TRIANGLE FREE GRAPHS AND THEIR CHROMATIC NUMBERS

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Abstract. In this paper, we will investigate the family of triangle free graphs, some of whose members have arbitrarily high chromatic numbers. Such a family of graphs will be constructed iteratively and proven to have the above mentioned properties. These graphs will also be shown to be universal to all triangle free graphs, allowing us to use it to deduce bounds on the chromatic number and independence number of any triangle free graph.

Contents

1. Introduction

The family of graphs $\mathcal{F}$, studied in this paper was first constructed in response to an exercise posted by Professor Lazio Babai during one of his discrete math lectures. He asked if one could show that for every $k \in \mathbb{N}$, there exists a triangle free graph G with chromatic number $\chi(G) = k$. Indeed, $\mathcal{F}$ does exhibit the above mentioned property.

However, after further studying this family of graphs, I realized that every triangle free graph is an induced subgraph of some member of $\mathcal{F}$. The bulk of this paper is spent proving this discovery, and some possible applications of this property to bound the chromatic number and the independence number of a given triangle free graph are mentioned towards the end.

I also discovered and proved some other properties of the members in $\mathcal{F}$, which I included in Section 4. Although the relevance of some of these properties is not immediate in this paper, I suspect that they would come in useful in further strengthening the result proved in Section 6.

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2. Definitions and Construction of $\mathcal{F}$

Here, $\mathcal{F}$ will be constructed. Since I am assuming that the reader has no knowledge of graph theory, we will thus start with several basic definitions that will be used often throughout this paper.

Definition 2.1. A *graph* G, is an ordered pair $(V(G), E(G))$ consisting of a finite set of *vertices* $V(G)$, and a set $E(G)$, of unordered pairs of distinct vertices in $V(G)$. A member of $E(G)$ is known as an *edge*. The edge $e_{i,j} = (v_i, v_j)$ is said to be *associated* with the vertices v_i and v_j. Two vertices are *adjacent* if there is an edge that is associated to both of them, and these two vertices are also called *neighbors*.

In this definition, graphs have finitely many vertices, no vertex is adjacent to itself, and every two distinct vertices have at most one edge associated to them. Elsewhere, such graphs might be called **finite simple graphs**. In this paper, we use this narrower definition, because the only graphs we are considering are of this kind.

Definition 2.2. A *k-coloring*, c, of a graph, G, is a function $c : V(G) \to \{1, 2, \ldots, k\}$. One can also see this as an assignment of k colors to its vertices. Such a coloring is said to be *proper* if no two adjacent vertices have the same color, and a graph is *k-colorable* if it has a proper k-coloring.

Definition 2.3. The *chromatic number* of a graph, $\chi(G)$, is the smallest $k \in \mathbb{N}$ such that G has a proper k-coloring.

Definition 2.4. A graph is said to be *triangle free* if no two adjacent vertices are adjacent to a common vertex.

Definition 2.5. A set of vertices $\{v_1, \ldots, v_n\}$ is *independent* if v_i is not adjacent to v_j for all $i, j \in [n]$.

Construction 2.6. We will now construct the members of $\mathcal{F}$ iteratively. First, let T_1 be the simple graph with one vertex. Now, given any graph, T_k, of this family, construct T_{k+1} by the following steps:

(1) Choose any independent set $\{v_1, \ldots, v_{k-1}\}$ of $k-1$ vertices.
(2) For this set of vertices, create a new vertex, u, and create an edge, e_i, between v_i and u for all $i \in [k-1]$. Do this for every independent set of $k-1$ vertices.
(3) Create a new vertex c and create an edge between c and each vertex u created in (2).

For this construction to make sense, there needs to exist some independent set of vertices of size $k-1$ in the graph T_k. We will show that this is true later.

Definition 2.7. The vertex, u, created in step (2) of Construction 2.6 is known as the *ascendent of* $\{v_1, \ldots, v_{k-1}\}$.

Definition 2.8. The *critical vertex* of the graph T_{k+1} is the vertex, c, created in step (3) of Construction 2.6.

Example 2.9. For clarity, I will demonstrate constructing T_2 from T_1, T_3 from T_2 and T_4 from T_3.

