

$$\mathrm{Vol}(M) = \frac{1}{d!}\int_M \omega^d, \tag{2.6}$$

where $d = \dim M$ and ω^d is the dth exterior power of the associated (1,1)-form of the Hermitian metric on M.

3. Hodge Theory and the Hodge Decomposition

Suppose we have a Hermitian matrix ds^2 with coframe φ_i. Then we also get an inner product on the tensor bundle $T^{*(p,q)}(M) = (\bigwedge^p T'^*(M)) \otimes (\bigwedge^q T''^*(M))$ by taking the basis induced by the coframe to be orthogonal and with each element having length 2^{p+q}. If we let Φ be the volume form induced by the metric by Wirtinger's theorem, then this gives us an inner product on $A^{p,q}(M)$ defined by

$$\langle \psi, \eta \rangle = \int_M \langle \psi(z), \eta(z)\rangle_z \Phi. \tag{3.1}$$

Using this inner product, we can construct adjoints to operators. For this purpose, the operator we care most about is $\overline{\partial}$. To do this, we first define the star operator $* : A^{p,q}(M) \to A^{n-p,n-q}(M)$. We want

$$\langle \psi, \eta \rangle \Phi = \psi \wedge *\eta \tag{3.2}$$

for all ψ. To get this, we say that if $\eta = \sum_{I,J} h_{IJ}\varphi_I \wedge \overline{\varphi}_J$, then

$$*\eta = 2^{p+q-n}\sum_{I,J} \epsilon_{IJ}\overline{h}_{IJ}\varphi_{I^c} \wedge \overline{\varphi}_{J^c} \tag{3.3}$$

where $I^c = \{1, \dots, n\} - I$ and similarly with J^c, and ϵ_{IJ} is the sign of the permutation in $S_{n+n'}$ that takes the $n + n'$-tuple $(1, \dots, n, 1', \dots, n')$ to the $n + n'$-tuple $(I_1, \dots, I_p, J_1, \dots, J_q, I^c_1, \dots, I^c_{n-p}, J^c_1, \dots, J^c_{n-q})$. All this is designed to get $* * \eta = (-1)^{p+q}\eta$.

We then define the adjoint of $\overline{\partial}^*$ by

$$\overline{\partial}^* = - * \overline{\partial} * \tag{3.4}$$

Using Stokes' theorem and the fact that on $A^{n,n-1}$, $d = \overline{\partial}$, we see that $\overline{\partial}^*$ is the adjoint to $\overline{\partial}$.

This adjoint is useful because you can show by variational methods that a representative ψ of a cohomology class in $H^{p,q}_{\overline{\partial}}$ has minimal norm if and only if $\overline{\partial}^*\psi = 0$.

We define the $\overline{\partial}$-Laplacian $\Delta_{\overline{\partial}} : A^{p,q}(M) \to A^{p,q}(M)$ by

$$\Delta_{\overline{\partial}}\eta = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial})\eta \tag{3.5}$$

Elements of the kernel of $\Delta_{\overline{\partial}}$ are called harmonic forms. Since

$$\langle \Delta_{\overline{\partial}}\eta, \eta \rangle = \langle \overline{\partial}^*\,\overline{\partial}\,\eta, \eta \rangle + \langle \overline{\partial}\,\overline{\partial}^*\,\eta, \eta \rangle = \|\,\overline{\partial}\,\eta\|^2 + \|\,\overline{\partial}^*\,\eta\|^2, \tag{3.6}$$

we see that a form η is harmonic if and only if $\overline{\partial}\,\eta = \overline{\partial}^*\,\eta = 0$. The space of harmonic (p,q)-forms on M is denoted $\mathcal{H}^{p,q}(M)$.

The most important facts about harmonic forms are contained in the

Theorem 3.7 (Hodge Theorem). *$\mathcal{H}^{p,q}(M)$ is finite-dimensional, and hence closed in $A^{p,q}(M)$, so there is an orthogonal projection operator $\mathcal{H} : A^{p,q}(M) \to \mathcal{H}^{p,q}(M)$. There is also a unique linear transformation, Green's operator, $G : A^{p,q}(M) \to A^{p,q}(M)$ such that $G\mathcal{H} = 0$, G commutes with $\overline{\partial}$ and $\overline{\partial}^*$, and $I = \mathcal{H} + \Delta_{\overline{\partial}}G$.*

Note that the last equation can be rewritten as $\psi = \mathcal{H}(\psi)+\overline{\partial}(\overline{\partial}^* G\psi)+\overline{\partial}^*(\overline{\partial}\, G\psi)$, called the Hodge decomposition for forms, which implies

$$A^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \overline{\partial}\, A^{p,q-1}(M) \oplus \overline{\partial}^*\, A^{p,q+1}(M). \tag{3.8}$$

The last two direct summands are orthogonal since

$$\langle \overline{\partial}\,\psi, \overline{\partial}^*\,\eta\rangle = \langle \overline{\partial}^2\,\psi, \eta\rangle = 0, \tag{3.9}$$

and looking at the first form of the Hodge decomposition for forms, we see that

$$\langle \mathcal{H}\psi, \Delta_{\overline{\partial}} G\eta\rangle = \langle \Delta_{\overline{\partial}}\mathcal{H}\psi, G\eta\rangle = 0, \tag{3.10}$$

since $\Delta_{\overline{\partial}}$ is obviously self-adjoint. The Hodge decomposition for forms thus gives us an orthogonal decomposition of $A^{p,q}(M)$.

Note that $\overline{\partial}(\mathcal{H} + \overline{\partial}(\overline{\partial}^*\, G)) = 0$, but if $\psi \in \overline{\partial}^*(A^{p,q+1}(M))$, then $\overline{\partial}\,\psi \neq 0$ unless $\psi = 0$, since if $\overline{\partial}\,\psi = 0$, with $\psi = \overline{\partial}^*\,\eta$, then $\langle \overline{\partial}\,\overline{\partial}^*\,\eta, \eta\rangle = 0$, so $\langle \overline{\partial}^*\,\eta, \overline{\partial}^*\,\eta\rangle = 0$, and hence $\psi = \overline{\partial}^*\,\eta = 0$.

This means that if ψ is $\overline{\partial}$-closed, then $\psi = \psi + \overline{\partial}(\overline{\partial}^*\, G\psi)$, i.e. any $\overline{\partial}$-closed form differs from a harmonic form by a $\overline{\partial}$-exact form, so we have

$$\mathcal{H}^{p,q}(M) \cong H^{p,q}_{\overline{\partial}}(M). \tag{3.11}$$

One very important consequence of the Hodge theorem is the

Theorem 3.12 (Kodaira-Serre Duality Theorem). $H^n(M, \Omega^n) \cong \mathbb{C}$ *and there is a nondegenerate map*

$$H^q(M, \Omega^p) \otimes H^{n-q}(M, \Omega^{n-p}) \to H^n(M, \Omega^n). \tag{3.13}$$

A Kähler manifold is defined to be a complex manifold with an Hermitian metric such that the associated (1,1)-form ω is d-closed. This is equivalent to approximating a Euclidean (flat) metric to order 2 pointwise.

