THE BANACH-TARSKI PARADOX

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Abstract. Stefan Banach and Alfred Tarski introduced the phrase: "a pea can be chopped up and reassembled into the Sun," a seemingly impossible concept. Using this theorem as motivation, this paper will explore the existence of non-measurable sets and paradoxical decompositions as well as provide a sketch of the proof of the paradox.

1. Introduction

Banach-Tarski states that a sphere in $\mathbb{R}^3$ can be split into a finite number of pieces and reassembled into two spheres of equal size as the original. Quite deserving of the name paradox, this theorem completely defies all geometric intuition about the volume of a sphere. How can a simple function such as rearrangement of the pieces double the volume? While it does seem impossible, when the idea of infinity is considered, the paradox becomes more intuitive. An example of two halves together equalling the same size as each half is the equal cardinality of the odd or even integers and all integers. Using this fact as a comparison, suddenly Banach-Tarski seems far more believable.

Physically, the Banach-Tarski Paradox cannot be achieved, because a solid sphere is comprised of a finite number of atoms. But in an Euclidean space of dimension 3 or higher, a sphere is infinitely dense and splitting it creates pieces which are also infinitely dense. Therefore, in actuality, it is not that surprising these pieces can be rotated and transformed to make two spheres of equal volume as the original. Take two subsets of $\mathbb{R}^3$ to represent the these spheres. Let set A be the original sphere, and B be the union of the two spheres from rearrangement. Banach-Tarski Paradox is then stated as follows, in its strong form.

Theorem 1.1. *For any 2 bounded subsets A and B of a Euclidean Space of dimension 3 or higher with non-empty interiors, there exists partitions of A and B into finitely many disjoint subsets*

$$A = A_1 \cup A_2 \ldots \cup A_k$$
$$B = B_1 \cup B_2 \ldots \cup B_k$$

such that for each $1 < i < k$, A_i and B_i are congruent.

Definition 1.2. Two subsets A and B are congruent if there is a distance-preserving bijection from A to B.

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However, before we continue onward with a discussion of the construction and proof of this theorem, a foundation of paradoxical sets, equidecomposition, and the axiom of choice must be built.

2. Non-measurable Sets

The proof of this theorem relies crucially on the Axiom of Choice.

Theorem 2.1. *For any sets A, B and binary relation* $P \subseteq A \times B, (\forall x \in A)(\exists y \in B)P(x, y) \implies (\exists f : A \to B)(\forall x \in A)P(x, f(x))$.

The axiom asserts that given an arbitrary number of decisions, each with at least one possible choice, then there exists a function that assigns a choice per decision. This is where debate about the axiom stems. Its consequences include many strange results such Banach-Tarski, but is not constructive since it never states which choices are made.

This type of property allows for the pieces of the sphere to be constructed from the choice sets $\{x_i\}$. As a consequence, these pieces have no Lebesgue measure. Without measure, these pieces can be distorted with transformations to produce sets which defy geometric logic.

3. Paradoxical Sets and Equidecomposition

Paradoxical sets are sets with paradoxical decompositions, which mean that they can be partitioned into two subsets which have a relation to the original set through a group. These sets are crucial to Banach-Tarski because the spheres we are considering must be paradoxical.

The easiest method of obtaining a paradoxical set is to use transformations to duplicate the set. These transformations are bijections of a single set and the simplest way to accomplish this is to use a group. A group G acts on a set X if to each $g \in G$ there corresponds a bijective function $X \longrightarrow X$, such that for any $g, h \in G$ and $x \in X, g(h(x)) = (gh)(x)$ and $1(x) = x$, where 1 is the identity of G.

Definition 3.1. Let G be a group acting on a set X and suppose that $E \subseteq X$. E is G-Paradoxical if for some m, n there exists $g_1, \ldots, g_m$ and $h_1, \ldots, h_n \in G$ and pairwise disjoint $A_1, \ldots, A_m$ and $B_1, \ldots, B_n \subseteq E$ such that $E = \cup g_i A_i = \cup h_j B_j$.

Namely, the two disjoint subsets $\cup A_i and \cup B_i$ map to the entirety of E through finitely many functions of G, a sort of "stretching" of the sets. In these terms, Banach-Tarski says that any ball in $\mathbb{R}^3$ is paradoxical with respect to the group of isometries of $\mathbb{R}^3$. The following is an example of a paradox that shows, with the Axiom of Choice, there are certain sets which are non-measurable.