In Figure 1, we start off first with T_1, the single vertex. Next, we choose a set of 0 vertices, or the empty set, and create an ascendent for it. Note that no vertices are adjacent to this ascendent because it is the ascendent of the empty set. Finally, we create a critical vertex and an edge between the ascendent and the critical vertex.

FIGURE 1. Constructing T_2 from T_1

Again, as shown in Figure 2, we perform the same operation. We begin with T_1 created in Figure 1, and for each set of one vertex, we create an ascendent and an edge between the vertex and its ascendent. After all that is done, we create a critical vertex and add an edge between each ascendent and the critical vertex.

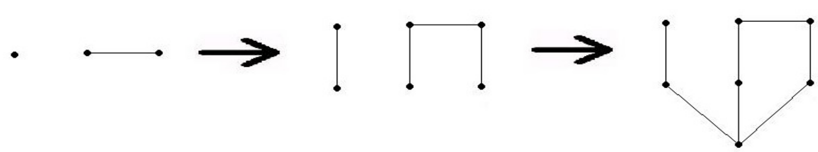

FIGURE 2. Constructing T_3 from T_2

Observe that the graph that we start with in Figure 3 is the same as the graph obtained at the end of the previous iterative step, i.e. the final graph in Figure 2. As before, we perform the same operation. First, choose an independent set of 2 vertices, and create an ascendent for this set. For both of these vertices, we then add an edge between each of these vertices and the ascendent created. Do this for every independent set of two vertices. Now, create a critical vertex, and an edge between each ascendent and the critical vertex. This gives us the final graph in Figure 3.

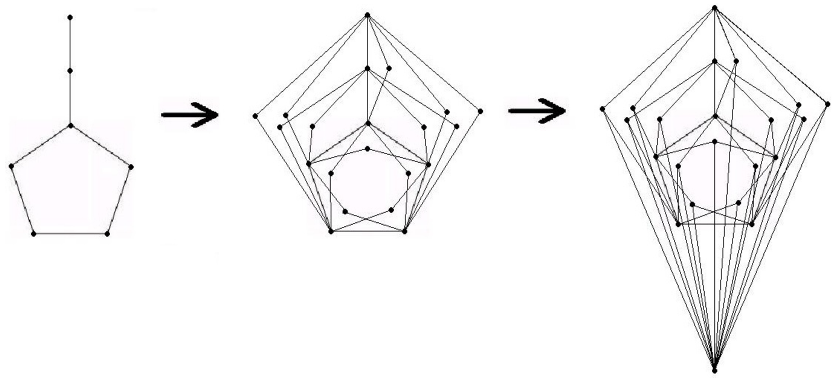

FIGURE 3. Constructing T_4 from T_3

3. Triangle Free and Chromatic Properties of $\mathcal{F}$

With these definitions, we are now suitably prepared to begin proving some interesting properties of $\mathcal{F}$. For a start, we will show that the family of graphs, $\mathcal{F} = \{T_k : k \in \mathbb{N}\}$ has only triangle free members, and that the chromatic numbers of the graphs in $\mathcal{F}$ is unbounded. In order to proceed further, we need to introduce yet another important concept in graph theory.

Definition 3.1. A graph H is an *induced subgraph* of another graph G if H can be obtained from G by removing a set of vertices, A, and all edges associated to A in G from G. For any vertex $v \in V(H)$, $H - v$ is the induced subgraph formed by removing v and all edges associated to v from H.

It is easy to see that for all $i \leq j$, T_i is an induced subgraph of T_j.

Lemma 3.2. *For all $k \in \mathbb{N}$, T_k is triangle free.*

Proof. This will be shown by induction. For our base case, it is clear that T_1 is triangle free. Suppose now that there exists $k \in \mathbb{N}$ such that T_k is triangle free. To construct T_{k+1}, we first create one ascendent for every independent set of $k-1$ vertices in T_k. Observe that no two neighbors of any of the ascendents are adjacent, so none of the ascendents created in this step are vertices of a triangle. Since T_k is triangle free, this means that the graph created up to this stage of the iteration process, i.e. $T_{k+1} - v$, where v is the critical vertex of T_{k+1}, is still triangle free. By construction, none of the ascendents of T_k are adjacent, and v is adjacent only to ascendents, so v is not a vertex of a triangle. Hence, T_{k+1} is also triangle free. □

Lemma 3.3. *For all $k \in \mathbb{N}$, $\chi(T_k) = k$.*

Proof. Again, we will prove this by induction. First, observe that $\chi(T_1) = 1$ because it has only one vertex. This gives us our base case. Now, suppose that $\chi(T_k) = k$. If we can use this to prove that $\chi(T_{k+1}) = k + 1$, then we will be done.