Kähler manifolds are important because if we define the operators Δ_d and Δ_∂ the same way we defined $\Delta_{\overline{\partial}}$, then on a Kähler manifold, we have

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial, \tag{3.14}$$

so all the various Laplacians are related.

If we define $\mathcal{H}^{p,q}_d(M)$ to be the space of d-harmonic (p,q)-forms and $\mathcal{H}^r_d(M)$ to be the space of d-harmonic r-forms, then on a Kähler manifold, we have

$$\mathcal{H}^r_d(M) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}_d(M), \tag{3.15}$$

and

$$\mathcal{H}^{p,q}_d(M) = \overline{\mathcal{H}^{q,p}_d(M)}. \tag{3.16}$$

The Hodge theorem also holds for d-closed forms, so if ψ is a d-closed form, then $\psi = \mathcal{H}_d(\psi) + dd^* G_d(\psi)$, so $H^{p,q}_d(M) \cong \mathcal{H}^{p,q}_d(M)$, and similarly, $H^r_{dR}(M) \cong \mathcal{H}^r(M)$.

These identities, along with de Rham's theorem, give us

$$H^r(M, \mathbb{C}) \cong H^r_{dR}(M) \cong \bigoplus_{p+q=r} H^{p,q}_d(M) \tag{3.17}$$

and

$$H^{p,q}_d(M) = \overline{H^{q,p}_d(M)}. \tag{3.18}$$

Together, these equations are the Hodge decomposition of cohomology.

Since $\Delta_d = 2\Delta_{\overline{\partial}}$, the two harmonic spaces for the two Laplacians are the same, and hence there is an isomorphism of cohomology groups

$$H_d^{p,q}(M) \cong H_{\overline{\partial}}^{p,q}(M). \tag{3.19}$$

Combining this with the Dolbeault isomorphism, there is an isomorphism

$$H_d^{p,q}(M) \cong H^q(M, \Omega^p). \tag{3.20}$$

Applying these identities to M a compact Riemann surface, we get

$$H^1(M, \mathbb{C}) = H_d^{1,0}(M) \oplus H_d^{0,1}(M) \cong H^0(M, \Omega^1) \oplus \overline{H^0(M, \Omega^1)}. \tag{3.21}$$

Henceforth, I will define $H^{p,q}(M) = H_d^{p,q}(M) \cong H_{\overline{\partial}}^{p,q}(m)$. Note that if M is an n-dimensional complex manifold, then we have

$$H^{2n-1}(M, \mathbb{C}) = H^{n,n-1}(M, \mathbb{C}) \oplus \overline{H^{n,n-1}(M, \mathbb{C})}, \tag{3.22}$$

because the dimension of the holomorphic cotangent bundle of M is n, so there are no holomorphic p-forms for $p > n$, and hence $H^{p,2n-1-p}(M) \cong H^{2n-1-p}(M, \Omega^p)$ is 0 for $p > n$, and using the fact that $\overline{H^{p,q}(M)} = H^{q,p}(M)$, we see that it is 0 for $p < n-1$, so the only nonzero terms in the Hodge decomposition of $H^{2n-1}(M, \mathbb{C})$ are $H^{n,n-1}(M)$ and $H^{n-1,n}(M) = \overline{H^{n,n-1}(M)}$.

4. Abelian Sums (II)

I now had the tools necessary to prove that ω and $\overline{\omega}$ generate $H_{dR}^1(C)$. First, we have to prove that ω is a holomorphic 1-form. This is a consequence of the more general fact that if S is a projective plane curve defined by the zero locus of the homogeneous polynomial $F(x, y, z)$, then if $f(x, y) = F(x, y, 1)$, then the differential

$$g(x, y) \frac{dx}{(\partial f / \partial y)(x, y)}, \tag{4.1}$$

where g is a polynomial of degree $\leq \deg(f)$ is a holomorphic 1-form.

Since C has genus one, $\dim H^1(C, \mathbb{C}) = 2g$, and $H^1(C, \mathbb{C}) \cong H^0(C, \Omega^1) \oplus \overline{H^0(C, \Omega^1)}$, we know that $\dim H^0(C, \Omega^1) = 1$, and we know that ω is a nontrivial holomorphic differential, so ω and $\overline{\omega}$ generate $H^1(C, \mathbb{C})$, which by de Rham's theorem is isomorphic to $H_{dR}^1(C, \mathbb{C})$, so by the previous argument, Λ is a lattice in $\mathbb{C}$.

After this, I read about one version of Abel's theorem, which says that if we define $\psi(L)$ as above, with codomain $\mathbb{C}/\Lambda$, then $\psi(L)$ is constant. First, we note that we can consider ψ as a map from $\mathbb{P}^{2*}$, the set of lines in $\mathbb{P}^2$, but $\mathbb{P}^{2*} \cong \mathbb{P}(\mathbb{C}^{3*}) \cong \mathbb{P}(\mathbb{C}^3) \cong \mathbb{P}^2$, since we can identify a line in $\mathbb{P}^2$ with the set of functionals on $\mathbb{C}^3$ which vanish on representatives in $\mathbb{C}^3$ of the points, and two linear functionals have the same kernel if and only if they differ by a scalar multiple.

The torus $\mathbb{C}/\Lambda$ has a holomorphic 1-form dz which comes from taking a global Euclidean coordinate z, which in turn descends from the natural coordinate on $\mathbb{C}$. $\psi^* dz$ is a global holomorphic 1-form on $\mathbb{P}^2$. But $H^1(\mathbb{P}^2, \mathbb{C}) = 0$ by cellular decomposition, and $H^1(\mathbb{P}^2, \mathbb{C}) = H^0(\mathbb{P}^2, \Omega^1) \oplus \overline{H^0(\mathbb{P}^2, \Omega^1)}$ by the Hodge decomposition and the Dolbeault isomorphism, so $H^0(\mathbb{P}^2, \Omega^1) = 0$, and hence $\mathbb{P}^2$ has no global holomorphic 1-forms other than 0, and thus $\psi^* dz = 0$. This means that ψ must be constant, since its derivative must vanish identically.

5. Jacobian Varieties

On a Riemann surface, a divisor is a locally finite linear combination of points. On a general complex manifold, a divisor is a locally finite linear combination of hypersurfaces. I later learned that this corresponded to the more general notion of a Weil divisor. For complex manifolds, this is the same thing as a global section of the quotient sheaf of meromorphic units by holomorphic units, $H^0(M, \mathcal{M}^*/\mathcal{O}^*)$, which in a more general setting corresponds to a Cartier divisor.

Meromorphic functions have zeroes and poles along hypersurfaces, so there is a function which sends the meromorphic function f to $\sum_V \text{ord}_V(f) \cdot V$. Any divisor of this form is called principal. There is also a map called the degree map on compact manifolds such that $\deg(\sum p_i V_i) = \sum p_i$. Since the manifold is compact, this sum is finite. Any principal divisor has degree 0.

