OPTIMAL BLUFFING FREQUENCIES

RICHARD YEUNG

Abstract. We will be investigating a game similar to poker, modeled after a simple game called La Relance. Our analysis will center around finding a strategic equilibrium.

A word of thanks- To my mentors, Matt and Jared, for brainstorm with me and revising draft after draft when I was making no sense.

Contents

1. Introduction

Consider the following simplified version of poker. Players I and II receive hands x and y, respectively, drawn uniformly and independently from the interval $[0, 1]$. Each player contributes an ante of 1 into the pot. Player I acts first either by folding and thus conceding the pot to Player II, or by betting a prescribed amount $\beta > 0$, which he adds to the pot. If Player I bets, then Player II acts either by folding and thus conceding the pot to Player I, or by calling and adding β to the pot. If Player II calls the bet of Player I, the hands are compared and the player with the higher hand wins the pot. That is, if $x > y$ then Player I wins the pot; if $x < y$ then Player II wins the pot. We do not have to consider the case $x = y$ since this occurs with probability 0.

The betting tree is shown below.

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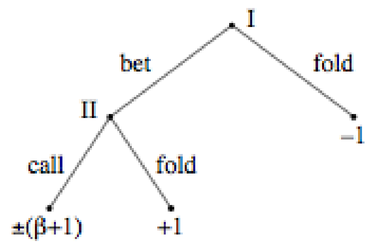


In the betting tree above, the $\pm$ indicates that Player I's return is $\beta + 1$ if his hand is higher than Player II's hand and $-(\beta + 1)$ otherwise.

We now define strategy for games similar to the game described above. Assume that regardless of the hand he receives, each player i has a finite set of options $O_i = \{O_{i,1}, \dots, O_{i,k_i}\}$. For example in the game above, Player I has 2 options: folding or betting.

Definition 1.1. A *strategy* for player i is a set of k_i functions $s_1^i, s_2^i, \dots, s_{k_i}^i : [0,1] \to [0,1]$ such that $\sum_{j=1}^{k_i} s_j^i(x) = 1$ for each $x \in [0,1]$. The value $s_j^i(x)$ gives the probability that player i chooses option $O_{i,j}$ given that he has hand x.

In our analysis, we will use a convenient form of this game.

Definition 1.2. The *strategic form* of a game is given by three objects:
(1) the set $N = \{1, 2, \dots, n\}$ of players,
(2) the sequence $O_1, \dots, O_n$ of option sets of the players, and
(3) the sequence $f_1, \dots, f_n : \left(\prod_{j=1}^n O_j\right) \times [0,1]^n \to \mathbb{R}$ of real-valued payoff functions for the players. The inputs are the options chosen by the players and the values of their hands.

This paper will predominately study a certain type of equilibria of the game La Relance, described below. For the sake of notation, let a_i denote the strategy of player i, i.e. $a_i = \{s_1^i, s_2^i, \dots, s_k^i\}$. And let A_i denote the set of all possible strategies for player i. Then we can define a function g which gives a player's expected payoff.

$$g_i(a_1, \dots, a_n) := \int_0^1 \int_0^1 \dots \int_0^1 f_i(a_1, \dots, a_n, x_1, \dots, x_n) dx_1 \dots dx_n.$$

Definition 1.3. Given the strategic form of a game with strategy sets $A_1, \dots, A_n$ and payoff functions $f_1, \dots, f_n$, a vector of strategy choices $(a_1, a_2, \dots, a_n)$ with $a_i \in A_i$ for $i = 1, \dots, n$ is said to be a *strategic equilibrium* of the game if for all $i = 1, 2, \dots, n$, we have

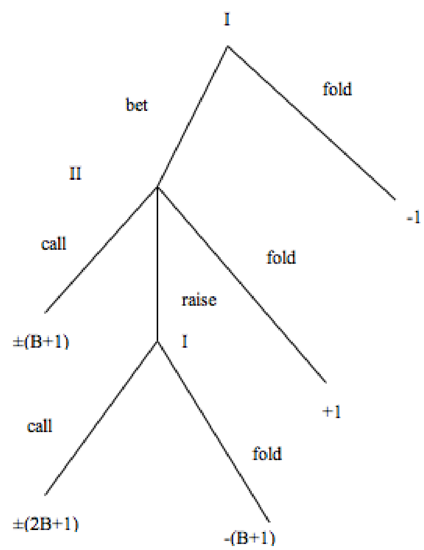
$$g_i(a_1, \dots, a_n) \geq g_i(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n)$$

for all $a \in A_i$.

The above equation says that if the players other than player i use their indicated strategies, then player i optimizes his return by playing strategy a_i. In other words, a particular selection of strategy choices of the players forms a strategic equilibrium if each player is using a best response to the strategy choices of the other players.

In [1], a strategic equilibrium for the game we introduced is outlined as follows. The optimal strategy for Player II is to fold if $y < b$ and to call if $y > b$, for some $b \in [0, 1]$. Player II chooses b to make Player I indifferent between betting and folding when Player I has some hand $x < b$. Player I's optimal strategy is not unique, but are of the form: if $x > b$, bet; if $x < b$, do anything provided the total probability of folding is b^2. Some insight into the reason why this is a strategic equilibrium is provided in [2], but this is not proved. Thus in section 2 we will explore the reasons that these strategies form a strategic equilibrium.