Consider a graph G with chromatic number $\chi(G) = k$. Choose any proper k-coloring, c, of G. Pick any color, a, in this coloring. Now, note that there exists at least one a colored vertex, v, such that for each of the $k-1$ remaining colors, v has a neighbour of that color. If this were not true, then we can change all the vertices that are colored a to some other color, which means that G is $\{k-1\}$-colorable, and this contradicts the condition that $\chi(G) = k$.

Since $\chi(T_k) = k$, it is clear that $\chi(T_{k+1}) \geq k$. We can prove that $\chi(T_{k+1}) > k$ by contradiction. Assume that $\chi(T_{k+1}) = k$. This means that we can color T_{k+1} by a proper k-coloring, c. But for this to be true, c also has to be a proper coloring of T_k. By the discussion in the previous paragraph, we know that for any color a of c, there exists an a colored vertex $v \in V(T_k)$ with neighbors that are colored in $k-1$ different colors. Choose a set, A, of $k-1$ differently colored neighbors of v, and observe that A is an independent set because if $v_1, v_2 \in A$ are adjacent, then v is their common neighbor. This violates the fact that T_k is triangle free.

By construction, A has an ascendent, u_a, and u_a has to be colored a because A already has $k-1$ colors, and every vertex in A is a neighbor of u_a. This can be repeated for any of the other $k-1$ colors, and so we end up having k differently colored ascendents. As a result, the critical vertex of T_{k+1} cannot be colored by the first k colors, so $\chi(T_{k+1}) > k$.

If we choose a proper k-coloring for T_k, by the same argument as before, it is patent that T_{k+1} can be colored in $k+1$ colors, so $\chi(T_{k+1}) \leq k+1$. We thus have $\chi(T_{k+1}) \leq k+1$ and $\chi(T_{k+1}) > k$, so $\chi(T_{k+1}) = k+1$.

□

Theorem 3.4. *Every member of $\mathcal{F}$ is triangle free and for any $k \in \mathbb{N}$, $\chi(T_k) = k$.*

Proof. This follows directly from Lemma 3.2 and Lemma 3.3.

□

4. Ascendents and the Critical Vertex

In this section, we will take a closer look at the classes of vertices created in our construction, namely the set of ascendents of T_k, which we call A_k, and the critical vertex of T_{k+1}, which we denote as c_{k+1}. These properties will be useful as an aid to obtain an estimate of the rate at which T_k grows as k increases.

Lemma 4.1. *Let A_k be the set of all ascendents created when constructing T_{k+1} from T_k. For all $k \in \mathbb{N} \setminus \{1\}$, $|A_k| \geq k+1$.*

Proof. When $k = 2$, $|A_2| = 3 \geq 2+1$ and when $k = 3$, $|A_3| = 14 > 3+1$, so it is clear that the lemma holds for these two cases. Using $k = 3$ as our base case, we will now prove by induction that $|A_k| > k+1$.for all $k \geq 3$. Suppose that for some $k \geq 3$, $|A_k| > k+1$. Since A_k is an independent set, any combination of k vertices in A_k yields an ascendent in A_{k+1}. Hence, $|A_{k+1}| \geq \binom{|A_k|}{k} > \binom{k+1}{k} = k+1$. This implies that $|A_{k+1}| \geq k+2$, so the lemma holds for all $k \in \mathbb{N} \setminus \{1\}$.

□

It is clear that Lemma 4.1 is not true for the case where $k = 1$. However, if we instead require only that $|A_k| \geq k$, then this will be true for all $k \in \mathbb{N}$. This has in fact been proven implicitly in Lemma 3.3. We proved that for any color a, used in any coloring of T_k, some independent subset $A \in V(T_k)$ of size $k-1$ has an ascendent that is colored a. This is true for each of the k colors that is needed to color T_k, so $|A_k| \geq k$.