On any curve, a degree 0 divisor can be written as $\sum p_i - q_i$, so in particular, any principal divisor can be written this way. If $\text{Div}^0(M)$ denotes the group of divisors of degree 0 on M, then we can define a map $\mu : \text{Div}^0(M) \to \mathbb{C}/\Lambda$ by $\mu : \sum p_i - q_i \mapsto \sum \int_{q_i}^{p_i} \omega$. Using an argument very similar to that showing ψ was constant, we can see that $\mu(\text{PDiv}(M)) = \{0\}$, where $\text{PDiv}(M)$ is the group of principal divisors on M.

The construction of the lattice Λ can be generalized to curves of arbitrary genus. To do this, if M is a curve, we first pick a basis for $H^1(M, \mathbb{Z})$, $\delta_1, \ldots, \delta_{2g}$ with the property that the intersection number of δ_i and δ_j is 0 unless $j = i + g$, in which case it is 1, or $j = i - g$, in which case it is -1. The $\delta_1, \ldots, \delta_g$ are called A-cycles, and the $\delta_{g+1}, \ldots, \delta_{2g}$ are called B-cycles. We also pick a basis $\omega_1, \ldots, \omega_g$ for $H^0(M, \Omega^1)$. The periods are the vectors

$$\Pi_i = (\int_{\delta_i} \omega_1, \ldots, \int_{\delta_i} \omega_g). \tag{5.1}$$

The proof that the period vectors are $\mathbb{R}$-linearly independent is identical to the genus one case. The periods thus define a lattice $\Lambda \subset \mathbb{C}^g$. The torus $\mathbb{C}^g/\Lambda$ is called the Jacobian variety of M, and denoted $\mathscr{J}(M)$.

The matrix Ω with columns the periods is called the period matrix. It is possible to prove certain equations about the periods that show that by choosing good bases for the homology and cohomology, the period matrix is of the form $\Omega = (I, Z)$, where $Z = {}^tZ$ and $\text{Im}\, Z$ is positive definite.

Another way of thinking of this is that integration gives a map from $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ to $H^0(M, \Omega^1)^* \cong \mathbb{C}^g$. The argument that the periods are linearly independent over $\mathbb{R}$ then shows that the image of $H^1(M, \mathbb{Z})$ is a lattice, and the Jacobian variety is then the quotient.

We can define a map $M \times M \to \mathscr{J}(M)$ by sending (w, z) to the residue class of the integration along γ functional, where γ is a path from w to z. Any two such paths differ by an element of $H_1(M, \mathbb{Z})$, so this map is well-defined. From this we get a function μ' from $\text{Div}^0(M)$ to $\mathscr{J}(M)$. We also get a map μ from M to $\mathscr{J}(M)$ by fixing a base point p_0. We can think of μ' as the composition of μ and the map which sends the point $p \in M$ to the divisor $p - p_0$. The map μ is called the Abel-Jacobi map. Abel's theorem gives us a lot of information about the former map.

Theorem 5.2 (Abel's Theorem)**.** *If* $\mu : \text{Div}^0(M) \to \mathscr{J}(M)$ *is the map defined above, then* $\ker \mu = \text{PDiv}(M)$.

Note that this means that if μ' is the map from M to $\mathscr{J}(M)$, then if $\mu'(p) = \mu'(q)$, $\mu(p-q) = 0$, so there is a meromorphic function on M which has a simple pole at q, and is holomorphic on $M - \{q\}$, but this would give us a non-constant degree 1 map to $\mathbb{P}^1$, which would have to be an isomorphism, so if $M \ncong \mathbb{P}^1$, then μ' is injective.

A divisor D is called effective if each of the nonzero p_i in $D = \sum p_i V_i$ is positive. This is denoted $D \geq 0$. We can define a complex manifold $M^{(d)}$, called the d-fold symmetric product of M, by taking the quotient of M^d by the natural action of the symmetric group S^d. In the case of curves, we can identify $M^{(d)}$ with the set of effective divisors on M of degree d.

By fixing a base point $p_0 \in M$, we get a map $\mu : M^{(d)} \to \mathscr{J}(M)$. The Jacobi inversion theorem describes one particular case of this map.

Theorem 5.3 (Jacobi Inversion Theorem)**.** *If M is a curve of genus g, then $\mu : M^{(g)} \to \mathscr{J}(M)$ is surjective and generically injective, i.e. the set of points on $M^{(g)}$ at which μ is not injective forms a subvariety of strictly smaller dimension.*

This means that $M^{(g)}$ and $\mathscr{J}(M)$ are in some sense "almost" isomorphic. Indeed, an open dense subset of the former is isomorphic to an open dense subset of the latter.

One immediate consequence of the Jacobi inversion theorem is that all genus one curves are tori. This is because we know that the Abel-Jacobi map is an injection by Abel's theorem, and since $M^{(1)} = M$, the Jacobi inversion theorem tells us that it is surjective, so it is an isomorphism.

In particular, this means that any projective plane cubic C has a group structure. Another way to see this is to take $H = \mathrm{Div}(C)/G$, where G is the subgroup generated by some chosen base point $p_0 \in C$ and by divisors of the form $p + q + r$, with $p, q, r \in C$ such that there is a line in $\mathbb{P}^2$ which intersects C at p, q, and r. The fact that the line intersects C at three points counting multiplicity is again just Bezout's theorem.

The elements of this group are points on C. Since points are divisors, and the equivalence relation we take doesn't identify distinct points, we see that points of C form a subset of H. That they are all of H is a consequence of the fact that any two points of C lie on a unique line. The element $-p$ of H corresponds to the third point on the line containing p_0 and p, and the element $p + q$ is the inverse of the third point of the line containing p and q.

The fact that the function ψ we defined above is constant says that if we have the same base point for the group structure as defined by the Abel-Jacobi map and the group structure as defined by the divisor construction, then the two are related by a translation.

Actually, we are cheating with this construction somewhat, since we are really constructing $\mathrm{Pic}^0(M)$, which is the dual torus to M, and implicitly using the fact that since all one-dimensional complex tori are principally polarized, they are self-dual.

Now we want to see whether Jacobian varieties can always be embedded in projective space. Actually, we're going to look at something more general – which complex tori can be embedded into projective space. This required more background though.

6. Line Bundles

By looking at the transition functions, you can see that the set of line bundles on a manifold M is in bijection with $H^1(M, \mathcal{O}^*)$. We can define a group structure on the set of line bundles too, with multiplication being the tensor product, and inversion being the dual bundle. The set of line bundles with this group structure is then isomorphic as a group to $H^1(M, \mathcal{O}^*)$. This group is called the Picard group, $\mathrm{Pic}(M)$.