In section 3, we will explore an extended version of this game. Assume that instead of only being given the option to call or fold, Player II is given the additional option of raising by putting a total of 2β into the pot. Player I is in turn given the option to either call the raise by putting an additional β into the pot or to fold, thus conceding the pot to Player II. The betting tree is produced below.



Using our conclusions and technology from section 2, we will find a strategic equilibrium for this new game in section 3 and prove that it is an equilibrium.

2. The Simple Game

Before we begin, let us first define two terms which we will use very often over the course of this paper: expected value and optimal strategy. These terms are adapted from poker and game theory and defined in a way that works with our definition of strategic equilibrium.

Definition 2.1. Assuming the opponent's strategy is chosen and known, a player's *expected value* or *expectation* for a strategy, given his hand x, is defined as the expected value of the player's payoff given this information. In the two-player poker-based game we are concerned with, the expected value of Player I's strategy

$a \in A_1$, given a strategy $a' \in A_2$ employed by player II, is given by

$$EV = \int_0^1 f_1(a, a', x, y) D(y)\, dy$$

where $D(y)$ is the probability density function of the value of the opponent's hand in this situation and f_1 is the payoff for Player I given each player's strategies and hands.

Definition 2.2. Assuming the strategy of the opponent is chosen and known, a player's *optimal strategy* is a strategy where for each hand that he receives, expected value can not be increased by choosing a different strategy.

We will approach The Simple Game by first determining the form of Player II's optimal strategy in 2.0.1. Given the form of Player II's optimal strategy, we will solve for Player I's optimal strategy in 2.0.2.

2.0.1. *Player II's Optimal Strategy.* First we want to show that Player II's optimal strategy must involve calling for all $y > y_0$ and folding the rest, for some $y_0 \in [0, 1]$. To do this we will prove a more general case.

Proposition 2.3. *If a player is last to act, with only the options of calling or folding, an optimal strategy is to call for all values above some $y_0 \in [0, 1]$, and fold everything else.*

Proof. Without loss of generality, assume Player II is last to act. Player I is betting or raising with some distribution of his hands. Let $D(x)$ be the probability density distribution of the value of Player I's hand given his previous action of betting (since otherwise he folded and Player II has no choice to make). For all $x \in [0, 1]$, we know that $D(x) \geq 0$. This means that the area under $D(x)$, $\int_0^t D(x)\, dx$ is the probability that Player I's value is below t, and is also monotone increasing over $[0, 1]$.

Assume that after Player I acts, there is an amount P in the pot. Player II's expected value from folding is always zero, independent of his hand. Player II must contribute β if he calls. Then the expected value of calling for Player II if he has a hand t is

$$EV(t) := (P + \beta) \int_0^t D(x)\, dx - \beta$$

Player II will call if his expected value of calling is greater than his expected value of folding. Thus he will call when the integral above is greater than or equal to 0.[1] Since $EV(t)$ is a monotone increasing continuous function of $t \in [0, 1]$, he calls for $t > t_0$ for some t_0. $EV(t) = 0$ for $t \in [t_1, t_0]$ and $EV(t) < 0$ for $t < t_1$. Thus he is indifferent to folding or calling for hands $t \in [t_1, t_0]$ and he folds hands $t < t_1$. Hence an optimal strategy is to call for all values above some $t_0 \in [0, 1]$ and fold everything else. $\square$

[1]Money in the pot is always considered as money to be won. It is irrelevant that part of the money in the pot was once contributed by player I.

2.0.2. *Player I's Optimal Strategy.* Given the form of Player II's strategy, we will find the optimal strategy for Player I. We will approach the analysis of Player I's optimal strategy by using expected value graphs.

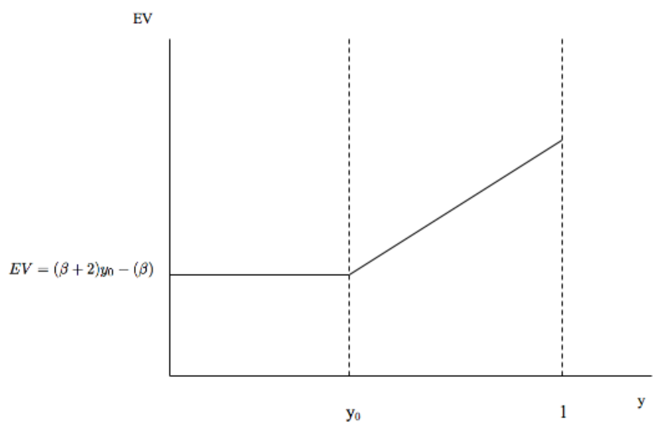
Definition 2.4. An *expected value graph* plots the expected value of an action as a function of the value of the player's hand.