Now that we have Lemma 4.1, several interesting corollaries are within reach.

Corollary 4.2. *For all $k \in \mathbb{N}$, $|A_k| \geq |V(T_k)|$.*

Proof. We will prove this by induction. Observe that $|A_1| = 1 \geq |V(T_1)|$, $|A_2| = 3 \geq |V(T_2)|$ and $|A_3| = 14 \geq 7 = |V(T_3)|$, so this corollary is true for the first three cases. Now, using $k = 3$ again as our base case, we will show by induction that this is true for all $k \geq 3$.

Suppose that for some $k \geq 3$, $|A_k| \geq |V(T_k)|$. This means that $|V(T_{k+1})| = |V(T_k)| + |A_k| + 1 \leq 2|A_k| + 1$. Also,

$$\begin{aligned} |A_{k+1}| &\geq \binom{|A_k|}{k-1} \\ &= \frac{(|A_k|)(|A_k|-1)\ldots(|A_k|-k+2)}{(k-1)!} \\ &\geq \frac{(|A_k|)(|A_k|-1)}{(2)(1)} \\ &= \frac{1}{2}(|A_k|^2 - |A_k|) \end{aligned}$$

The first inequality holds because A_k is an independent set of vertices with $|A_k| \geq k+1$, as shown in Lemma 4.1. In particular, $|A_k| \geq k-1$, so every subset of $A-k$ of size $k-1$ yields an ascendent in A_{k+1}. The second inequality is true because by Lemma 4.1 again, $|A_k|-2 \geq k-1$, $|A_k|-3 \geq k-2$, ..., $|A_k|-k+2 \geq 3$.

To prove the corollary, all we need is that $\frac{1}{2}(|A_k|^2 - |A_k|) \geq 2|A_k| + 1$. However, this can be shown easily by finding the roots of the quadratic equation $x^2 - 5x - 2 = 0$. The only positive root of this equation is $\frac{5+\sqrt{33}}{2} \leq 13 = |A_3|$, so $\frac{1}{2}(|A_k|^2 - |A_k|) \geq 2|A_k| + 1$ for all $k \geq 3$ since the last line of Lemma 4.1 implies that $|A_{k+1}| \geq |A_k|$. □

Corollary 4.3. *For all $k \in \mathbb{N}$, $|V(T_k)| \geq 2^{k-1}$.*

Proof. This will again be shown by induction. It is clear that $|V(T_1)| = 1 \geq 2^0$, so the base case holds. Now suppose that for some $k \in \mathbb{N}$, $|V(T_k)| \geq 2^{k-1}$. This gives us

$$\begin{aligned} |V(T_{k+1})| &= |V(T_k)| + |A_k| + 1 \\ &\geq 2|V(T_k)| \\ &= 2(2^{k-1}) \\ &= 2^k \end{aligned}$$

The first inequality follows from Corollary 4.2, and the second equality from the inductive hypothesis.

□

This gives us a lower bound on how fast $|V(T_k)|$ grows with k.

Corollary 4.4. $|A_k| \geq 2^{k-1}$.

Proof. This follows directly from Corollary 4.2 and Corollary 4.3.

□

We now have an improvement in the result in Lemma 4.1, as Corollary 4.4 gives us bigger lower bound on the size of A_k for $k \geq 3$.

Lemma 4.5. *For all $k \in \mathbb{N} \setminus \{1\}$ and for all vertices $v \in V(T_k)$, there exists an independent set $B_v^k \subset V(T_k)$ s.t. $|B_v^k| = k$ and $v \in B_v^k$.*

Proof. First, observe that for $k \geq 2$, we can decompose $V(T_k)$ into three components, $V(T_{k-1})$, A_{k-1}, which is the set of ascendents of T_{k-1}, and $\{c_k\}$, the critical vertex of T_k. We thus need to show that for all vertices v in each of these three components, B_v^k as defined in the statement of this lemma exists.

Now, let us consider A_{k-1}. As mentioned before, this is an independent set, and we showed in Lemma 4.1 that $|A_{k-1}| \geq k$. Hence, for all vertices $v \in A_{k-1}$, we can simply choose B_v^k to be a subset of A_{k-1} that is of size k and contains v.

