University of Chicago, REU 2009: K-Theory

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The first half of the notes for the 2009 REU were texed by Rolf Hoyer and Clair Tomesch and are posted separately. These notes deal with K-theory proper and are edits of notes texed in 2006 by Mohammed Abouzaid and Masoud Kamgarpour, with some added material texed by Clair. The lectures in 2009 did not cover the material in the same order as in 2006 and were less repetitive, but the notes retain the repetitions of the 2006 version. The goal is to show by use of a leisurely introduction to K-theory that there are no unital products without zero divisors on $\mathbb{R}^n$ unless $n = 1, 2, 4, 8$.

LECTURE 1

Tuesday, July 17, 2006 TEXed by Mohammed Abouzaid.

1. Vector bundles

First, we explain vector bundles. The idea is that we all know what vector spaces are. That's linear algebra. We also know about topological spaces. We would like to blend the two together. So we start with a topological space (say a subset of Euclidean space), and for each point in that space, we will have "over it" a vector space.

Say the topological space is B. We would like a surjective continuous map

$$E \xrightarrow{p} B$$

(E is the total space, B is the base space) with the property that for each point, the fibre $p^{-1}(b)$ has a fixed isomorphism to $\mathbb{R}^n$ or $\mathbb{C}^n$. That's not enough, but here's a picture.

Think of the Moebius strip. The Moebius strip will be E. There is a central circle. The map p in this case, is simply projection to that central circle.

On the other hand, we could also have just the product

$$S^1 \times \mathbb{R}.$$

These are examples of two different bundles over the same space.

We always have an example of a "trivial bundle"

$$B \times \mathbb{R}^n \to B.$$

In a sense, this is the most important example since we require that every bundle be locally trivial:

Definition 1.1. A **local trivialization** for

$$E \xrightarrow{p} B$$

is an open cover of B, $\mathcal{O} = \{U\}$, and for each U, a homeomorphism

$$\phi_u : U \times \mathbb{R}^n \hookrightarrow p^{-1}(U)$$

such that the diagram

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[rr, "\phi_U"] \arrow[dr, "\pi"] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

commutes.

Recall also, that we have a canonical identification of $p^{-1}(b)$ with $\mathbb{R}^n$ such that the the restriction of ϕ_U to $\{b\} \times \mathbb{R}^n$ is a linear map.

To simplify life, we will assume that B is compact.

DEFINITION 1.2. A topological space B is **compact** if every open cover has a finite subcover.

We would like our choice of local trivialization to be compatible. Consider the composition

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{\phi_U} p^{-1}(U \cap V) \xrightarrow{\phi_V^{-1}} (U \cap V) \times \mathbb{R}^n$$

Since all of these maps have to be isomorphisms of vector spaces on each fibre, we obtain a map

$$U \cap V \to GL_n(\mathbb{R}).$$

DEFINITION 1.3. A **vector bundle** is a surjective continuous map

$$E \xrightarrow{p} B$$

which is locally trivial for a cover $\mathcal{O}$ such that the corresponding maps

$$U \cap V \to GL_n(\mathbb{R})$$

are continuous.

Note that $GL_n(\mathbb{R})$ is a subset of the set of $n \times n$ matrices, and hence is topologized as a subset of $\mathbb{R}^{n^2}$.

Later, we will see an important theorem stating that every vector bundle embeds in a trivial bundle.

2. Tangent bundles

Given a manifold, we can study embeddings

$$M \hookrightarrow \mathbb{R}^q.$$

The Whitney embedding theorem guarantees that if M is n dimensional, we can choose q to be $2n + 1$.

For example, we can take the n-dimensional sphere

$$S^n \to \mathbb{R}^{n+1}.$$

In this case, at every x, we can study the set of v which are orthogonal to x. This is an n-dimensional vector space at every point, called the tangent space.

So have a subset of $S^n \times \mathbb{R}^{n+1}$ consisting of

$$\{(x, v) | v \perp x\}$$

which we call the tangent space of S^n. We write this as $\tau(S^n)$, which is a vector bundle over S^n.

We can also consider the set of points which are parallel to $x \in S^n$. We can do this for every manifold which embeds in $\mathbb{R}^n$ to obtain a tangent and normal bundle which both embed in the trivial bundle $M \times \mathbb{R}^q$. We can topologize them as subsets, and check that they satisfy the appropriate axioms.

One way of checking that these are bundles, is to use upper and lower hemispheres as the appropriate cover.

In this course, we're not going to study bundles one at a time. Rather, we will consider all bundles over a fixed base space, and study the appropriate structure.

First, we need

DEFINITION 1.4. An **isomorphism** of vector bundles E and E' over the same base B is a map

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"] & & E' \arrow[dl, "g'"] \\ & B & \end{tikzcd}$$

which is a linear isomorphism on each fibre.