By using expected value graphs, composing the graphs will give us Player I's optimal strategy. The optimal strategy will be to take the action with the highest expected value for each hand. The expected value of folding is always zero. Let us consider Player I's expected value graph of betting. For all values $x < y_0$, he must add β to the pot when betting, but will pick up the pot of $\beta + 2$ when not called. Player II will call $(1 - y_0)$ of the time, and fold y_0 of the time. Hence the expected value of betting $x < y_0$ is

$$EV = (\beta + 2)y_0 - (\beta)$$

Player I's expected value of betting $x > y_0$ is slightly more complex. The player still puts in β every time he bets, but now has two ways of winning - Player II can fold, or Player II can call with a weaker hand. When Player II calls with a weaker hand, which occurs $(1 - y_0)(\frac{x-y_0}{1-y_0})$ of the time, Player I wins a pot of $2\beta + 2$. Thus the expected value of betting $x > y_0$ is given by

$$EV = (2\beta + 2)(\frac{x - y_0}{1 - y_0})(1 - y_0) + (\beta + 2)(y_0) - \beta$$



The expected value graph of betting is shown above, $ev_b(x)$. Note that without knowing values for y_0 in terms of β, it is impossible to tell where the expected value of folding graph, $ev_f(x) = 0$, intersects this graph (if anywhere). Thus we can not yet solve for the optimal strategy of Player I. However $ev_b(1) = \beta(1 - y_0) + 2 > 0$, so $ev_b(x)$ lies entirely above $ev_f(x)$ or the two graphs intesect. It can also be seen that the expected value function, $ev_b(x)$, is non-decreasing over $[0, 1]$. Therefore if Player I has a non-negative expectation when betting with some $x \in [0, 1]$, he has a non-negative expectation by betting everything in the interval $[x, 1]$. We will call such a number x_0, and state an optimal strategy of Player I in the following form:

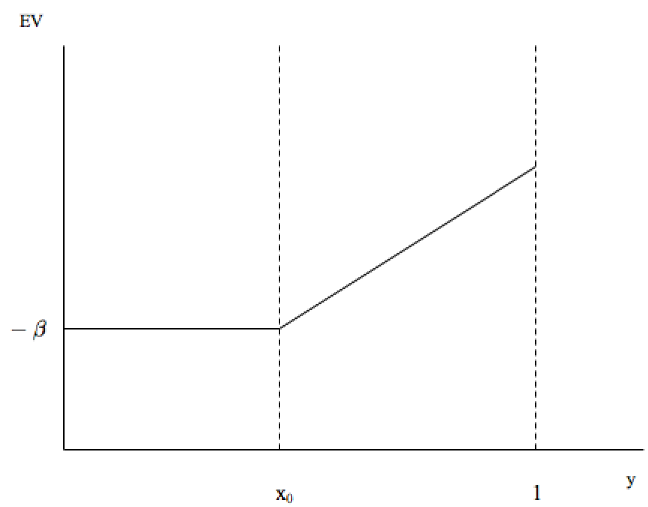
Player I will bet with all $x > x_0$[2], for some $x_0 \in [0, 1]$ such that $ev_b(x) \geq 0$ $\forall x > x_0$.

[2]We do not consider $x = x_0$, since the probability of having this hand is zero.

2.0.3. *A Slight Detour.* Knowing the form of Player I's strategy let us take a slight detour, and use Player II's expected value graphs to show that $y_0 > x_0$.

Player II's expected value of folding is zero, so let us consider his expected value graph of calling. He must add β the the pot when calling, and will never win with $y < x_0$, but will win the pot of $2\beta+2$ when his hand is higher than Player I's hand. Thus for $y < x_0$ the expected value of calling is $-\beta$, and for $y > x_0$, the expected value of calling is given by

$$EV = (2\beta + 2)(\frac{y - x_0}{1 - x_0}) - \beta$$



Note that this function is also strictly non-decreasing, so once again, we are interested in where Player II starts gaining a greater expectation by calling than folding, i.e, where $ev_f(y)$ and $ev_c(y)$ intersect. Since the expected value of calling with any $y < x_0$ is $-\beta$, we can reason that $y_0 > x_0$. Let us take this fact to return to solve for Player I's optimal strategy.

2.0.4. *Returning to Route.* When we were trying to find Player I's optimal strategy, one of our major obstacles for solving $ev_b(x) = ev_f(x) = 0$, was not knowing which part of the piecewise function to equate to zero. Now we know that $x_0 < y_0$, and hence either there is no intersection point or the intersection we are looking for occurs in the constant portion of the piecewise function $ev_b(x)$. In the first case, $ev_b(x) > ev_f(x) = 0$ for all x so $x_0 = 0$ and Player I will bet all hands $x > 0$. Looking at $ev_c(x)$, we see that Player II calls for $y > \frac{\beta}{2\beta+2}$; hence $y_0 = \frac{\beta}{2\beta+2}$. But then $ev_b(0) = -\frac{\beta^2}{2\beta+2} < 0 = ev_f(0)$. This contradicts our assumption that Player I bets all hands $x > 0$. Thus we can conclude that $ev_b(x)$ and $ev_f(x)$ do intersect and that the intersection we are looking for occurs in the constant portion of $ev_b(x)$. This yields two pieces of information about our strategic equilibrium. The first is the value of y_0. Since the constant portion of the piecewise equals the expected value of folding, we can equate it to zero and solve for y_0.

$$0 = (\beta + 2)y_0 - (\beta)$$
$$y_0 = \frac{\beta}{\beta + 2}$$

The second piece of information we gain, is that Player I is actually indifferent between choosing any value in $[0, y_0]$ to be x_0. By indifferent, we mean that given a chosen y_0 by Player II, Player I's choice of x_0 does not affect his *total expected value* The choices are all equally optimal. This does not, however, mean that any $x_0 \in [0, y_0]$ is a solution to the strategic equilibrium.