Next, we consider c_k. By our construction, c_k is not adjacent to any of the vertices in $V(T_{k-1})$, and in particular, is not adjacent to the vertices in A_{k-2}. We know that A_{k-2} is an independent set, and again by Lemma 4.1, $|A_{k-2}| \geq k-1$. We thus choose $B_{c_k}^k = A_{k-2} \cup \{c_k\}$, and observe that $B_{c_k}^k$ is an independent set of size k that includes c_k.

Finally, we need to prove the same for the vertices in T_{k-1}. This can be done inductively. For the base case, i.e. $k = 2$, this is true because T_1 contains only a single vertex which has degree 0 in T_2, so we can choose $B_{V(T_1)}^2$ to be the duple made of the vertex in T_1 and any other vertex in T_2. To prove the inductive step, suppose that for some $k \in \mathbb{N}$, $k \geq 2$ and for all vertices $v \in V(T_{k-1})$, B_v^k exists. From what was discussed in the previous two paragraphs, this also means that for all vertices $w \in V(T_k)$, B_w^k also exists. Moreover, in T_{k+1}, c_{k+1} is not adjacent to w, so we can choose $B_w^{k+1} = B_w^k \cup \{c_{k+1}\}$ for all $w \in T_k$.

□

Proposition 4.6. *For all $k \geq 2$, A_k is a largest independent set of vertices in T_{k+1}.*

Proof. We will prove this by contradiction. Let P be a largest independent set of vertices in T_{k+1}, and suppose that $|P| > |A_k|$. Observe that for all $k \geq 2$, $V(T_k) \cup \{c_{k+1}\}$ is not an independent set , and by Corollary 4.2, any proper subset of $V(T_k) \cup \{c_{k+1}\}$ is at most as large as A_k. This means that P cannot be a subset of $V(T_k) \cup \{c_{k+1}\}$, so P contains some vertices in A_k.

It is obvious that P is not a proper subset of A_k, so P contains vertices in both A_k and $V(T_k)$. (P does not contain c_{k+1} because c_{k+1} is adjacent to every vertex in A_k.) Let $R = A_k \cap P$, let $Q = V(T_k) \cap P$ and let Q' be the set of vertices in A_k that are adjacent to some vertex in Q. If $|Q| \geq k$, then every subset of Q of size $k-1$ gives us an ascendent, so $|Q'| \geq \binom{|Q|}{k-1} \geq |Q|$. If $|Q| \leq k-1$, then pick any vertex $v \in Q$. By Lemma 4.5, there exists an independent subset of $V(T_k)$ that is of size k and contains v, so v is adjacent to at least $\binom{k-1}{k-2}$ ascendents. Hence, $|Q'| \geq \binom{k-1}{k-2} = k-1 \geq |Q|$.

In both cases, $|Q'| \geq |Q|$. Moreover, $Q' \cup R \subset A_k$ is an independent set and $|Q' \cup R| \geq |P|$ because $Q' \cap R = \emptyset$, so $|P| = |Q| + |R| \leq |Q'| + |R| = |Q' \cup R|$. This contradicts the assumption that $|P| > |A_k|$.

□

Corollary 4.7. *For any $k \in \mathbb{N}$, the critical vertex c_{k+1} of the graph T_{k+1} is a vertex with maximal degree.*

Proof. It is easy to see that this is true when $k = 1$. For the rest of the proof, we will only consider the case where $k \geq 2$. Since T_{k+1} is triangle free, for any vertex

$v \in V(T_{k+1})$, the neighbors of v have to be an independent set. By Proposition 4.6, A_k is a largest independent set of vertices in T_{k+1}, and c_{k+1} is adjacent to every vertex in A_k, so this corollary has to be true.

□

5. Universality of $\mathcal{F}$

Here, we will prove that $\mathcal{F}$ is a universal family of triangle free graphs, in the sense that every triangle free graph is an induced subgraph of some member of $\mathcal{F}$. But before we do that, let us introduce another definition.

Definition 5.1. The *degree of a vertex*, v, in a graph, G, is the number of edges in G associated to v, and is denoted by $d_G(v)$. Also, we write $\Delta(G) = \max_{v \in V(G)} d_G(v)$, and call this the *maximum degree of the graph* G.