Theorem 3.2. S^1 *is countably* SO_2*-paradoxical (Paradoxical with a countable number of pieces).*

Proof. Consider RSO_2, the subgroup of SO_2 generated rotations of rational multiples of 2π. Let H be a choice set for the cosets of SO_2/RSO_2. Then let $M = \{\sigma(1,0) : \sigma \in H\}$. Note that RSO_2 is countable. It is them possible to enumerate it with ρ_i. Then let $M_i = \rho_i(M)$. This makes $\{M_i\}$ a countable partition of S^1.

So each set in $\{M_2, M_4, M_6, \ldots\}$ may be individually rotated to make $\{M_1, M_2, M_3, \ldots\}$ whose union is S^1. Similarly, this can be done with $\{M_i : i \text{ odd}\}$. Hence, S^1 is countably SO_2-paradoxical. □

Corollary 3.3. *There does not exist a countably additive rotation-invariant measure of total measure 1 defined for all subsets of S^1.*

Corollary 3.4. *There does not exist a countably additive, translation-invariant measure of total measure 1 defined on all subsets of $\mathbb{R}^n$ and normalizing $[0,1]^n$. Thus, there is a subset of $[0,1]$ which is not Lebesgue-measurable.*

Continuing, we now move to the topic of equidecomposability of these paradoxical sets. The prime example for this is the one of a broken circle. It will be shown next that a broken circle $S^1 \setminus \{\text{pt}\}$ can be partitioned into two sets, then rejoined to create a complete circle.

Definition 3.5. Let G act on a set X, and let $A, B \subseteq X$. A and B are G-equidecomposable if A and B can be partitioned into $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ such that each $A_i$is congruent to B_i. ($\exists g_i \in G$ such that $g_i(A_i) = B_i$).

In words, this means two subsets $A, B \in \mathbb{R}^n$ are equidecomposable if they can be partitioned into the same finite number of pieces. A can be reassembled to make B and vice versa. Further, if $A \sim B$ and A is paradoxical, then B is paradoxical as well.

Theorem 3.6. $S^1 \setminus \{\text{pt}\}$ *is equidecomposable to* S^1

Proof. Since we are considering a broken circle, let the group used be the isomentry group of $\mathbb{R}^2$ where S^1 is $\{x : |x| = 1\}$. Consider $\mathbb{R}^2$ identified with $\mathbb{C}$. Let the broken point (pt) be $1 = e^i 0$. Also, let $A = \{e^i n : n \in \mathbb{N}\}$ and $B = (S^1 \setminus \{\text{pt}\}) \setminus A$ be all other points. The points $e^i n$ are unique, because 2π is irrational. Let B be fixed while rotating A by exactly one radian. This will shift each point in A by one radian and the circle will be complete. □

4. Paradoxical Groups

There are many specifics to determine whether a group is paradoxical, but for this particular topic, we won't go into those details. We will, instead, take for granted the known fact that a free group on two generators is a paradoxical group. In fact, it is equidecomposable with two copies of itself, under its own group action. To show this, the simplest method is using the Schroder-Bernstein Theorem which implies the following

Theorem 4.1. *Suppose group G acts on X. If each of $A, B \subseteq X$ is G-equidecomposable with a subset of the other, then A is G-equidecomposable with B.*

Consider this free group of two generators to be a group of rotations of the sphere.

Theorem 4.2. *There exist rotations S, T of the unit sphere in $\mathbb{R}^3$ that are independent. These rotations generate the free group F on two generators.*

To use this theorem, let us define our group as the rotations of angle arccos(1/3) about perpendicular axes:

$$S = \begin{bmatrix} \frac{1}{3} & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$$

Then, it can be shown that any word W on S, T, S^{-1}, T^{-1} sends (1,0,0) to a vector $(a, \sqrt{b}, c)/3^n$, where a, b, and c are integers, $b > 0$, n is the length of the word W, and b is not divisible by 3. These rotations are also independent because (1,0,0) is never sent to itself by a nontrivial word.

With this, which we call our free group F, we can proceed to a sketch of the construction of the paradox.

5. Construction of the Paradox

To construct the paradox, we need to show that the unit ball B^3 is equidecomposable with two unit balls under the isometry group of $\mathbb{R}^3$. This is easily achieved if a spherical shell S^2 is first considered. Then, the ball can be viewed as a collection of these spherical shells. This covers all points of the ball, except the center. We begin by removing the center of such a sphere.