Definition 2.5. The *total expected value* of a strategy, EV_T, is the definite integral over $[0, 1]$ of the composite expected value graph[3]

2.0.5. *The real x_0.* We must find the value of x_0 for which Player II can not adjust to gain a greater total expected value by changing his value of y_0 to something other than $\frac{\beta}{\beta+2}$. In other words, we must find a x_0 such that $\frac{\partial EV_T}{\partial y_0} = 0$ when $y_0 = \frac{\beta}{\beta+2}$

Let us first find Player II's Total Expected Value as a function of y_0.

$$EV_T = \int_{y_0}^{1} (2\beta + 2)(\frac{y - x_0}{1 - x_0}) - \beta$$
$$= \frac{(-1 + y_0)(1 + y_0(1 + \beta) - x_0(2 + \beta))}{-1 + x_0}$$

Then we solve for the partial of EV_T with respect to y_0.

$$\frac{\partial EV_T}{\partial y_0} = \frac{(-1 + y_0)(1 + \beta)}{-1 + x_0} + \frac{1 + y_0(1 + \beta) - x_0(2 + \beta)}{-1 + x_0}$$

We then let $\frac{\partial EV_T}{\partial y_0} = 0$ and $y_0 = \frac{\beta}{\beta+2}$, to solve for x_0.

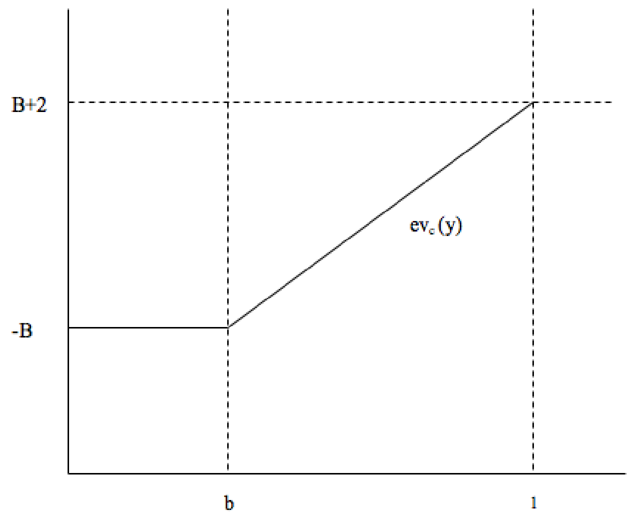
$$0 = (-1 + \frac{\beta}{\beta + 2})(1 + \beta) + 1 + \frac{\beta}{\beta + 2}(1 + \beta) - x_0(2 + \beta)$$
$$0 = \frac{\beta^2 - x(2 + \beta)^2}{2 + \beta}$$
$$x_0 = (\frac{\beta}{\beta + 2})^2$$

3. The Extended Game

We will begin by assuming Player I's optimal strategy and solving for Player II's optimal strategy. Then assuming the optimal strategy we find for Player II, we will prove that the strategy we assumed for Player I is optimal - thus proving we have a strategic equilibrium.

[3] A composite expected value graph plots the maximum of a player's expected value over all his possible actions, as a function of the player's hand.

3.1. Player II's Optimal Strategy. Assume that Player I's strategy at a strategic equilibrium is to bet all $x > b$ and call with all $x > d$, where $b, d \in [0, 1]$ and $b < d$. Let us construct Player II's expected value function for calling.

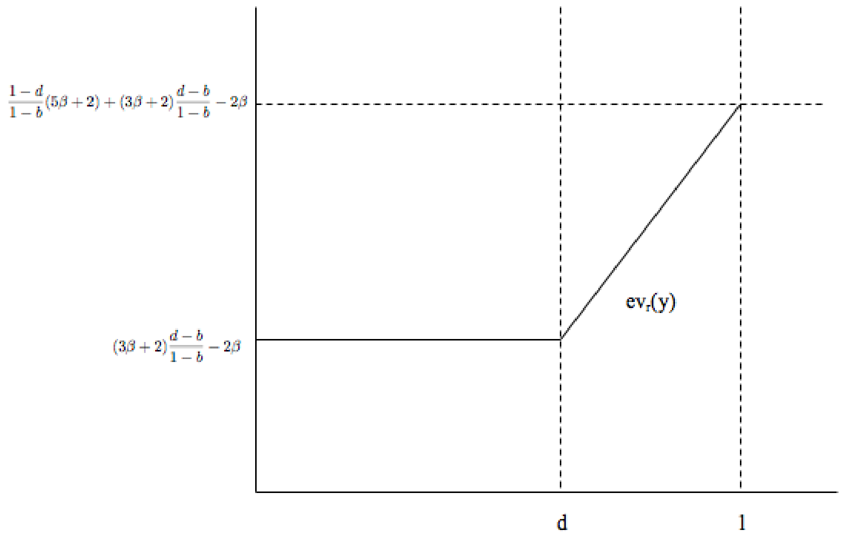


Knowing that Player I is only betting with $x > b$, calling with any $y < b$ gives Player II a zero percent chance of winning the pot. Thus he loses the amount β he puts in to call every time he does so. The expectation of calling with any $y \in [b, 1]$, however, is given by the function

$$ev_c(y) = (2\beta + 2)\frac{y - b}{1 - b} - \beta. \tag{3.1}$$

Player II must put β into the pot every time he calls, and has a $\frac{y-b}{1-b}$ chance of winning a $(2\beta + 2)$ dollar pot.