Lemma 5.2. *For any graph G of maximal degree $\Delta(G) = k$, let $F = \{v_1, \ldots, v_n\}$ be a maximal independent set of vertices of G such that for all $i \in [n]$, $d_G(v_i) = k$. Let G^* be the induced subgraph of G formed by removing F from $V(G)$. Then $\Delta(G^*) < k$ and for all $i \in [n]$, the neighbors of v_i in G are in $V(G^*)$.*

Proof. Since F is independent, for all $i \in [n]$, the neighbors of v_i are not in F. Hence, they are not removed from $V(G)$ in the construction of G^*, so the neighbors of v_i are in $V(G^*)$. This proves the second half of the lemma.

By construction, G^* is an induced subgraph of G, so it is obvious that $\Delta(G^*) \leq \Delta(G) = k$. Now, suppose that $\Delta(G^*) = k$. This means that there exists a vertex $w \in V(G^*)$ with $d_{G^*}(w) = k$, which implies that $d_G(w) = k$ also because $\Delta(G) = k$. Since $w \in V(G^*)$, $w \notin F$, so some neighbor of w in G, call it w', is in F. If this were not so, then $F \cup \{w\}$ is an independent set, which contradicts the maximality of F. Hence, $w' \notin V(G^*)$, so $d_{G^*}(w) \leq d_G(w) - 1 = k - 1$. This contradicts the statement that $d_{G^*}(w) = k$.

□

For the purposes of this paper, we shall introduce the following two definitions.

Definition 5.3. For any set of vertices $A \subset V(G)$, where G is a graph, a *common* of A in G is a vertex that is adjacent to every vertex in A and not adjacent to any vertex in $V(G) \setminus A$. The *common number* of A in G is the number of commons of A in G, and can be written as $c_G(A)$.

Definition 5.4. A *foil* of a graph G is defined as a set F in Lemma 5.2. Let N_{v_i} be the set of neighbors of v_i that are in G. The *multiplicity* of the foil F is the number $M_F = \max_{v_i \in F} c_G(N_{v_i})$.

Theorem 5.5. *Every triangle free graph G is isomorphic to an induced subgraph in some $T_k \in \mathcal{F}$.*

Proof. We will prove this by induction on $\Delta(G)$. If $\Delta(G) = 0$, then we simply choose the graph $T_k \in \mathcal{F}$ such that $|A_k| \geq |V(G)|$. We proved in Lemma 4.1 that this is always possible. Observe that in Construction 2.6, in no instance is any edge created between the vertices in A_k, so A_k is an independent set. As such, the graph $(A_k, \emptyset)$ is an induced subgraph of T_k, and G is also an induced subgraph of $(A_k, \emptyset)$, so G is an induced subgraph of T_k. This is our base case, and we now need only to prove the inductive step.

Suppose that the proposition holds for any triangle free graph with degree less than or equal to p, and let G be a triangle free graph with $\Delta(G) = p + 1$. By Lemma 5.2, we have that $\Delta(G^*) < p + 1$, so by the inductive hypothesis, G^* is an induced subgraph of some $T_k \in \mathcal{F}$. Let F be the foil removed to create G^* from G, and assume for now that $M_F = 1$.

We now have two cases. In the first case, we have that $p + 2 \geq k$. Since T_k is an induced subgraph of T_{p+2}, G^* is also an induced subgraph of T_{p+2}. In constructing T_{p+3} from T_{p+2}, we create an ascendent for every independent set of $p+1$ vertices. By Lemma 5.2, we know that for each vertex $v_i \in F$, $N_{v_i} \subset V(G^*)$, so N_{v_i} yields an ascendent in A_{p+3}, which we will call v_i'. Let G' be the induced subgraph of T_{p+3} with $V(G') = V(G^*) \cup \{v_1', \ldots, v_n'\}$, and observe that G is isomorphic to G'.

Now, let us consider the second case, when $p + 2 < k$. In particular, $p < k$. Since G^* is an induced subgraph of T_k, it is also and induced subgraph of T_{k+1}, but $A_{k+1} \cap V(G^*) = \emptyset$. Moreover, by Lemma 4.1, we know that $|A_{k+1}| \geq k + 2$, and A_{k+1} is an independent set of vertices, so in constructing T_{k+2} from T_{k+1}, every subset of A_{k+1} which is of size k yields an ascendent. Let the set of all such ascendents be L, and by what was mentioned above, $|L| \geq \binom{k+2}{k} \geq k + 2$.