It turns out that the inverse function is automatically continuous, and induces the inverse isomorphism of vector spaces when restricted to each fibre. We will only be looking at isomorphism classes of vector bundles. We will use ξ for a vector bundle, and $[\xi]$ for its class.

If B is connected, the dimension of the vector bundle is constant. However, if B is not connected, there are two different notions. Either we study n-plane bundles, in which the dimension is kept fixed, or, more generally, we can study vector bundles, where the dimension is allowed to vary in the different components. For simplicity, we will always assume connectivity of B.

Definition 1.5. $\mathcal{E}_n(B)$ is the set of equivalence classes of n-plane bundles over B.

One may complain that this may not be a set, but we will come to this set-theoretic point later.

Definition 1.6. $\mathrm{Vect}(B)$ is the set of isomorphism classes of vector bundles over B.

If B is path connected, then

$$\mathrm{Vect}(B) = \coprod_{n \geq 0} \mathcal{E}_n(B),$$

since every vector bundle has a fixed dimension.

Question: What are the isomorphism classes of 0-dimensional bundles?

Answer: There is only one such bundle, consisting of

$$\mathrm{id} : B \to B,$$

we call this $[\epsilon_0]$.

This should be thought of as the analogue of the trivial vector space whose only element is 0. This is a general principle. Anything you can do to vector spaces, I can do to vector bundles.

For example, given ξ and χ, let us write $\xi_b = \xi^{-1}(b)$ for the fibre at a point b. This is standard notation where the projection is given the same name as the vector bundle. We can now define the Whitney sum of ξ and χ to be be the vector bundle $\xi \oplus \chi$ whose fibre is given by

$$(\xi \oplus \chi)_b = \xi_b \oplus \chi_b.$$

We will come back later to why this operation is well defined. But we can check that this operation is compatible with isomorphism classes. Further, the 0-dimensional bundle acts as the zero for this operation. The result is therefore an abelian monoid, modulo the details we haven't checked.

We can now apply the Grothendieck construction to this monoid, and obtain

$$K(B)$$

which is the Grothendieck group of B. When we write $K(B)$, we mean the Grothendieck group of complex vector bundles over B, which we sometimes also write $KU(B)$.

There is also a Grothendieck group of real vector bundles

$$KO(B).$$

But we need more than a group. Rather, we would like to have a ring. First, we need to discuss tensor products.

3. Tensor Products; a review

Let R be a commutative ring, in particular, it could be a field, which is the case we're interested in. Suppose we have M, N two R modules (commutativity implies that we don't have to worry about left and right modules). In other words, we have abelian groups together with a multiplication

$$R \times M \to M$$

which is associative and bilinear. In general, a bilinear map

$$M \times N \xrightarrow{f} P$$

is a map which is linear in both variables separately.

$$\begin{aligned} f(m,n) + f(m',n) &= f(m+m',n) \\ f(rm,n) &= rf(m,n). \end{aligned}$$

We will now define the tensor product $M \otimes_R N$ via its universal property:

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "i"] & P . \\ M \otimes_R N \arrow[ur, dotted, "\tilde{f}"] & \end{tikzcd}$$

In order to check that this makes sense, we must provide a construction.

Let F be the free R module on the set $M \times N$. Elements of F are formal sums $\sum r_i(m_i, n_i)$ where all but finitely many entries in the sum vanish. We define

$$M \otimes_R N = F/\sim$$

where the equivalence relation is generated by

$$\begin{aligned} (m,n) + (m',n) - (m+m',n) &= 0 \\ (rm,n) - r(m,n) &= 0 \\ (m,n) + (m,n') - (m,n+n') &= 0 \\ (m,rn) - r(m,n) &= 0. \end{aligned}$$

Note the fact that our relations are symmetric in M and N.

Example 1.7. Let V and W be finite dimensional vector spaces of $\mathbb{F}$. In order to be concrete, we choose bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$

of V and W respectively. Consider a vector space $V \otimes W$ spanned by a basis $\{v_i \otimes w_j\}_{1 \leq i \leq m}^{1 \leq j \leq n}$ and define a map

$$\begin{tikzcd} (V \times W) \arrow[d, "\iota"] \\ V \otimes W \end{tikzcd}$$

where

$$\iota(v_i, w_j) = v_i \otimes w_j$$

on basis elements, and is extended by bilinearity to every element of $V \times W$. Given any bilinear function to P, we can construct a linear map $\tilde{f}$ such that the diagram

$$\begin{tikzcd} (V, W) \arrow[r, "f"] \arrow[d, "\iota"] & P \\ V \otimes W \arrow[ur, dotted, "\tilde{f}"] \end{tikzcd}$$

commutes.