The exponential short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{6.1}$$

with the first map being the inclusion and the second being $f \mapsto e^{2\pi i f}$ gives a map of cohomology $H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z})$. The first group is the group of line bundles, so we have a map from the group of line bundles to $H^2(M, \mathbb{Z})$. The image of a line bundle L under the map is called the (first) Chern class of L, and denoted $c_1(L)$. This definition isn't the most general one possible, but it will suffice for our purposes. There is an injection $H'^2(M, \mathbb{Z}) \to H^2_{dR}(M, \mathbb{R})$ of the torsion-free part of $H^2(M, \mathbb{Z})$ into $H^2_{dR}(M, \mathbb{C})$ which comes from the exact sequence of sheaves $0 \to \mathbb{Z} \to \mathbb{R}$ composed with the isomorphism of de Rham's theorem. If we have a Hermitian metric on L, then $c_1(L)$ is the (1,1)-form associated with the metric.

There is a short exact sequence of sheaves

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0 \tag{6.2}$$

which induces a long exact sequence of cohomology groups

$$\ldots \to H^0(M, \mathcal{M}^*) \to H^0(M, \mathcal{M}^*/\mathcal{O}^*) \to H^1(M, \mathcal{O}^*) \to \ldots. \tag{6.3}$$

The first map sends a nonzero meromorphic function to its divisor, $f \mapsto (f)$, while the second sends a divisor to a line bundle, $D \mapsto [D]$. By Poincaré duality, there is a differential form η_D corresponding to any divisor. The Chern class of the line bundle $[D]$ is then equal to η_D.

If we have a trivialization of line bundle L over the open set U_α, then we have an isomorphism between $\mathcal{O}(L)(U_\alpha)$, the holomorphic sections of L over U_α, and $\mathcal{O}(U_\alpha)$. A section is a function $\sigma : U_\alpha \to L_{U_\alpha}$. Composing this with the trivialization φ_α, we get a function from U_α to $U_\alpha \to \mathbb{C}$. Composing this with the projection onto the second coordinate gives a holomorphic function on U_α. Conversely, a holomorphic function on U_α can be multiplied with the identity map on U_α to get a map from U_α to $U_\alpha \times \mathbb{C}$, and applying the inverse of the trivialization gives us a section of L on U_α.

Using this isomorphism, a section of L over U is a collection of holomorphic functions s_α defined on $U_\alpha \subset U$ such that $s_\alpha = g_{\alpha\beta} s_\beta$ on $U_\alpha \cap U_\beta$. If we weaken our requirements by saying that the s_α need only be meromorphic, then we get a meromorphic section of L, which is the same thing as a section of the sheaf $\mathcal{O}(L) \otimes_{\mathcal{O}} \mathcal{M}$. If s, s' are two meromorphic sections of L, then their ratio is a meromorphic function, since

$$\frac{s_\alpha}{s'_\alpha} = \frac{g_{\alpha\beta} s_\beta}{g_{\alpha\beta} s'_\beta} = \frac{s_\beta}{s'_\beta}. \tag{6.4}$$

The meromorphic functions representing a meromorphic section on a given open set differ only by multiplication by a nowhere zero holomorphic function, so taking the divisors of these functions, we can find the divisor of any meromorphic section. Conversely, divisors are defined locally by nonzero meromorphic functions, and these local defining functions give a meromorphic section of the line bundle corresponding to that divisor.

Let $\mathcal{L}(D)$ denote the space of meromorphic functions f such that $D + (f) \geq 0$. Also, let $|D|$ denote the set of divisors on M linearly equivalent to D, that is, the set of effective divisors D' such that $D' - D$ is principal.

Let s_0 be a global meromorphic section of L, where $L = [(s_0)]$. Then for any holomorphic section s of L, $f_s = \frac{s}{s_0}$ is an element of $\mathcal{L}(D)$, and for any $f \in \mathcal{L}(D)$, $f \cdot s_0$ is holomorphic, so $\mathcal{L}(D)$ is isomorphic to $H^0(M, \mathcal{O}(L))$ under the multiplication by s_0 map.

One example of a line bundle is the canonical bundle. If M is an n-dimensional complex manifold, then the canonical bundle on M is $K_M = \bigwedge^n T_M'^*$, the nth exterior power of the holomorphic cotangent bundle. Holomorphic sections of K_M correspond to holomorphic n-forms, and meromorphic sections correspond to meromorphic n-forms. Divisors of holomorphic sections of the canonical bundle are called canonical divisors.

If M is compact, then $D' \in |D|$ means that there is some f such that $D = D' + (f)$. Since both D and D' are effective, $f \in \mathcal{L}(D)$. If there are two different functions f and g on M which have the same divisor, then the divisor associated to the quotient f/g is 0, and hence f/g must be a global holomorphic function, and hence constant. This means that two such functions differ by a nonzero scalar multiple, so we have

$$|D| \cong \mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}(H^0(M, \mathcal{O}(L))). \tag{6.5}$$

A linear system of divisors is a set of divisors corresponding to some subspace of this projective space. A complete linear system is a linear system of the form $|D|$. The base locus of a linear system is defined to be the intersection of all the divisors. An divisor D' which is in the base locus of a linear system is called a fixed component of the linear system.

The base points of a linear system are those points where all sections in the corresponding subspace of $H^0(M, \mathcal{O}(L))$ vanish. Let E be a subspace of $H^0(M, \mathcal{O}(L))$ such that the corresponding linear system has no base locus. Choose a basis $s_0, \ldots, s_N$ for E. For $p \in M$, we can find an open set U_α containing p and a trivialization $\varphi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$. The isomorphism between $\mathcal{O}(L)(U_\alpha)$ and $\mathcal{O}(U_\alpha)$ then gives us functions $s_{0,\alpha}, \ldots, s_{N,\alpha}$. If we choose another trivialization, φ_β, then on $U_\alpha \cap U_\beta$, we have $s_{i,\alpha}/s_{i,\beta} = g_{\alpha\beta}$. This means that the vectors $(s_{0,\alpha}(p), \ldots, s_{N,\alpha}(p))$ and $(s_{0,\beta}(p), \ldots, s_{N,\alpha}(p))$ differ by $g_{\alpha\beta}(p)$, so we have a well-defined function $\imath_E : M \to \mathbb{P}^N$ defined by $\imath_E(p) = [s_0(p), \ldots, s_N(p)]$, so long as not all the s_i vanish at some point. The fact that they do not all vanish is simply a restatement of the assumption that E has no base locus. The map into projective space induced by the complete linear system coming associated with all of $\mathbb{P}(H^0(M, \mathcal{O}(L)))$ is denoted $\imath_L$.

A line bundle L is called positive if there is a positive (1,1)-form which is a representative of the Chern class of L, taken as a de Rham cohomology class. We can get a positive line bundle on $\mathbb{P}^n$ as follows. First take the trivial vector bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$. Next consider the subbundle which associates to each point of $\mathbb{P}^n$ its fiber under the natural projection map. This is a line bundle by the definition of

$\mathbb{P}^n$, called the universal line bundle. It has a Hermitian form coming from the inner product on $\mathbb{C}^{n+1}$. The dual bundle, denoted H, called the hyperplane bundle also has a Hermitian form, which is positive definite. This hermitian metric is actually Kähler, and is called the Fubini-Study metric. One consequence of this is that any algebraic variety (a submanifold of some projective space) is Kähler. Another is that we can pull back H to a positive line bundle on any algebraic variety, so every algebraic variety has a positive line bundle.