Next, let us construct Player II's expected value function for raising, $ev_r(y)$.



The expected value of raising any $y < d$ is essentially a "bluff," and when called, Player II has a zero percent chance of winning the pot. Thus for all $y \in [0, d]$,

$$ev_r(y) = (3\beta + 2)\frac{d-b}{1-b} - 2\beta.$$

Player II must put in the pot 2β every time he raises, and the only way he makes money is by Player I folding and giving up the pot, which happens $\frac{d-b}{1-b}$ of the time.

For all $y \in [d, 1]$,

$$ev_r(y) = \left(\frac{1-d}{1-b}\right)\left(\frac{y-d}{1-d}\right)(4\beta+2) + (3\beta+2)\left(\frac{d-b}{1-b}\right) - 2\beta.$$

When Player I calls, which happens $\frac{1-d}{1-b}$ of the time, there is a $\frac{y-d}{1-d}$ chance that Player I is calling with $x < y$. There is also a $\frac{d-b}{1-b}$ chance of Player I folding, which allows Player II to pick up a smaller pot.

The motivation for creating expectation functions is that for each $y \in [0, 1]$, Player II's optimal strategy is to pick the option with the highest expectation. Thus when the expectation functions are combined into one graph, the intercepts will be the values of y with which Player II is indifferent between two actions, and thus are the constants which serve as boundaries between one option and another.

We will show in proposition 3.2, that Player II's optimal strategy involves raising with hands in an interval $[r, 1]$, where $r \in [0, 1]$ is defined as the value where $\forall y > r$, the expectation of raising is greater than the expectation of either folding or calling. i.e $ev_r(y) > ev_c(y)$ and $ev_r(y) > ev_f(y) = 0\ \forall y > r$.

Proposition 3.2. *There exist an $r > d$ where $\forall y \in [r, 1]$, Player II gains the greatest expected value by raising.*

Proof. Let us compare $ev_c(1)$ and $ev_r(1)$:

$$
\begin{aligned}
ev_r(1) &= \left(\frac{1-d}{1-b}\right)\left(\frac{1-d}{1-d}\right)(4\beta+2) + (3\beta+2)\left(\frac{d-b}{1-b}\right) - 2\beta \\
&= \left(\frac{1-d}{1-b}\right)(\beta) + (3\beta+2) - 2\beta \\
&= \left(\frac{1-d}{1-b}\right)(\beta) + (\beta+2),
\end{aligned}
$$

while

$$
\begin{aligned}
ev_c(1) &= (2\beta+2)\left(\frac{1-b}{1-b}\right) - \beta \\
&= (\beta+2).
\end{aligned}
$$

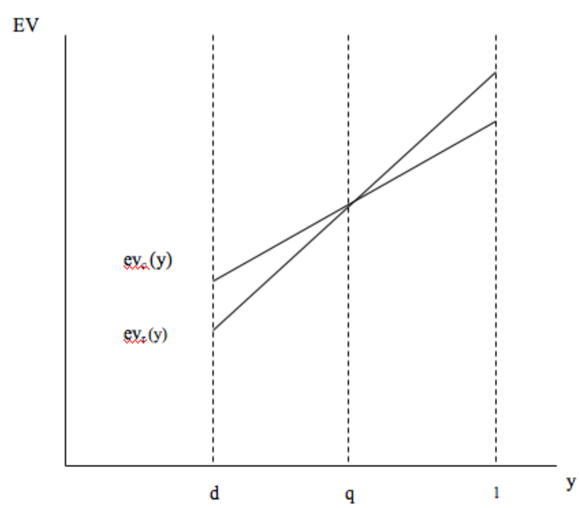
Thus we know that $ev_r(1) > ev_c(1)$. Let us now compare $ev_r(d)$ and $ev_c(d)$:

$$ev_r(d) = (3\beta+2)\left(\frac{d-b}{1-b}\right) - 2\beta,$$

while

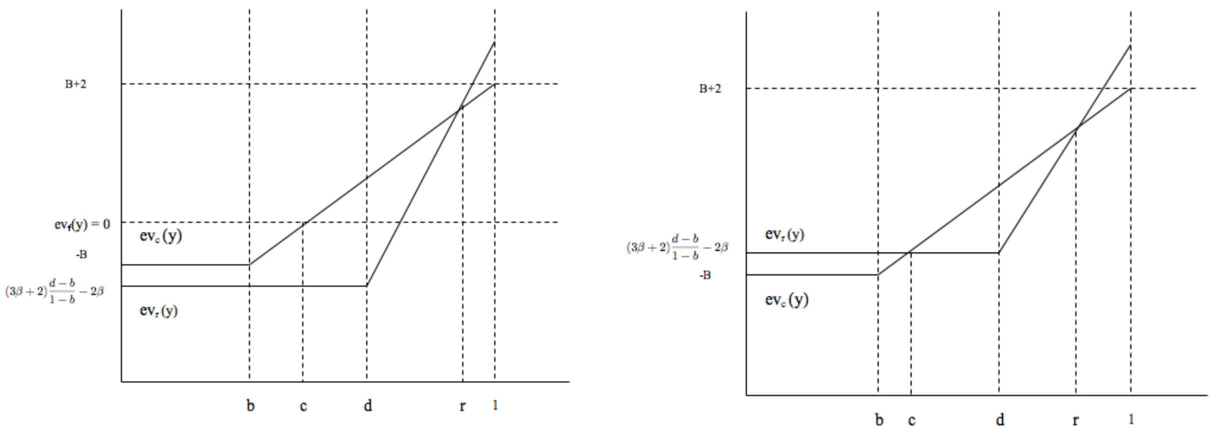
$$ev_c(d) = (2\beta+2)\left(\frac{d-b}{1-b}\right) - \beta$$