This proves the existence (and uniqueness) of the tensor product of vector spaces.

We have a distributivity property

$$(V \oplus V') \otimes W \cong (V \otimes W) \oplus (V' \oplus W),$$

which can be proved either by the universal property, or by the construction.

Note that

$$0 \otimes W = 0$$

while

$$\mathbb{F} \otimes W \cong W.$$

So direct sum looks like addition, and tensor product looks like multiplication. Formally, we can say that the set of isomorphism classes of vector spaces over a fixed field is a semi ring under these two operations, however, this is a boring object, since isomorphism classes of vector spaces are determined by their dimension.

However, if we start with an arbitrary ring, and we restrict to good (projective) finitely generated modules, we obtain

$$K_0(R)$$

the K-theory group of the ring R. The subscript is referring to the existence of higher K-theory groups. This is in fact the beginning of the rich subject of algebraic K-theory.

4. Topological K-theory

We now repeat the same procedure with vector bundles. First, we define the tensor product of vector bundles. Given ξ and ξ', we define $\xi \otimes \xi'$ to be the new vector bundle whose fibre is

$$(\xi \otimes \xi')_b = \xi_b \otimes \xi'_b.$$

So $\mathrm{Vect}(B)$ is a semi ring under Whitney sum and tensor product with

$$0 = [\epsilon_0]$$
$$1 = [\epsilon_1].$$

where

$$\epsilon_1 : B \times \mathbb{R} \to B,$$

is just given by projection to the first factor.

Unfortunately, we have been keeping B fixed this whole time. We need an aside

Remark 1.8 (Aside on Categories). A category is a field of mathematics. It has objects, say, spaces or groups, and maps between a pair of objects. We write X, Y, $\cdots$ for the objects, and $\mathcal{C}(X,Y)$ for the set of maps between these objects.

We have composition maps

$$\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

which satisfy associativity, and we have chosen identities

$$\mathrm{id}_X \in \mathcal{C}(X,X).$$

Now a functor acts on categories

$$F : \mathcal{C} \to \mathcal{D}$$

assigning an object FX of $\mathcal{D}$ to every object C of $\mathcal{C}$, and acts on morphisms in one of two different ways. A functor can either be covariant

$$F : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY),$$

or contravariant

$$F : \mathcal{C}(X,Y) \to \mathcal{D}(FY, FX).$$

We would like to say that K is a contravariant functor from topological spaces to groups.

Given a continuous map

$$f : B' \to B$$

we will construct a map of rings

$$K(f) : K(B) \to K(B').$$

Further, as in the proof of fundamental theorem of algebra, we would like for this construction be to homotopy invariant

$$f \cong g \Rightarrow K(f) = K(g).$$

Given a vector bundle over B, and a map $f : B' \to B$, we will construct the pullback of E. Roughly speaking,

$$E' = \{(b', e) | f(b') = p(e)\} \subset B' \times E.$$

We claim that this bundle fits in a commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B' & \xrightarrow{f} & B \end{array}.$$

LECTURE 2

Wednesday, July 19, 2006 [1]

Recall that we defined vector bundles to be surjective maps

$$E \to B$$

whose fibres

$$E_b \equiv p^{-1}(b)$$

are identified with either $\mathbb{R}^n$ or $\mathbb{C}^n$. Further, we require a local triviality condition relative to a cover $\mathcal{O}$ of B.

We saw examples such as the Moebius strip, and the tangent and normal bundles of S^n. However, our goal is to consider

$$\mathcal{E}_n^O(B),$$

and

$$\mathcal{E}_n^U(B),$$

the sets of equivalence classes of (respectively) real and complex vector bundles over B. We also introduced

$$\mathrm{Vect}(B),$$

the set of equivalence classes of bundles of arbitrary (finite) dimension. $\mathrm{Vect}(B)$ is closed under direct sums and tensor products, and is in fact a semi-ring. We defined

$$K(B) = KU(B)$$

to be the Grothendieck group of this semi-ring of complex vector bundles, and

$$KO(B)$$

to be the Grothendieck group of real vector bundles.

From the point of view of mathematics in general, KU is more important, but for an algebraic topologist, KO is a richer object.

We also introduced the notion of categories. One important convention is that we will write an element of $\mathcal{C}(X, Y)$, a map from X to Y, as an arrow

$$X \to Y.$$

[1] TEXed by Mohammed Abouzaid.

1. A Category of vector bundles

Consider the category Vect whose objects are vector bundles over arbitrary spaces. We do not take the quotient by equivalence.

A morphism between the vector bundles p and p' is a commutative diagram of continuous maps

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B'. \end{tikzcd}$$

There are two choices here. We can either require that g be a linear map on each fibre, or, more restrictively, that g induce an isomorphism on each fibre.