In fact, the converse of this statement is also true.

Theorem 6.6 (Kodaira Embedding Theorem)**.** *If $L \to M$ is a positive line bundle, then there is a k_0 such that for $k \geq k_0$, $\imath_{L^k}$ is an embedding, where L^k is the kth tensor power of L.*

The Chern class of L will contain a closed positive (1,1)-form with integral cohomology class, called a Hodge form, by definition of positive line bundle. Conversely, suppose we have a Hodge form, ω. In the exact sequence $H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O})$, we have $H^2(M, \mathcal{O}) \cong H^{0,2}_{\bar{\partial}}(M)$, so nonzero elements of this group are represented by pure (0,2)-forms. Since ω is a pure (1,1) form, it is sent to 0, but by the exactness of this sequence, that means that ω represents the Chern class of some line bundle which is thus positive, so by the Kodaira embedding theorem, M is an algebraic variety.

We can use the Kodaira embedding theorem to show that $\mathbb{P}^m \times \mathbb{P}^n$ is an algebraic variety. If π_1 and π_2 are the projections onto the first and second coordinates respectively, then $\pi_1^* H \otimes \pi_2^* H$ is a positive line bundle, hence some power of it must give rise to an embedding of $\mathbb{P}^m \times \mathbb{P}^n$ into projective space. In fact, this bundle itself is very ample, which means that it gives rise to an embedding into projective space. This map is called the Segré map.

One example of the Segré map is the map $\tau : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ defined by $\tau(([w_1, w_2], [z_1, z_2])) = [w_1 z_1, w_1 z_2, w_2 z_1, w_2 z_2]$. This formula for the map easily generalizes to give us an explicit formula for the Segré map from $\mathbb{P}^m \times \mathbb{P}^n$ to $\mathbb{P}^{(m+1)(n+1)-1}$. More generally, if X and X' are algebraic varieties with Hodge forms ω and ω', then $X \times X'$ is an algebraic variety with Hodge form $\pi_1^*\omega + \pi_2^*\omega'$, where π_1 and π_2 are the projections onto the first and second coordinate respectively.

7. Abelian Varieties

In this section, we want to study Abelian varieties – complex tori which are also algebraic varieties, in order to prove that all Jacobian varieties are algebraic varieties. However, before we do this, we will start off with some general facts about complex tori. They are not necessary to what follows, but I feel they are interesting in their own right. The first fact is that complex tori are the only complex compact connected Lie groups. The second is that any map between complex tori is a group homomorphism followed by a translation.

Now we will look more specifically at Abelian varieties. First, we note that if $M = V/\Gamma$ is a torus, where V is an n-dimensional complex vector space and Γ is a lattice, then $T'_\mu(M) \cong V$, so any Hermitian inner product on V induces a translation-invariant Kähler metric on M.

Under this metric, harmonic forms are translation-invariant because translations, being nullhomotopic, induce the identity map on $H^1(M, \mathbb{C})$, to which $\mathscr{H}(M)$ is isomorphic. Since an invariant form is determined by its value on the tangent space at a point, the space of all invariant (p, q)-forms is $\bigwedge^p V^* \otimes \bigwedge^q \overline{V}^*$. A dimension counting argument then shows that all invariant forms are harmonic, so

$$H^r(M, \mathbb{C}) \cong \mathscr{H}^r(M) \cong \bigoplus_{p+q=r} (\bigwedge^p V^* \otimes \bigwedge^q \overline{V}^*). \tag{7.1}$$

This means that the one-forms $dz_1, \ldots, dz_n$, and $d\overline{z}_1, \ldots, d\overline{z}_n$ on V freely generate $H^*(M, \mathbb{C})$ as a graded ring.

We can get another basis for the cohomology of M by noticing that since V is the universal cover of M, elements of $H_1(M, \mathbb{Z})$ can be lifted to distinct paths in V starting at the origin, so identifying a loop with its endpoint in V, we get $H_1(M, \mathbb{Z}) = \Lambda$. This identification then gives us a basis $dx_1, \ldots, dx_{2n}$ for the cohomology on M which come from a complex basis $e_1, \ldots, e_n$ for V and an integral basis $\lambda_1, \ldots, \lambda_{2n}$ for Λ respectively.

The Kodaira embedding theorem says that M is an Abelian variety if and only if there is a positive (1,1)-form of integral cohomology class ω. The cohomology class of this form is called a polarization of the Abelian variety M. By looking at how this form would look in terms of the two bases, we get conditions on the relation between the two that tell us when a torus is an Abelian variety.

Proposition 7.2 (Riemann Conditions)**.** *Let Ω be the $n \times 2n$ change of basis matrix from λ_i to e_i. Then M is an Abelian variety if and only if we can choose λ_i and e_i such that Ω is of the form $\Omega = (\Delta_\delta, Z)$, where Δ_δ is a diagonal matrix with integer coefficients such that for each α for which the equation makes sense, $\delta_\alpha \mid \delta_{\alpha+1}$, and Z is a symmetric matrix with $\operatorname{Im} Z$ positive definite.*

In terms of these bases, the Hodge form is $\omega = \sum \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}$. If all the δ_α are equal to 1, then the polarization corresponding to ω is called a principal polarization.

Riemann's bilinear equations told us that for a Jacobian variety, we could choose a basis for the integral and de Rham cohomologies such that the corresponding change of basis matrix was of the form $\Omega = (I, Z)$ with Z symmetric and $\operatorname{Im} Z$ positive definite. The Riemann conditions then tell us that any Jacobian variety is an Abelian variety, indeed a principally polarized Abelian variety. Recalling that $\mathscr{J}(S)$ was isomorphic to $H^0(S, \Omega^1)^*/H^1(S, \mathbb{Z})$, the polarizing form ω is the intersection form $H_1(S, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(S, \mathbb{Z}) \to \mathbb{Z}$. This is an element of $H^2(\mathscr{J}(S), \mathbb{Z})$, and hence by the inclusion map, of $H^2_{dR}(\mathscr{J}(S))$ too.

We know that

$$H^1(V, \mathcal{O}) \cong H^{1,0}_{\overline{\partial}}(V) \cong H^{1,0}(V) \subset H^1(V, \mathbb{C}) = 0, \tag{7.3}$$

and $H^2(V, \mathbb{Z}) = 0$, since $V \cong \mathbb{C}^n$ is contractible, so the long exact sequence in cohomology associated to the exponential sheaf sequence gives us $\cdots \to 0 \to H^1(V, \mathcal{O}^*) \to 0 \to \cdots$, so every line bundle over V is trivial. This means that given a line bundle $L \to M$, we can pull back L by the projection π to get a trivial bundle, $\pi^* L$.