This means that in T_{k+2}, there is an independent set of $k+2$ vertices (the set L), each of which is not adjacent to any of the vertices in $V(G^*)$. Hence, when creating T_{k+3} from T_{k+2}, every independent set of $p+1$ vertices in $V(T_{k+1})$ can be combined with $k - p$ vertices from L to form an independent set of $k + 1$ vertices, which has an ascendent in A_{k+2}. In particular, we can do this for N_{v_i} for any $v_i \in F$, and the ascendent created this way, call it v_i', is a common of N_{v_i} in G^*. Now, define G' in the same way as the first case, and we have that G' is an induced subgraph of T_{k+3} and that G is isomorphic to G'.

In both cases, G might not be isomorphic to G' we do not assume that $M_F = 1$, because we have only added one vertex to G^* for each N_{v_i} to create G'. However, this can be easily remedied. Let $p + 3 = m$ in the first case and $k + 3 = m$ in the second case, i.e. G' is an induced subgraph of T_m in both cases. Moreover, observe that $p + 2 < m$, so by replacing G^* with G' and F with $F' = \{v \in F : c_G(N_v) > 1\}$ in the argument in previous paragraph, we can add one more common in G^* for each N_v with $v \in F'$. Do this M_F times, and we obtain an isomorphism of G that is in some graph in $\mathcal{F}$.

□

As proven in Lemma 3.2, every graph in $\mathcal{F}$ is triangle free, so there is no way that a non-triangle free graph can be an induced subgraph of a member of $\mathcal{F}$. The triangle free restriction in Theorem 5.5 is thus necessary.

6. Chromatic number of triangle free graphs

Definition 6.1. The *independence number* of a graph G is the size of a largest independent set of vertices of G. This can be written as $\alpha(G)$.

In this section, I will highlight a possible approach to use $\mathcal{F}$ to obtain a relationship between some properties of triangle free graphs, their chromatic number and their independence number. A critical ingredient for this though, is is a formula for the exact size of A_k, which I have not found.

Proposition 6.2. *Every triangle free graph G of independence number $\alpha(G) \geq m$ is not an induced subgraph of T_k for all $k \in \mathbb{N}$ with $|A_k| < m$.*

Proof. By Proposition 4.6, A_k is the largest independent set of vertices in $V(T_k)$. Hence, since $\alpha(G) \geq m > |A_k|$, the largest independent of vertices in G is greater than that of T_k, which means that G is not an induced subgraph of T_k.

□

Proposition 6.2 gives us the reason we need to know the exact size of each A_k. With this information we can obtain the smallest k such that a particular triangle free graph G is an induced subgraph of T_k. Moreover, this can also give us the exact number of vertices of T_k; it is evident that $|T_k| = \sum_{n=1}^{k-1} |A_k| + k$.

Proposition 6.3. *Every triangle free graph G of chromatic number $\chi(G) \geq m$ is not an induced subgraph of T_k for all $k < m$.*

Proof. If G is an induced subgraph of T_k, then $\chi(G) \leq \chi(T_k) = k < m$. This contradicts the condition that $\chi(G) \geq m$.

□

It seems possible to use Proposition 6.2 and Proposition 6.3 to deduce some properties of triangle free graphs with a known chromatic number and a known independence number. Conversely, given these properties, we should be able to find an upper bound on the chromatic number and the independence number of a particular triangle free graph.

Take any arbitrary graph G with $\Delta G = k_0$. As shown in Lemma 5.2, we can obtain G_1^* with $\Delta(G_1^*) = k_1 < k_0$ by taking the induced subgraph of G with vertices $V(G) \setminus F_0$, where F_0 is a foil of G. Let F_1 be a foil of G_1^*, and let G_2^* be the induced subgraph of G_1^* with vertices $V(G_1^*) \setminus F_1$. We can perform this iteratively until we obtain G_n^* with $\Delta(G_n^*) = 0$.

