SHORT ABSTRACTS: 2009 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

APPRENTICE PROGRAM

Apprentices attend courses offered in the full program during week 1. A morning program of introductory material will be given in weeks 2–5. That's right: an extra week of morning classes will be given). Apprentices are also welcome to attend all courses in later weeks. Several of the courses in the full program are designed to be accessible to apprentice participants.

Weeks 2–5: Laci Babai and Miklos Abert

Linear algebra and the spectral theory of graphs

The course will develop the usual topics of linear algebra and illustrate them on highly unusual and often striking applications to combinatorics. Emphasis will be on creative problem solving and discovery. The basic topics include permutations, determinants, linear transformations, the characteristic polynomial, Euclidean spaces, orthogonalization, the Spectral Theorem. The principal application area to be highlighted is the connection between the combinatorial structure and the spectrum of graphs, including expansion, random walks, chromatic number, and more.

FULL PROGRAM

Whatever faculty have posted in past abstracts, the material actually presented has generally diverged from that originally planned so far in advance. Therefore, only quite abbreviated indications are being posted. There are more faculty participants than in past years. Tentative week by week schedules have been posted on the web page. As noted there, we expect other talks not yet scheduled.

1. Laci Babai. Finite and transfinite combinatorics. Weeks 4–8.
2. Miklos Abert. Infinite groups. Weeks 1–4.
3. Gregory Lawler. Martingales and fractal dimension. Weeks 1–2.
4. Maddhav Nori. Summation of series. Weeks 1–4.
5. Benson Farb. Topics in geometry. Weeks 2–5.
6. Howard Mazur. Billiards in polygons. Weeks 5–6.
7. Peter May. Algebraic and topological K-theory. Weeks 1, 3, 5–8.

1. Combinatorics

Weeks 4 – 8: Laci Babai

Finite and Transfinite Combinatorics: Adventures in Paul Erdős's World.

Beyond the countable there lies a vast and very different world, yet questions analogous to those of finite combinatorics can be asked. Here is an example of the sometimes striking contrast: (1) (Erdős) Given positive integers g and k there exists a finite graph of chromatic number k with no cycle of length less than g. (2) (Erdős - Hajnal) If the chromatic number of a graph is uncountable then the graph contains a cycle of length 4.

Following Paul Erdős, we shall discuss these two worlds in parallel. The finite models require methods such as the probabilistic method and the linear algebra method.

We shall learn the nuts and bolts of working with the transfinite: hierarchies of infinite "numbers" (cardinals and ordinals), transfinite induction, ultraproducts, compactness.

2. Infinite groups

Weeks 1 – 4: Miklos Abert

A possible solution to many of the world's problems would be to double oranges by cutting and moving the pieces around. This can be done but one needs a very strangely shaped knife. On the other hand, no matter the knife, one can not double pancakes. We will discuss this and other curious things, using basic notions of infinite group theory, like amenability, group actions, some ergodic theory and invariant measures. Random walks will also come into the picture. Previous knowledge is not assumed. As usual, there will be many challenging problems of all levels.

3. Martingales and fractal dimension

Weeks 1 – 2: Gregory Lawler

Week One: Martingales

A martingale is a mathematical model of a fair game. Many interesting facts can be derived from the optional stopping theorem which states roughly "You can't beat a fair game." I will start by giving a counterexample to the theorem (!), and then discuss what kind of assumptions are needed in order for the theorem to hold.

Week Two: Fractal Dimension

I will discuss rigorous definition(s) of the idea that a set has a fractional dimension. I will look in detail on the example of the Cantor set (with dimension log 2/log 3) and then consider some random Cantor sets. The most mathematically satisfying definition is that of Hausdorff dimension which is related to a quantity known as Hausdorff measure.

4. Summation of series

Weeks 1–4: Madhav Nori

Abstract: Many interesting functions are obtained by summing series of complex numbers. These include trigonometric functions, elliptic functions, theta functions, Eisenstein series, the Riemann zeta function. Most of these will be discussed, and also how the Heisenberg group enters the picture.

5. Topics in geometry

Weeks 2 – 5: Benson Farb

We will study selected topics in geometry and/or topology. Topics might include: knot theory, mapping class groups (a subject at the intersection of low-dimensional topology, symplectic geometry, algebraic geometry, combinatorial group theory, ...), groups acting on trees, triangulations.

We will focus on interesting problems. Open problems accessible to undergraduates will also be given.

6. Billiards in polygons

Weeks 5–6: Howard Masur

Dynamical systems is an important subject in mathematics. One has a self mapping of a space or a 1-parameter family of mappings called a flow. One wishes to study the orbit of a point under the map or flow. An appealing example of a dynamical system is billiards in a domain in the plane. A particle moves at unit speed bouncing off the boundary with angle of reflection equal to the angle of incidence. A basic example is where the domain is a square. Analyzing this example we are led to the basic notion of ergodicity. We will prove Weyl's theorem of the ergodicity of the flow in irrational directions. We will look at other billiard tables to see similarities and difference to the dynamics in the square.

7. Algebraic and topological K-theory

Weeks 1, 3, 5–8: Peter May

Loosely speaking, I mean by K-theory anything that starts with the idea of constructing algebraic structures by grouping together Klasses (the German spelling) of objects of a given kind. The starting constructions are really very naive, a generalization of how one constructs the integers from the natural numbers or the rational numbers from the integers. I'll give many examples. One is the Burnside ring A(G). It is constructed from isomorphism Klasses of finite sets with an action of a given finite group G; sample calculations are fun. I'll also explain the construction of the algebraic K-group of a ring R and the ideal class group of a Dedekind ring. Perhaps I will prove the finiteness of the class group when R is a number ring. I also hope to explain in outline how topological K-theory works to prove that the only real division algebras are the real numbers, the complex numbers, the quaternions, and the octonions. In odd dimensions greater than 1 and in even dimensions greater than 8, there are no real division algebras. The essential point is the beautiful and surprising interaction between various branches of mathematics.