For any $\lambda \in \Lambda$, the fibers of $\pi^* L$ at z and $z + \lambda$ are identified, so for any global trivialization φ of $\pi^* L$, the map $\varphi_{z+\lambda} \circ \varphi_z^{-1} : \mathbb{C} \to \mathbb{C}$ is a linear automorphism, which can be identified with an element of $\mathbb{C}^\times$, and varying this with z gives us a nowhere zero holomorphic map e_λ, with the compatibility condition $e_{\lambda'}(z+\lambda)e_\lambda(z) = e_\lambda(z+$

$\lambda')e_{\lambda'}(z) = e_{\lambda+\lambda'}(z)$. Conversely, given such functions, called multipliers, $V \times \mathbb{C}/\sim$, where $(z, \xi) \sim (x+\lambda, e_\lambda(z)\xi)$ gives a line bundle on the torus V/Λ, so the multipliers correspond to line bundles. Not all the multipliers are needed to determine a a line bundle; rather, we only need to specify those corresponding to some basis of Λ.

By projecting V first onto $(\mathbb{C}^\times)^n$ before onto M, we can find a global trivialization of π^*L such that $e_{\lambda_\alpha} \equiv 1$ for $\alpha = 1, \ldots, n$. Given any invariant positive integral (1,1)-form ω, we can choose a basis z_α for V such that if we take multipliers $e_{\lambda_\alpha} \equiv 1$ and $e_{\lambda_{n+\alpha}} = e^{2\pi i z_\alpha}$ for $\alpha = 1, \ldots, n$, then we get a line bundle L with Chern class $[\omega]$.

Since translation is nullhomotopic, the Chern class of a line bundle is translation-invariant. If a line bundle L is given by multipliers $e_{\lambda_\alpha} \equiv 1$, $e_{\lambda_{\alpha+n}} = e^{2\pi i z_\alpha}$, for $\alpha = 1, \ldots, n$, which I will henceforth call nice multipliers, then any translate of L is given by multipliers $e_{\lambda_\alpha} \equiv 1$ and $e_{\lambda_{\alpha+n}} = c_\alpha e^{2\pi i\alpha}$, and conversely, any line bundle given by such multipliers is a translate of L. It can be shown that any line bundle with Chern class zero can be realized by constant multipliers, and since the tensor product on line bundles has the effect of multiplying multipliers, this means that given a line bundle L' with Chern class $[\omega]$, if L is a line bundle given by nice multipliers with the same Chern class as L', then $L' \otimes L^*$ is a line bundle with Chern class zero, so it can be given by constant multipliers, so the multipliers of L' are multiples of the multipliers of L, and hence L' is a translate of L.

If e_{λ_i} is a collection of nice multipliers for a line bundle L, then given a trivialization φ of π^*L which induces the e_{λ_i} as multipliers and a holomorphic section $\tilde{\theta}$ of L, the trivialization gives us a holomorphic map θ on V satisfying $\theta(z+\lambda_\alpha) = \theta(z)$ and $\theta(z + \lambda_{\alpha+n}) = e^{2\pi i z_\alpha}\theta(z)$.

For any positive line bundle, we can find a translate, L, which is given by multipliers $e_{\lambda_\alpha} \equiv 1$ and $e_{\lambda_{\alpha+n}} = e^{-2\pi z_\alpha - \pi Z_{\alpha\alpha}}$, where Z comes from the polarization induced by the Chern class of our line bundle. Global sections correspond to entire holomorphic functions θ on V such that $\theta(z + \lambda_\alpha) = \theta(z)$ and $\theta(z + \lambda_{\alpha+n}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}\theta(z)$.

Because the θ functions are periodic in the λ_α, they can be expanded as a power series of the form

$$\theta(z) = \sum_{l \in \mathbb{Z}^n} a_l z'^l, \tag{7.4}$$

where the $z'_\alpha = e^{2\pi i \delta_\alpha^{-1} z_\alpha}$ are chosen such that $z'_\alpha(w + \lambda_\alpha) = z'_\alpha$.

The second conditions then give us the equations

$$a_{l+\lambda_\alpha} = e^{2\pi i \langle l, \Delta_\delta^{-1}\lambda_{n+\alpha}\rangle + \pi i Z_{\alpha\alpha}} \cdot a_l, \tag{7.5}$$

so the theta function is determined by the choice of a_l where $0 \le l_\alpha < \delta_\alpha$. Conversely, we can show that any choice of these coefficients gives a convergent power series. Since the group of θ functions is isomorphic to $H^0(M, \mathcal{O}(L))$, we have just shown that the dimension of $H^0(M, \mathcal{O}(L))$ is $\prod \delta_\alpha$. If L induces a principal polarization on M, then $H^0(M, \mathcal{O}(L))$ is one-dimensional, and is generated by the section corresponding to the Riemann theta function, which is defined by

$$\theta(z) = \sum_{l \in \mathbb{Z}^n} e^{\pi i \langle l, Zl\rangle} \cdot e^{2\pi i \langle l, z\rangle}. \tag{7.6}$$

Along with the usual relations for the Riemann theta function, we also have $\theta(z) = \theta(-z)$ by the symmetry of the defining equation.

Note that since sections of positive line bundles are given by theta functions, the Kodaira embedding theorem tells us that we can embed M into projective space using the theta functions defined by some power of any positive line bundle. In this case, the Kodaira embedding theorem can be improved upon to give us an explicit exponent.

Proposition 7.7. *If $L \to M$ is any positive line bundle on an Abelian variety, then $H^0(M, \mathcal{O}(L^k))$ has no base points for $k \geq 2$ and is an embedding for $k \geq 3$.*

If we take L to be a line bundle which induces a principal polarization, then $H^0(M, \mathcal{O}(L^3))$ will be 3^n-dimensional, so we can embed any n-dimensional Abelian variety into $\mathbb{P}^{3^n-1}$.

We know as a corollary of the Kodaira embedding theorem that the product of two Abelian varieties M and M' (of dimensions m and n respectively) is also an Abelian variety. By composing the embedding of each into projective space with the Segré map, if the embeddings of M and M' are into $\mathbb{P}^{3^m-1}$ and $\mathbb{P}^{3^n-1}$, then the new embedding of the $m+n$-dimensional torus is into $\mathbb{P}^{3^{m+n}-1}$. Nevertheless, this bound is not sharp, because any algebraic variety of dimension n can be embedded in $\mathbb{P}^{2n+1}$.

I now want to show that all meromorphic functions on the torus come from rational functions of theta functions. The fact that certain rational functions of theta functions descend to meromorphic functions on the torus is a consequence of the fact proved above that the ratio of two meromorphic sections is a well-defined meromorphic function. The fact that these are all the meromorphic functions requires some general facts about algebraic varieties.