Lemma 5.1. *B^3 is equidecomposable with $B^3/\{0\}$ where 0 is the center of the ball.*

Proof. Take a small circle which passes through 0 and is contained in B^3. Then let ρ be a 1 radian rotation of the circle. The it follows that $0, \rho 0, \rho^2 0, \rho^3 0, \ldots$ are all distinct. If ρ is then applied to this set of points again, the set remains the same except 0 is now gone. This creates a equidecomposition of B^3 with $B^3/\{0\}$. One piece is $0, \rho 0, \rho^2 0, \rho^3 0, \ldots$ going to $\rho 0, \rho^2 0, \rho^3 0, \ldots$ under rotation ρ, the other is the remaining points in the ball going to itself. □

Now that the center is out of the way, we must remove the other countably many points in the shell which are problematic.

Lemma 5.2. *Let D be the set of points S^2, our shell, that are fixed by some nontrivial element of F (constructed earlier). Then S^2 is equidecomposable with $S^2 \setminus D$.*

Proof. D is countable because F is a countable set of words and each rotation of F places only two points of S^2 on the rotational axis. Hence, there exists an axis that passes through the center, but not any points of D. Moreover, there exists a rotation ρ about the orbit such that $D, \rho D, \rho^2 D, \rho^3 D, \ldots$ are all disjoint. As in the proof of the previous lemma, when the rotation is applied to the set which is the union of $D, \rho D, \rho^2 D, \rho^3 D, \ldots$, D itself is removed. Thus, an equidecomposition of S^2 and $S^2 \setminus D$ is created. $\square$

Now, continuing with our shell $S^2 \setminus D$ which has no troublesome points, we use the Axiom of Choice to duplicate it. $S^2 \setminus D$ can be split into two sets, each of which is equidecomposable with S^2/D. To do this we note that distinct rotations send points of S^2/D to distinct images.

First, let us define an orbit. We will call $Fx = \{fx | f \in F\}$ the orbit of such a point $x \in S^2 \setminus D$. Two points are in the same orbit if and only if there exists a rotation which transports the first to the second. These orbits partition $S^2 \setminus D$ since they are equivalence classes under the relation $x \sim y \iff y \in Fx$. Using the Axiom of Choice, we will choose one member from each orbit to form a set M. Then all points $x \in S^2 \setminus D$ can be written as $x = fm$ for some $f \in F and m \in M$. It follows then that these sets partition $S^2 \setminus D$ also.

Because F is equidecomposable with two copies of itself (from Schroder-Bernstein Theorem), it can be partitioned into the subsets F_1, F_2 both of which are F-equidecomposable with F. Let $\phi_i : F_i \longrightarrow F$ where $i = 1, 2$ be F-equidecompositions. Consider these to be maps from $f to \phi_i f$. The sets $F_1 M, F_2 M$ partition $FM = S^2 \setminus D$. And there exist equidecompositions $\bar{\phi} : F_i M \longrightarrow FM = S^2 \setminus D$ from $fm \mapsto (\phi_i f)m$.

To prove this claim, let $A_{ik} \subseteq F_i$ be a piece that goes to $B_{ik} \subseteq F$ under ϕ_i, by $\phi_{ik} \in F$. There there exists a corresponding piece of $\bar{\phi}_i$ which sends the piece $A_{ik}M$ to $B_{ik}M$. And so $S^2 \setminus D$ has been successfully partitioned into two sets equidecomposable with the original.

If each shell of the sphere is treated in this way simultaneously, we end up with an equidecomposition of $B^3 \setminus \{0\}$ with two copies of itself. And using the Core Lemma in the beginning of this section, we find that B^3 is actually equidecomposable with two copies of itself. This completes the construction of the paradox.

A few notes about the paradox follow. The theorem only holds for dimension 3 or higher because there are no free non-abelian groups in the lower dimensions to make the paradox possible. It has also been proven that the paradox can be accomplished with as few as five pieces.

6. Conclusions

So we have examined the Banach-Tarski Paradox inside out. Even after all this analysis, the idea of doubling the volume of a ball but not using more matter is still counterintuitive. But when proposed with the thought that each piece is not Lebesgue measurable and properties such as volume are undefined on them, things become much easier to put into perspective.

However, this is not the end of the paradox. If we take a look at the result and apply it to probability theory, we see even stranger things such as the existence of events with undefined probability, but that is a problem for another day. For now, we will be content knowing that we have made peace with such a logic defying idea.

References

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