We know that $\frac{d-b}{1-b} < 1$ and hence $ev_r(d) - ev_c(d) = \beta\left(\frac{d-b}{1-b} - 1\right) < 0$. Thus we know that $ev_r(d) < ev_c(d)$. Below is a diagram showing what we know about how the composited expected value graph looks.



We have labeled the intersect of $ev_r(y)$ and $ev_c(y)$ to be q. For all $y \in [q, 1]$, we know that $ev_r(y) > ev_c(y)$, but do not yet know whether or not $ev_r(y) > ev_f(y)$. What we do know, is that Player II's optimal strategy cannot be to fold every single hand. Therefore $ev_f(y)$ must intersect $ev_r(y)$ at some point before 1. Player II's raising cutoff, r, will be the greater of the two intersections. Since we already know that $q > d$, regardless of where $ev_f(y)$ intersects $ev_r(y)$, we know that $r > d$. $\square$

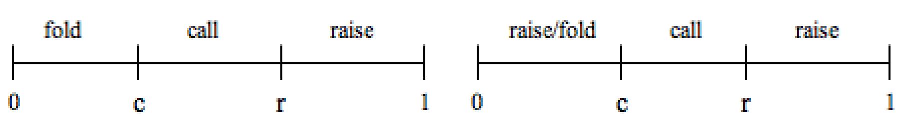
We now know that $ev_r(y)$ and $ev_c(y)$ intersect once over the interval $[d, 1]$. It is not immediately apparent whether or not $ev_r(y)$ and $ev_c(y)$ intersect again over the interval $[0, d]$. Both cases shown in the figure below are possible, depending on the values of β, b and d.



Finding whether or not an intersection in $[0, d]$ is possible is important to our analysis because it determines the form of Player II's optimal strategy. The figure below shows the two possible optimal strategies, they are ranges within the interval [0,1] where Player II takes a certain action depending on the range his value y is in.

The interval on the left will always be the strategy if there is no second intersection. The lack of a second intersection means that $ev_c(y) > ev_r(y), \forall y < d$. Thus Player II gains the greatest expectation by either calling or folding all $y < r$. This is shown in the figure above in the graph on the left.

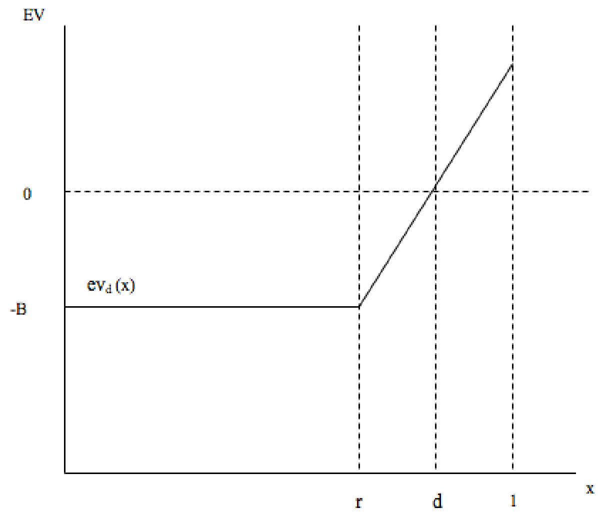
The figure and interval on the right assume there is a second intersection. Thus for an interval $[0, c]$ Player II will never call. We will show that the interval and graph on the right depict Player II's optimal strategy. In fact, we will show that for hands $y \in [0, c]$ Player II will raise a nonempty set of hands, and fold a nonempty set of hands.



Proposition 3.3. *Player II's optimal strategy must involve raising a non-empty set of hands* $y \in [0, c]$

Proof by contradiction. Assume for the sake of contradiction that the alternative is true, that Player II's strategy involves not raising any $y \in [0, c]$.

Let us construct Player I's expected value graph for calling, $ev_d(x)$.



Since Player II is only raising with $y > r$, Player I can expect to lose every time he calls with $x < r$. As we assumed in the beginning of this section [4], Player I's optimal strategy for calling is to call with $x > d$ for some $d \in [0, 1]$ and to fold with $x < d$. By inspection, we see that d is the cutoff where $ev_d(x) > 0$ for all $x > d$.