First, note that ratios of homogeneous polynomials of the same degree in the homogeneous coordinates of some $\mathbb{P}^n$ give rise to well-defined meromorphic functions on $\mathbb{P}^n$. This is a consequence of the fact that homogenous polynomials of degree d are in bijection with the sections of the dth power of the hyperplane bundle, H^d. These are actually all the meromorphic functions on $\mathbb{P}^n$. Rational functions on $\mathbb{P}^n$ descend to meromorphic functions on algebraic subvarieties so long as the rational function doesn't have a pole along the subvariety, and we have the following important theorem:

Theorem 7.8. *For any given embedding of an algebraic variety M into projective space, all the meromorphic functions on M are restrictions of rational functions on the projective space.*

Since we have shown that any Abelian variety can be embedded into projective space with theta functions, this theorem tells us that any meromorphic function is a rational function of theta functions. Indeed, any meromorphic function on an n-dimensional Abelian variety is the ratio of two homogeneous polynomials of the same degree in 3^n different θ functions.

We can consider all the theta functions on a variety to form a graded ring for some suitable grading (not $\mathbb{N}$), since the sum of two theta functions corresponding to the same line bundle gives another theta function corresponding to that line bundle and the product of two theta functions is again a theta function for the tensor product of the corresponding line bundles. The grading is then the monoid of positive and trivial line bundles with the tensor product, where we say that the theta functions associated to the trivial bundle are the constant functions.

There is also a more algebraic way to think about Abelian varieties. First, since we have identified the tangent space on any point of M with V, we can identify V^* with $H^0(M, \Omega^1)$. Kodaira-Serre duality and the Dolbeault isomorphism then tell us that $V \cong H^n(M, \Omega^{n-1}) \cong H^{n-1,n}(M)$. Also, since we have said $\Lambda \cong H_1(M, \mathbb{Z})$, so Poincaré duality tells us that $\Lambda \cong H^{2n-1}(M, \mathbb{Z})$. If we take $V_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, then $V_{\mathbb{R}} \cong H^{2n-1}(M, \mathbb{R})$, and if we take $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, then $V_{\mathbb{C}} = H^{2n-1}(M, \mathbb{C})$, and using the Hodge decomposition, we can get $V_{\mathbb{C}} = H^{n-1,n}(M) \oplus \overline{H^{n-1,n}(M)} \cong V \oplus \overline{V}$. A Hodge form would be given by a skew-symmetric bilinear form $Q : \Lambda \otimes_{\mathbb{Z}} \Lambda \to \mathbb{Z}$. If we take the tensor product of Λ with $\mathbb{C}$ over $\mathbb{Z}$, then we can identify this space with $V \oplus \overline{V}$ by the above argument, so we get $Q : (V \oplus \overline{V}) \otimes_{\mathbb{Z}} (V \oplus \overline{V})$, with the conditions $Q(v, v') = 0$ for $v, v' \in V$, and $-iQ(v, \overline{v}) > 0$ for $0 \neq v \in \overline{V}$.

In the special case of the Jacobian variety, we saw that the Jacobian of a curve S could be defined by $\mathscr{J}(S) = (H^0(S, \Omega^1)^*/\gamma(H_1(S, \mathbb{Z}))$, where γ is the integration map, but since we know that γ is nondegenerate, we can identify the image of $H_1(S, \mathbb{Z})$ with its dual, $H^1(S, \mathbb{Z})$, and by Kodaira-Serre duality and the Dolbeault isomorphism, we get $H^0(S, \Omega^1)^* \cong H^{0,1}(S)$, so $\mathscr{J}(S) = H^{0,1}(S)/H^1(S, \mathbb{Z})$, and the bilinear form Q which gives the polarization comes from the cup product on $H^1(S, \mathbb{Z})$.

8. Intermediate Jacobians

If M is any algebraic variety, then we can define $\mathrm{Pic}^0(M)$ to be the subgroup of $\mathrm{Pic}(M)$ consisting of those line bundles whose Chern class is zero. By the exponential sheaf sequence we get the exact sequence of cohomology

(8.1)
$$H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}^*) \to H^1(M, \mathbb{Z}) \to H^1(M, \mathcal{O}) \to \mathrm{Pic}(M) \to H^2(M, \mathbb{Z}).$$

By definition, $\mathrm{Pic}^0(M)$ is the kernel of the last map, so by exactness, it is the image of $H^1(M, \mathcal{O})$. Since M is compact, $H^0(M, \mathcal{O}) \cong \mathbb{C}$ and $H^0(M, \mathcal{O}^*) \cong \mathbb{C}^{\times}$. The map between them is the exponential map, which is surjective, so by exactness, the map from $H^0(M, \mathcal{O}^*)$ to $H^1(M, \mathbb{Z})$ is the zero map, and so the map from $H^1(M, \mathbb{Z})$ to $H^1(M, \mathcal{O})$ is injective, and hence we can identify $\mathrm{Pic}^0(M)$ with $H^1(M, \mathcal{O})/H^1(M, \mathbb{Z})$.

If M is an Abelian variety, then by the previous identifications, we have

$$\mathrm{Pic}^0(M) = \overline{V}^*/\Lambda^*, \tag{8.2}$$

so $\mathrm{Pic}^0(M)$ is also a complex torus. In fact, the polarization on M induces a polarization on $\mathrm{Pic}^0(M)$, so $\mathrm{Pic}^0(M)$ is an Abelian variety, called the dual Abelian variety of M, and denoted $\widehat{M}$.

If M is any compact Kähler manifolds, then we can associate a number of tori to M, called Griffiths' intermediate Jacobians. First, we define

$$V_q = H^{q-1,q}(M) \oplus \cdots \oplus H^{0,2q-1}(M) \tag{8.3}$$

for $1 \leq q \leq \dim M$. By the Hodge decomposition, $H^{2q-1}(M, \mathbb{C}) = V_q \oplus \overline{V}_q$. If Λ_q is the image of $H^{2q-1}(M, \mathbb{Z})$ in V_q, then the qth intermediate Jacobian is defined to be $\mathscr{J}_q(M) = V_q/\Lambda_q$. For $q = 1$, we have $\mathscr{J}_1(M) = H^{0,1}(M)/H^1(M, \mathbb{Z})$ is the Picard variety $\mathrm{Pic}^0(M)$. We define the Albanese variety to be $\mathrm{Alb}(M) = \mathscr{J}_n(M) = H^{n-1,n}(M)/H^{2n-1}(M, \mathbb{Z}) \cong H^{1,0}(M)^*/H_1(M, \mathbb{Z})$ by Poincaré and Kodaira-Serre duality. Notice that the Picard and Albanese varieties are dual complex tori. If M is an algebraic variety, then $\mathrm{Alb}(M)$ is an Abelian variety, so $\mathrm{Pic}^0(M)$ is too. We know that $\mathrm{Pic}^0(\mathrm{Alb}(M)) = H^{0,1}(\mathrm{Alb}(M))/H^1(\mathrm{Alb}(M),\mathbb{Z}) \cong \overline{V}^*_n/\Lambda^*_n$, but $V_n = H^{n-1,n}(M)$, so by the Hodge decomposition and Kodaira-Serre duality, $\overline{V}^*_n \cong H^{0,1}(M)$, while $\Lambda^*_n \cong H^1(M,\mathbb{Z})$, so $\mathrm{Pic}^0(\mathrm{Alb}(M)) \cong H^{0,1}(M)/H^1(M,\mathbb{Z}) = \mathrm{Pic}^0(M)$, and similarly, $\mathrm{Alb}(\mathrm{Pic}^0(M)) = H^{n-1,n}(\mathrm{Pic}^0(M))/H^{2n-1}(\mathrm{Pic}^0(M),\mathbb{Z}) = V_1/\Lambda_1 \cong \mathrm{Pic}^0(M)$. Actually the same argument shows that for any Abelian variety M, $\mathrm{Alb}(M) \cong M$.