The composition of morphisms

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"] & E' \arrow[r, "g'"] \arrow[d, "p'"] & E'' \arrow[d, "p''"] \\ B \arrow[r, "f"] & B' \arrow[r, "f'"] & B'', \end{tikzcd}$$

is simply given by composing g with g' and f with f'

$$\begin{tikzcd} E \arrow[r, "g' \circ g"] \arrow[d, "p"] & E'' \arrow[d, "p''"] \\ B \arrow[r, "f' \circ f"] & B''. \end{tikzcd}$$

In any category, we can formulate the notion of Cartesian product. Given X and Y objects, the Cartesian product is an object $X \times Y$ together with maps to X and Y satisfying the universal property

$$\begin{tikzcd} X & X \times Y \arrow[l] \arrow[r] & Y \\ & Z. \arrow[ul] \arrow[u, dotted] \arrow[ur] & \end{tikzcd}$$

Not every category has Cartesian products. However, since Cartesian products exist for sets, it is sometimes possible to construct a Cartesian product by first taking the Cartesian product of sets, then equipping it with the appropriate structure.

We claim that Vect is a category with Cartesian products. Indeed, we can define the Cartesian product of

$$\begin{tikzcd} D \arrow[d, "p"] & E \arrow[d, "q"] \\ A & B \end{tikzcd}$$

to be the map

$$\begin{tikzcd} D \times E \arrow[d, "p\times q"] \\ A \times B, \end{tikzcd}$$

where the linear structure on the fibres is given by the product of the linear structures, and local triviality follows by taking the cover of $A \times B$ which consists of products of the open sets of A and B which are used to prove local triviality of p and q.

Note that in order for this object to be a Cartesian product, we must allow the maps of vector bundles

$$\begin{tikzcd} D \arrow[d, "p"] & D \times E \arrow[l] \arrow[r] \arrow[d, "p\times q"] & E \arrow[d, "q"] \\ A & A \times B \arrow[l] \arrow[r] & B \end{tikzcd}$$

which are NOT isomorphisms on each fibre. This is a general principle that requires us to enlarge our categories in order to be able to obtain a richer structure. Note that we must verify that the universal property is satisfied for this construction.

We now define pullback in an arbitrary category. Given two maps p and f in any category, we define their pullback P to be the object equipped with maps to E and A satisfying the a universal property

$$\begin{tikzcd} Q \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ & A \arrow[r, "f"] & B. \end{tikzcd}$$

If we let B be a point, then P is the Cartesian product. In the category of sets, we can map B to a point, and consider the diagram

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr, dotted] \arrow[dd] & & E \arrow[dd, "p"] \arrow[dddr] & \\
& A \times E \arrow[ur] \arrow[dl] & & \\
A \arrow[rr, "f"] \arrow[drrr] & & B & \\
& & & \star,
\end{tikzcd}$$

in particular, the universal property of the Cartesian product implies that P is equipped with a map to $A \times E$. In the category of sets, we can in fact define the pullback to be

$$\{(a, e) | f(a) = p(e)\} \subset (A, E).$$

We must now prove the existence of pullbacks in Vect. Given a vector bundle p and a map of topological spaces f, we define $f^*(E)$ to be the pullback of topological spaces

$$\begin{tikzcd}
f^*(E) \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd} .$$

Lemma 2.1. *f^*E is a vector bundle.*

Proof. Note that

$$q^{-1}(a) = \{(a, e) | f(a) = p(e)\} = p^{-1}(f(a)).$$

So, in particular, the fibres of q are vector spaces. As to local triviality, we consider a cover $\mathcal{O} = \{U\}$ of open sets $U \subset B$. Now $f^{-1}(U)$ is an open set in A, and we can construct a trivialization

$$\begin{tikzcd}
f^{-1}(U) \times \mathbb{R}^n \arrow[rr] \arrow[dr] & & q^{-1} f^{-1}(U) \arrow[dl] \\
& f^{-1}(U) &
\end{tikzcd}$$

by using the trivialization of p. $\square$

Note that this means that given a map f from A to B, we obtain a well-defined map

$$\mathcal{E}_n(B) \xrightarrow{f^*} \mathcal{E}_n(A)$$

once we check that an isomorphism of vector bundles over B gives an isomorphism of their pullbacks over A.

Let $\mathcal{U}$ be the category of topological spaces (secretly, compact and connected). And for each n, we have a functor $\mathcal{E}_n$

$$\mathcal{U} \xrightarrow{\mathcal{E}_n} \text{Sets} .$$

In particular, we are assigning not only a set $\mathcal{E}_n(B)$ to every topological spaces B, but also a map of sets

$$f^*\colon\ \mathcal{E}_n(B) \to \mathcal{E}_n(A)$$

to every map

$$f\colon\ A \to B.$$

Note that the order was reversed in this procedure. Further, we must prove that the functor preserves composition (although it reverses the order in which it's taken).