Definition 6.4. We call the set $B = \{F_0, F_1, \ldots, F_{n-1}, G_n^*\}$, where F_i and G_n^* are defined in the paragraph above, a *decomposition* of G. The *decomposition number* of B is defined as

$$\begin{aligned} D(B,G) \quad = \quad & \max\{\Delta(G) + 1 + M_{F_0}, \max\{\Delta(G_1^*) + 1 + M_{F_1}, \ldots, \max\{\Delta(G_{n-2}^*) \\ & +1 + M_{F_{n-2}}, \max\{\Delta(G_{n-1}^*) + 1 + M_{F_{n-1}}, k + 1 + M_{F_{n-1}}\} + 1 \\ & + M_{F_{n-2}}\} \cdots + 1 + M_{F_1}\} + 1 + M_{F_0}\} \end{aligned}$$

k here is the number such that $|A_k| \geq |G_n^*|$ and $|A_{k-1}| < |G_n^*|$.

Alternatively, we can write the decomposition number more concisely as

$$\begin{aligned} D(B,G) \quad = \quad & \max\{\Delta(G) + 1 + M_{F_0}, \Delta(G_1^*) + 2 + M_{F_0} + M_{F_1}, \ldots, \Delta(G_{n-1}^*) + n \\ & + \textstyle\sum_{i=0}^{n-1} M_{F_i}, k + n + 1 + \sum_{i=0}^{n} M_{F_i}\} \end{aligned}$$

However, the version given in Definition 6.4 gives us a more intuitive understanding of the decomposition number, as will be demonstrated later.

It is important to note that the decomposition number of a graph G depends not only on the graph, but also the decomposition chosen. In other words, a graph, if decomposed in different ways, might have different decomposition numbers.

Proposition 6.5. *Let B be the decomposition of a triangle free graph G, then G is an induced subgraph of $T_{D(B,G)}$.*

Proof. Since $|G_n^*| \leq |A_k|$, we know that G_n^* is an induced subgraph of T_k, so by what we have mentioned in Theorem 5.5, the graph G_{n-1}^* is an induced subgraph of $T_{\Delta(G_{n-1}^*)+1+M_{F_{n-1}}}$ if $\Delta(G_{n-1}^*)+1 \geq k$, or it is an induced subgraph of $T_{k+2+M_{F_{n-1}}}$ if $\Delta(G_{n-1}^*)+1 < k$. Hence, G_{n-1}^* is an induced subgraph of $T_{m_{n-1}}$, where $m_{n-1} = \max\{\Delta(G_{n-1}^*)+1+M_{F_{n-1}}, k+1+M_{F_{n-1}}\}$.

Again, by the reasoning in Theorem 5.5, we have that G_{n-2}^* is an induced subgraph of $T_{m_{n-2}}$, where $m_{n-2} = \max\{\Delta(G_{n-2}^*)+1+M_{F_{n-2}}, m_{n-1}+1+M_{F_{n-2}}\}$. Perform this iteratively, and we get the desired result.

□

Corollary 6.6. *Let B be a decomposition of a triangle free graph G. Then $\chi(G) \leq D(B,G)$ and $\alpha(G) \leq |A_{D(B,G)}|$. Conversely, if G is a graph such that $\chi(G)$ and $\alpha(G)$ are known, then every decomposition B, of G is such that $D(B,G) \geq \max\{\chi(G), k\}$, where k is such that $|A_{k-1}| < \alpha(G) \leq |A_k|$.*

Proof. This follows immediately from Proposition 6.2, Proposition 6.3 and Proposition 6.5.

□

Note that the upper bound of the chromatic number given in Proposition 6.6 is trivial. In the rewrite of the decomposition number in the paragraph after Definition 6.4, it is easy to see that $D(B,G) \geq \Delta(G) \geq \chi(G)$ by the same argument used in Lemma 3.3.

7. Final Comments

There are still many problems with the result proven in Proposition 6.6. Firstly, in order for the result to be used, a closed form expression of $|A_k|$ needs to be found. Moreover, the statement of the Proposition 6.6 in its current form is still too weak to be useful. By the current definition of the decomposition number, the upper bound of the independence number given in Proposition 6.6 is trivial in most cases, because it grows too quickly. This is especially so in graphs with a large number of foils. Furthermore, the upper bound for the chromatic number is trivial in all cases, as discussed in a previous comment. A better definition of the decomposition number might be able to tighten these two upper bounds.