Since $ev_d(x) < 0$ for all $x < r$, we know that $d > r$. This contradicts the fact that $r > d$. □

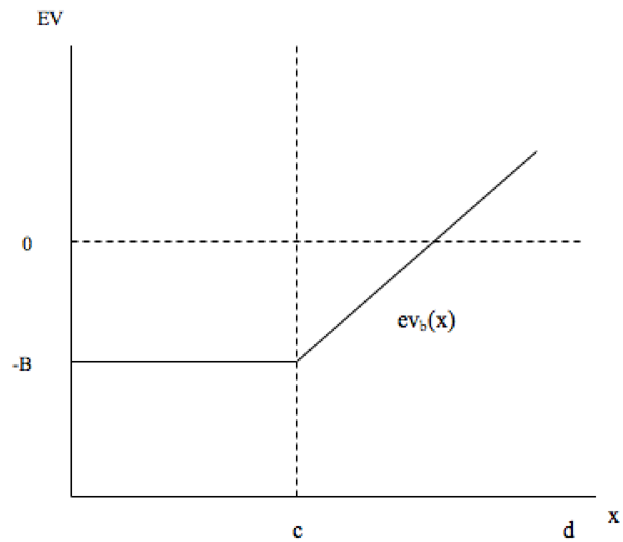
Thus we know that Player II must raise some set of hands in $[0, c]$.

Proposition 3.4. *Player II's optimal strategy must involve folding a non-empty set of hands $y \in [0, c]$.*

Proof by contradiction. Assume for the sake of contradiction that the alternative is true, that Player II's strategy involves not folding any $y \in [0, c]$.

Let us construct Player I's expected value graph for betting, $ev_b(x)$.

[4]And shown by Proposition 2.0.1



Player I will bet with $x > b$ only when the expectation from betting is greater than the expectation of folding. [5] Since Player II is never folding, Player I can only expect to win the pot when Player II calls with a hand less than that of Player I. So Player I will only begin betting with hands greater than c. In other words, $ev_b(x)$ intersects $ev_f(x)$ at a point b, where $b > c$.

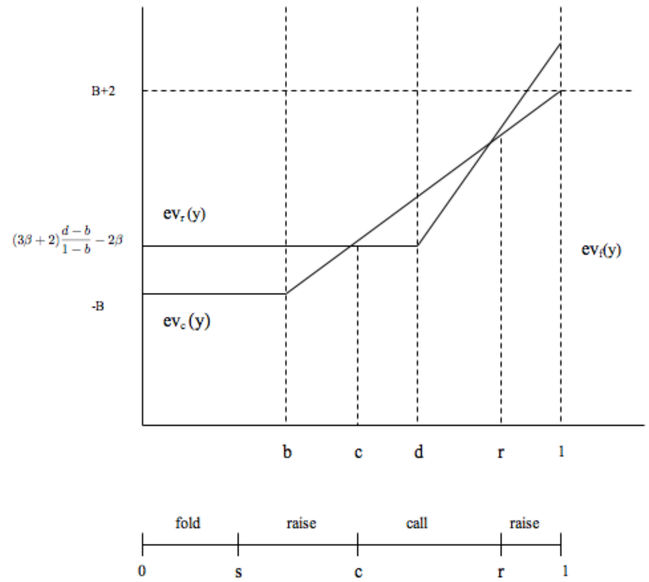
However, it can never be optimal for Player II to call with hands less than b, for he will always lose. Therefore $c > b$. □

Thus Player II's optimal strategy is to either raise or fold $y \in [0, c]$. We do not yet know the amounts with which it is optimal for Player II to raise or fold, but for convenience, we can define a value $s \in [0, c]$ where Player II folds all $y \in [0, s]$ and raises with all $y \in [s, c]$. [6]

The expected value graphs and action interval are shown below.

[5]To be more exact, Player I will bet when *either* the expectation of $bet/folding$ or the expectation of *bet/calling* is greater than the expectation of folding. But because $b < d < r$, Player I will always lose more when calling here.

[6]Before this point, we did not know where $ev_f(y)$ is.



Because we now know that it is optimal for Player II to either raise or fold $y \in [0, c]$, we know that $ev_r(y) = ev_f(y) = 0$ for all $y \in [0, c]$.

$$0 = (3\beta + 2)\left(\frac{d-b}{1-b}\right) - 2\beta$$

$$d = \left(\frac{2b + 2\beta + b\beta}{2 + 3\beta}\right). \tag{3.5}$$

Recall that we defined r as the greater of the intersections $ev_c(y)$ with $ev_r(y)$ and $ev_f(y)$ and $ev_r(y)$. We now know that the former is the greater of the two intersections. So let us solve for r.

Recall $ev_c(y)$ and $ev_r(y)$, defined for $y \in [d, 1]$, found in Section 3.1,

$$ev_r(y) = \left(\frac{1-d}{1-b}\right)\left(\frac{y-d}{1-d}\right)(4\beta + 2) + (3\beta + 2)\left(\frac{d-b}{1-b}\right) - 2\beta$$

and

$$ev_c(y) = (2\beta + 2)\left(\frac{y-b}{1-b}\right) - \beta.$$

To find r, we find y such that $ev_r(y) = ev_c(y)$:

$$\left(\frac{1-d}{1-b}\right)\left(\frac{y-d}{1-d}\right)(4\beta + 2) + (3\beta + 2)\frac{d-b}{1-b} - 2\beta = (2\beta + 2)\frac{y-b}{1-b} - \beta$$

$$(y-d)(4\beta + 2) - (y-b)(2\beta + 2) = -(d-b)(3\beta + 2) + (1-b)\beta$$

$$2\beta y = \beta + \beta d$$

$$r = \frac{1+d}{2}. \tag{3.6}$$

Interestingly, this gives us r as an average of d and 1.