We can now make our construction of Whitney sums rigorous. Indeed, given two bundles p and q over A and B,

$$\begin{tikzcd} D \arrow[d, "p"] & E \arrow[d, "q"] \\ A & B \end{tikzcd}$$

the Cartesian product

$$\begin{tikzcd} D \times E \arrow[d, "p\times q"] \\ A \times B, \end{tikzcd}$$

can be thought of as an **external** direct sum of vector bundles. If our bundles are bundles over the same base space we simply pull back this direct sum over the diagonal

$$\begin{tikzcd} D \oplus E \arrow[r] \arrow[d] & D \times E \arrow[d, "p\times q"] \\ B \arrow[r, "\Delta"] & B \times B. \end{tikzcd}$$

This define the Whitney sum rigorously. We can define tensor products in the same way, by first defining an **external tensor product** of vector bundles, then pulling back this construction along the diagonal. In general, to check continuity, it is easiest to think of vector bundles as a set of continuous maps

$$U \cap V \xrightarrow{\phi_{U\cap V}} GL_n(R)$$

satisfying an appropriate co-cycle condition. Any continuous operation on $GL_n(R)$ yields a natural construction on vector bundles.

Recall that if B is connected,

$$\mathrm{Vect}(B) = \coprod_{n\geq 0} \mathcal{E}_n(B).$$

Note that we have a set valued functor $\mathcal{E}_n$ for each n. By taking their "disjoint union" we obtain a set valued functor

$$\mathcal{U} \xrightarrow{\mathrm{Vect}} \mathrm{Sets}\ .$$

Now, $\mathrm{Vect}(B)$ had the structure of a semi ring. In fact, given a map $f\colon A \to B$, its pullback

$$f^*\colon\ \mathrm{Vect}(B) \to \mathrm{Vect}(A)$$

is a map of semi-rings. This entails checking that

$$f^*(p \oplus q) \cong f^*(p) \oplus f^*(q)$$

which can be checked easily at the level of fibres. The same should be done for the tensor product.

We also have the Grothendieck construction which is a functor from semi-rings to rings. Indeed, given ϕ, a map of semi-rings, there exists a unique map $\tilde{\phi}$ making the diagram

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "i"] & S \arrow[d, "j"] \\ K(R) \arrow[r, "\tilde{\phi}"] & K(S) \end{tikzcd}$$

commute. The existence (and uniqueness) of $\tilde{\phi}$ is simply a consequence of the universal property of $K(R)$. In particular, if A and B are spaces, and

$$f\colon\ A \to B$$

is a continuous map, we obtain a map of rings,

$$K(B) \xrightarrow{K(f)} K(A)$$

in particular, K is a functor from spaces to rings.

Theorem 2.2. *If $f \simeq f'\colon\ A \to B$ then*
- $\mathcal{E}_n(f) = \mathcal{E}_n(f')$
- $\mathrm{Vect}(f) = Vect(f')$
- $K(f) = K(f')$.

This is a surprising (though not difficult to prove) fact. The conclusion is the existence of isomorphisms of vector bundles, which is a rigid notion, while the assumption, that of the existence of a homotopy, is a much weaker condition. As a consequence, we will be able to prove important results in topology. For example, Milnor's proof that there are different smooth structures on the n-sphere used cobordism theory, which is closely related to K-theory. Indeed, the classification of such structures, uses the homotopy groups of spheres, and relies on constructions analogous to K-theory.

Recall that a homotopy from f to f' is simply a map

$$h\colon\ A \times I \to B$$

such that $h(a,0) = f$ and $h(a,1) = f'$.

Proof of Theorem 2.2. The idea is to look at $h^*(E)$, which is a bundle over $B \times I$.

Claim 2.3. For any bundle E' over $A \times I$, there exists a map g of vector bundles

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d] & E' \arrow[d] \\ A \times I \arrow[r, "r"] & A \times I, \end{tikzcd}$$

where $r(b,t) = (bx, 1)$.

Let us assume the claim holds. If we consider the pull-back, we obtain a diagram

$$\begin{tikzcd} E' \arrow[rr, "g"] \arrow[dd] \arrow[dr, dotted] & & E' \arrow[dd] \\ & r^*(E') \arrow[ur] \arrow[dl] & \\ A \times I \arrow[rr, "r"] & & A \times I. \end{tikzcd}$$

Now, simply because these are maps of bundles, we can study the fibres, and conclude that the map

$$E' \to r^*E'$$

is an isomorphism.