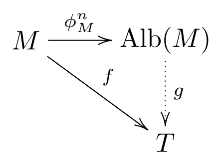
By picking a base point $p_0 \in M$, we get a map $\phi^n_M : M \to \mathrm{Alb}(M)$, called the Abel-Jacobi map, by a simple generalization of the one-dimensional case. Actually, this construction can be generalized to give a map $\phi^k_M : \mathcal{Z}^k(M)_{hom} \to \mathcal{J}_k(M)$, where the domain is the group of $n-k$-cycles homologous to 0. There's also a generalization of Abel's theorem:

Theorem 8.4 (Abel's theorem)**.** *If D is a divisor homologous to 0, then $\phi^1_M(D) = 0$ if and only if D is a principal divisor.*

And of the Jacobi inversion theorem:

Theorem 8.5 (Jacobi inversion theorem)**.** *For sufficiently large k, the map from $M^{(k)}$ to $\mathrm{Alb}(M)$ induced by ϕ^n_M is surjective.*

Also, the Albanese map has the universal property that for any map $f : M \to T$, where T is a complex torus, such that $f(p_0) = 0$, there is a unique map g such that the following diagram commutes:



A consequence of this universal property is that if M embeds into any complex torus, then the Albanese map is an embedding. This naturally leads to the question of which complex manifolds can be embedded into complex tori. This is obviously possible for all irrational curves and for all complex tori, but I have no idea about which others, if any.

9. Curves and their Jacobians

We can get an embedding of any compact Riemann surface S except $\mathbb{P}^1$ into some projective space $\mathbb{P}^N$ by embedding it in its Jacobian variety and then embedding that into $\mathbb{P}^N$. By the theorem stated above about meromorphic functions on algebraic varieties, any meromorphic function on S is the restriction of a rational function on $\mathbb{P}^N$, but this rational function also restricts to a meromorphic function on $\mathscr{J}(S)$, so any meromorphic function on S is the restriction of a meromorphic function on its Jacobian variety. Note that this gives us a bound on the transcendence degree of the field of meromorphic functions on S (albeit an awful one, since this degree is actually just one, the dimension of S).

Any Jacobian variety $\mathscr{J}(S)$ has a natural principal polarization, $[\omega]$, as we showed above. We can find a line bundle L whose Chern class is $[\omega]$. Since it is a principal polarization, some translate of L will have a global holomorphic section $\tilde{\theta}$ represented by the Riemann theta function. The divisor of $\tilde{\theta}$ is denoted Θ and called the theta divisor. Since $\tilde{\theta}$ is holomorphic, Θ is effective, and so we can identify it with a subset of $\mathscr{J}(S)$. For any $x \in \mathscr{J}(S)$, we define $\Theta_x \subset \mathscr{J}(S)$ to be $\Theta + x$, the translate of Θ by x.

Let μ be the one-dimensional Abel-Jacobi map. We can consider $\mu(S)$ as a submanifold of $\mathscr{J}(S)$ isomorphic to S for $S \ncong \mathbb{P}^1$, since we know that in this case, μ is injective. We can show that for any $x \in \mathscr{J}(S)$, either $\mu(S) \subset \Theta_x$ or the intersection number of $\mu(S)$ and Θ_x is precisely g, the genus of S. If $\mu(S) \not\subset \Theta_x$, then we can write $\mu^*\Theta_x = z_1(x) + \ldots + z_g(x)$. If K is a canonical divisor on S, then we can define the constant κ to be $\kappa = -1/2\mu(K)$. We then have the following proposition:

Proposition 9.1. *If $\mu(S) \not\subset \Theta_x$, then*

$$\sum_{i=1}^{g} \mu(z_i(x)) + \kappa = x. \tag{9.2}$$

Define $W_d \subset \mathscr{J}(S)$ to be $W_d = \mu(S^{(d)})$. The Jacobi inversion theorem says that for $d \geq g$, $W_d = \mathscr{J}(S)$. In fact, for no lower d is this map surjective. This is one of the consequences of

Theorem 9.3 (Riemann's theorem)**.**

$$\Theta = W_{g-1} + \kappa \tag{9.4}$$

This is important because Θ is defined purely in terms of the Abelian variety $\mathscr{J}(S)$, while κ and W_{g-1} are defined by S. This relation allows us to get information about the theta divisor using knowledge of S. For example, if S has genus at least five, then the set of singular points of Θ has dimension at least $g - 4$.

The fact that Jacobian varieties carry a lot of information about curves is perhaps made most clear by

Theorem 9.5 (Torelli's theorem)**.** *If S and S' are algebraic curves such that $(\mathscr{J}(S), [\omega_S]) \cong (\mathscr{J}(S'), [\omega_{S'}])$ as principally polarized Abelian varieties, then $S \cong S'$.*

Torelli's theorem proves that the map from the moduli space of curves of genus g to g-dimensional principally polarized Abelian varieties is injective. Is it surjective? In general, no. We know that the moduli space of a genus g surface has dimension $3g - 3$. A principally polarized Abelian variety is characterized by the symmetric matrix Z with $\operatorname{Im} Z > 0$, but a matrix is positive definite if and only if its eigenvalues are positive, and the eigenvalues of a matrix depend continuously upon the matrix, so the set of such Z is an open subset of the set of symmetric matrices, which has dimension $1/2g(g+1)$ for a $g \times g$ matrix, so for $g > 3$, the map of moduli spaces cannot be surjective. The problem of how to determine whether a principally polarized Abelian variety is a Jacobian variety is called the Schottky problem.

Note that the generalization of this statement to ask whether a given Abelian variety is the Albanese or Picard variety for some algebraic variety is trivial, since for any Abelian variety M, $M \cong \operatorname{Alb}(M) \cong \operatorname{Pic}^0(\operatorname{Pic}^0(M))$. Since not all the intermediate Jacobians are Abelian varieties, this still leaves the question of whether all tori are the intermediate Jacobian of some complex manifold. We could also generalize in another direction, and ask whether every Abelian variety or complex torus is the intermediate Jacobian of some complex manifold of bounded dimension. If not, it would also be interesting to see how the minimal dimension of Abelian variety which is not the intermediate Jacobian of some manifold of dimension $\leq d$ varies with d.

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