We know that Player II is calling with an interval $[c, r]$, where c is the value of y when $ev_c(y) = 0$. From equation 3.1, $ev_c(y)$ for all $y > b$ is given by

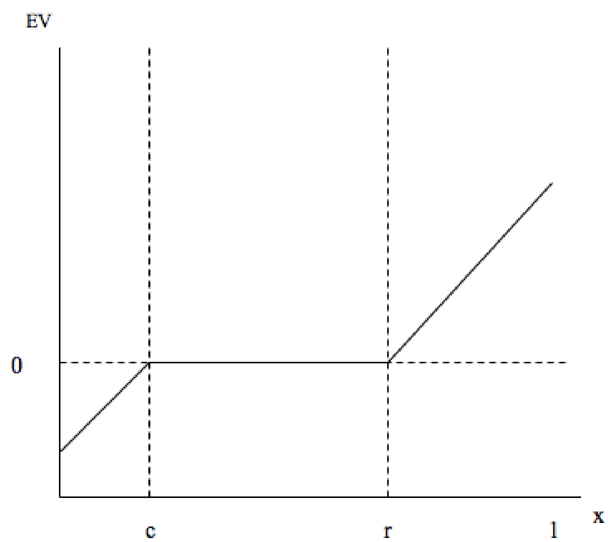
$$ev_c(y) = (2\beta + 2)\frac{y - b}{1 - b} - \beta.$$

Equating $ev_c(y)$ to zero, we find that

$$b = \frac{2c - \beta + 2c\beta}{2 + \beta} \tag{3.7}$$

3.2. **Player I's Optimal Strategy.** Assume Player II's optimal strategy is to fold with all $y \in [0, s]$, raise with all $y \in [s, c] \cup [r, 1]$, and to call with all $y \in [c, r]$, where $s, c, r \in [0, 1]$ and $s < c < r$.

3.2.1. *Solving for d.* Let us consider Player I's expected value function of calling, $ev_d(x)$, when he has already bet and Player II has raised.



The expected value of calling with $x \in [c, r]$ is constant, since calling with these values is essentially trying to "catch a bluff." Player I wins the pot if Player II raised with $y \in [s, c]$, and loses if II raised with $y \in [r, 1]$. The expected value for calling with $x \in [c, r]$ is given by,

$$ev_d(x) = \frac{c - s}{1 - r + c - s}(4\beta + 2) - \beta$$

We know from Proposition 3.2 that $d \in [c, r]$. [7] From the fact that Player I's optimal strategy is to call with all $x > d$, d must also be where $ev_d(x) = 0$. Thus we can equate the above equation to zero and solve for c as a function of r and s.

$$0 = \frac{c - s}{1 - r + c - s}(4\beta + 2) - \beta$$

$$c = \frac{2s + \beta - r\beta + 3s\beta}{2 + 3\beta} \tag{3.8}$$

[7] The proposition showed that $d < r$. We know that $d > c$ because otherwise, we would have two lines intersecting at two points. An impossibility.

From the fact that Player II is calling with $y \in [c, r]$, we know that there must exist some $b < c$, where Player I is betting all $x > b$. We also know that $d \in [c, r]$, so there must be some interval $[b, d]$ where Player I cannot find an option with a greater expectation than to $bet/fold$.

Player I's expected value of $bet/fold$ does not change in $[0, c]$. But we know that he does not bet the whole interval and he does not fold the whole interval. Betting the whole interval would mean he is never folding, which we have shown is impossible in Section 2.0.4. Folding the whole interval, on the other hand, would give Player II no reason to call at c. Hence Player I is calling with some amount in $[0, c]$ and folding with some amount in $[0, c]$. This also means that within $[0, c]$, the options $bet/fold$ and $fold$ have the same expectation.

Player I's expectation for $bet/fold$ in $[0, c]$ is given by

$$ev_{bf}(x) = (\beta + 2)s - \beta$$

We equate $ev_{bf}(x)$ to zero to solve for s.

$$s = \frac{\beta}{\beta + 2} \tag{3.9}$$

4. Putting it all together

We now have four unknowns and four equations. Recall from equations 3.5, 3.6, 3.7, 3.8, and 3.9 that

$$\begin{aligned} d &= \frac{2b + 2\beta + b\beta}{2 + 3\beta} \\ r &= \frac{1 + d}{2} \\ b &= \frac{2c - \beta + 2c\beta}{2 + \beta} \\ c &= \frac{2s + \beta - r\beta + 3s\beta}{2 + 3\beta} \\ s &= \frac{\beta}{\beta + 2} \end{aligned}$$

Some example solutions.

When $\beta = 1$,

$$s = \frac{1}{3} d = \frac{41}{81} \qquad r = \frac{61}{81} b = \frac{43}{243} \qquad c = \frac{31}{81}$$

When $\beta = 3$,

$$s = \frac{3}{5} d = \frac{489}{665} \qquad r = \frac{577}{665} b = \frac{1389}{3325} \qquad c = \frac{423}{665}$$

References

[1] Ferguson, Thomas. Game Theory.< http://www.math.ucla.edu/ tom/gamescourse.html>.

[2] R. Bellman (1952) On games involving bluffing, *Rendiconti del Circolo Math. di Palermo* Ser.2, Vol. 1 139-156