Let us consider the case where $E' = h^*E$; i.e. it is pulled back from the homotopy. In this case,

$$h|_{A\times\{1\}} = f'$$

so

$$r^*E' = (f')^*E \times I \to A \times I.$$

In particular, the factor I does not enter in the construction of r^*E'. But the claim implies that h^*E is isomorphic to this vector bundle. Since the same property hold for the restriction to $A \times \{0\}$, we conclude that

$$f^*(E) \cong f'^*(E).$$

□

Preview: On Friday, we will construct spaces $BO(n)$ and $BU(n)$, and we will show that for real vector bundles,

$$\mathcal{E}_n \cong [B, BO(n)]$$

where the right hand side refers to homotopy classes of maps from B to $BO(n)$. An analogous statement holds for complex vector bundles.

LECTURE 3

Friday, July 21, 2006 [1]

We will eventually say a bit about the proof of the Bott Periodicity theorem

$$K(X \times S^2) \cong K(X) \otimes K(S^2).$$

Using the fact that a vector bundle over a point is just a vector space, and vector spaces are determined up to isomorphism by their dimension, we see that

$$K(\star) = \mathbb{Z}.$$

If we choose a point $\star \in X$, the inclusion induces

$$K(X) \xrightarrow{\epsilon} K(\star)$$

which is just recording the dimension of the fibre at $\star$. We call the kernel of ϵ the **reduced** K-theory $\tilde{K}(X)$.

It is a crucial fact that

$$\tilde{K}(S^2) \cong \mathbb{Z}.$$

This is used in the proof of Bott periodicity, of which it is also an immediate consequence, as we see by taking $X = *$. It can be proved concretely as is done in Atiyah's book, by using the cover of S^2 by the northern and southern hemisphere. We can also use a more-homotopy theoretic approach.

Consider X and Y two topological spaces. Consider the set

$$\mathrm{Maps}(X, Y)$$

of continuous maps from X to Y. We can partition this set into equivalence classes of **homotopic maps**, where

$$f \simeq g$$

if there exists a map

$$h\colon\ X \times I \to Y$$

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such that

$$h \circ i_0 = f$$
$$h \circ i_1 = g,$$

where $i_0(x) = (x, 0)$, and $i_1(x) = (x, 1)$. One should check that this is an equivalence relation by proving transitivity, reflexivity, etc.

We write $[X, Y]$ for the quotient of the set of maps by this equivalence relation.

This allows us to define the homotopy category of spaces, whose

- Objects are topological spaces,
- Morphisms are homotopy classes of maps, i.e the set of morphisms from X to Y is $[X, Y]$.
- Composition is given by choosing representatives of our homotopy classes, then composing them as ordinary maps of spaces, and finally taking the homotopy class of the composite. One should check that this is independent of the choice of representatives.

We are about to prove that

$$\mathcal{E}_n^{\mathbb{R}}(X),$$

the set of isomorphism classes of real n-plane bundles over X, can be identified with

$$[X, BO(n)],$$

where $BO(n)$ is a space we are about to construct. A stronger fact is that the functors $\mathcal{E}_n$ and $[\_, BO(n)]$ are naturally isomorphic functors.

Given two contravariant functors

$$F, G\colon \mathcal{C} \to \mathcal{D}$$

a natural transformation

$$\eta : F \to G$$

is a " map between these two functors." More precisely, for every object X of $\mathcal{C}$, we have a map

$$\eta_X\colon F(X) \to G(X)$$

such that for every morphism $f\colon X \to Y$, the diagram

$$\begin{tikzcd} F(Y) \arrow[r, "F(f)"] \arrow[d, "\eta_Y"] & F(X) \arrow[d, "\eta_X"] \\ G(Y) \arrow[r, "G(f)"] & G(X) \end{tikzcd},$$

commutes.

So if we let $G(X) = \mathcal{E}_n^{\mathbb{R}}(X)$, and $F(X) = [X, BO(n)]$, we obtain contravariant functors from the homotopy category of spaces to sets. To say that $\mathcal{E}_n^{\mathbb{R}}$ is a functor means that given a continuous map

$$f\colon\ X \to Y,$$

then we can pull-back bundles from Y to X

$$\begin{tikzcd}
f^*(E) \arrow[r, "g"] \arrow[d] & E \arrow[d, "p"] \\
X \arrow[r, "f"] & Y
\end{tikzcd} .$$

and hence define a map

$$\begin{aligned}\mathcal{E}_n(f)\colon\ \mathcal{E}_n(Y) &\to \mathcal{E}_n(X)\\ [E] &\mapsto [f^*E].\end{aligned}$$

We are now ready to define $BO(n)$. Recall that an inner product on a real vector space V is a bilinear, symmetric, non-degenerate pairing. Recall that two vectors are orthogonal if their inner product vanishes. A subset $\{b_i\}$ of V is said to be orthonormal if

$$\langle b_i, b_j\rangle = \delta_{i,j}.$$

In other words, inner products among the vectors b_i and b_j vanish unless $i = j$, in which case the inner product is equal to one.

Let

$$V_n(\mathbb{R}^q) \subset \mathbb{R}^{nq}$$

denote the set of n-tuples of orthonormal vectors in $\mathbb{R}^q$. In particular, $V_n(\mathbb{R}^q)$ inherits a natural topology from the above inclusion. This is the Stiefel manifold (variety). Note that $V_n(\mathbb{R}^q)$ is empty if $n > q$.

Given an n-tuple of orthonormal vectors, we obtain a basis for an n dimensional subspace of $\mathbb{R}^q$ which is simply their span. In particular

$$V_2(\mathbb{R}^3)$$

is simply the set of possible orthonormal bases for planes in $\mathbb{R}^3$. Let us define

$$G_n(\mathbb{R}^q)$$

to be the set of all n-planes in $\mathbb{R}^q$. When $n = 1$, we simply recover the projective space $\mathbb{RP}^{q-1}$ of lines in $\mathbb{R}^q$. We have a map

$$V_n(\mathbb{R}^q) \xrightarrow{\pi} G_n(\mathbb{R}^q)$$

which takes every orthonormal set to the plane that it spans. We can specify the quotient topology on $G_n(\mathbb{R}^q)$ which makes this map continuous. With this topology, we call $G_n(\mathbb{R}^q)$ the Grassmannian, or the Grassmann manifold.

On $G_n(\mathbb{R}^q)$, we have a natural subundle of the trivial bundle

$$\begin{tikzcd} G_n(\mathbb{R}^q) \times \mathbb{R}^q \arrow[d] \\ G_n(\mathbb{R}^q). \end{tikzcd}$$

which as a set is given by

$$E(\gamma_n^q) = \{(x, v) | v \in x\}.$$

Let γ_n^q denote the projection

$$\begin{aligned} E(\gamma_n^q) &\to G_n(\mathbb{R}^q) \\ (x, v) &\mapsto x. \end{aligned}$$

From the point of view of n-plane bundles, the choice of q is arbitrary. However, if we embed $\mathbb{R}^q$ into $\mathbb{R}^{q+1}$, we obtain a map

$$G_n(\mathbb{R}^q) \to G_n(\mathbb{R}^{q+1})$$

letting q go to infinity, we obtain a space

$$BO(n) \equiv G_n(\mathbb{R}^\infty).$$

The right hand side is topologized as an increasing union. This is not the topology that an analyst would give to $\mathbb{R}^\infty$. Note that in fact, we only care about the **homotopy type** of the space $BO(n)$.

Let us now explain why $F(X) = [X, BO(n)]$ is a functor. This is in fact a general fact.

LEMMA 3.1. *If $\mathcal{C}$ is any category, and Y is an object of $\mathcal{C}$, then*

$$F(X) \equiv \mathcal{C}(X, Y)$$

is a functor from $\mathcal{C}$ to Sets*.*

PROOF. Given

$$f\colon X \to X',$$

precomposition with f yields a map

$$\mathcal{C}(X', Y) \to \mathcal{C}(X, Y).$$

Associativity of composition of morphisms in $\mathcal{C}$ yields the desired properties of F. $\square$

In general, functors of this type are called **representable**. The idea is that Y represents the functor F. In our case, Y will be $BO(n)$.

Note that the bundles $E(\gamma_n^q)$ glue together to give a bundle

$$EO(n) \xrightarrow{\gamma_n} BO(n)$$

such that the fibre over every point of $BO(n)$ (which represents an n-plane in $\mathbb{R}^\infty$), is simply that n-dimensional vector space.

We define a natural transformation

$$\begin{aligned} \Phi\colon\ [X, BO(n)] &\to \mathcal{E}_n(X) \\ [g\colon\ X \to BO(n)] &\mapsto [g^*(\gamma_n)]. \end{aligned}$$

Recall that $[g^*(\gamma_n)]$ is the isomorphism class of the pull-back of the universal bundle $EO(n)$ over $BO(n)$. This idea of representing functors is a central idea of mathematics.

Now, given any map

$$f\colon\ X \to X',$$

we obtain a commutative diagram

$$\begin{tikzcd} {[X', BO(n)]} \arrow[r, "f^* \equiv [f, \mathrm{id}]"] \arrow[d, "\Phi_{X'}"] & {[X, BO(n)]} \arrow[d, "\Phi_X"] \\ \mathcal{E}_n(X') \arrow[r, "f^* \equiv \mathcal{E}_n(f)"] & \mathcal{E}_n(X). \end{tikzcd}$$

The commutativity of this diagram states that pulling back $EO(n)$ to X' using a map g, then pulling that back to X using f, yields a bundle which is isomorphic to the pullback of $EO(n)$ by the composite $f \circ g$. This establishes the fact that Φ is a natural transformation. In fact, we are simply using the general fact that

$$\begin{tikzcd} (f^*(g^*E) \cong (g \circ f)^*E \arrow[rr] \arrow[d] & & E \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z, \end{tikzcd}$$

which can be checked from the construction of the two bundles on the top left corner, or by the universal property of pullbacks.

This natural isomorphism of functors gives us two different methods for studying vector bundles. It remains to prove three things.

- First, if $f \simeq g\colon\ X \to Y$ and $p\colon\ E \to Y$ is an n-plane bundle, then $f^*E \cong g^*E$.
- Next, Φ is surjective.
- Lastly, Φ is injective.

This will complete the proof that

$$\Phi\colon\ [X, BO(n)] \to \mathcal{E}_n(X)$$

is a natural isomorphism. We begin by proving surjectivity.

Proof of Surjectivity. The idea is to use a variant of the Whitney embedding theorem. Say M is a manifold embedded in $\mathbb{R}^q$, e.g. the sphere S^{q-1}. We can translate the tangent plane at each point to a plane at the origin, which gives a map

$$\begin{array}{ccc} \tau(M) & \longrightarrow & E(\gamma_n^q) \\ \downarrow & & \downarrow \\ M & \longrightarrow & G_n(\mathbb{R}^q). \end{array}$$

which is called the Gauss map of the tangent bundle. If we take every point to its normal plane, we obtain the Gauss map of the normal bundle. The idea is to generalize this to arbitrary bundles.

For simplicity, let us assume that our space is compact. This allows us to work with finite covers. Suppose we have such a bundle E over X. Let $\mathcal{O} = \{U_1, \ldots, U_m\}$ denote the cover over which we have trivializations

$$\begin{tikzcd} U_i \times \mathbb{R}^n \arrow[rr] \arrow[dr] & & p^{-1}(U_i) \arrow[dl] \\ & U_i & \end{tikzcd}.$$

Now, recall that $E(\gamma_n^q) \subset G_n(\mathbb{R}^q) \times \mathbb{R}^q$. So we would like a continuous map of the total space

$$\hat{g}\colon\ E \to \mathbb{R}^q$$

such that the restriction of $\hat{g}$ to every fibre is an injective linear map. Using the planes given by the images of the fibres, this would define a map $X \to G_n(\mathbb{R}^q)$, and, by construction, we would have a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E(\gamma_n^q) \\ \downarrow & & \downarrow \\ X & \longrightarrow & G_n(\mathbb{R}^q), \end{array}$$

having the property that the top map is an isomorphism on fibres, which establishes the fact that E is isomorphic to the pullback of the universal bundle.

We need one fact from point set topology.

LEMMA 3.2 (Urysohn's Lemma). *Under appropriate conditions on the topological space B, there exists a map*

$$\lambda_i : B \to I$$

such that $\lambda_i^{-1}((0,1]) = U_i$.

Let $q = m \times n$, i.e: the size of the fibre times the number of open sets in the cover. We think of $\mathbb{R}^q$ as

$$\mathbb{R}^n \oplus \mathbb{R}^n \cdots \oplus \mathbb{R}^n$$

with the ith factor corresponding to U_i. We now define $\hat{g}$ by

$$g = (g_1, \ldots, g_m) \colon\ E \to \mathbb{R}^n \oplus \mathbb{R}^n \cdots \oplus \mathbb{R}^n$$

where

$$g_i(e) = \lambda_i(p(e)) \cdot \pi_2(\phi_{U_i}^{-1}(e))$$

where π_2 is the projection onto the $\mathbb{R}^n$ factor of

$$U_i \times \mathbb{R}^n.$$

Note that g_i is clearly a monomorphism on each fibre in $p^{-1}(U_i)$ since it simply identifies every fibre with the corresponding copy of $\mathbb{R}^n$ coming from the trivialization ϕ_{U_i}. Since every point lies in some U_i, we conclude that g is globally a monomorphism on fibres. This completes the proof of surjectivity of Φ. $\square$

PROOF OF INJECTIVITY. We must prove that $\Phi[f] = \Phi[f']$ implies that f and f' are homotopic. Let

$$E = f^*(\gamma_n^q) \cong f'^*(\gamma_n^q) = E'.$$

We have an isomorphism in the diagram

$$\begin{tikzcd}
E' \arrow[r, "\alpha"] \arrow[dr] & E \arrow[r] \arrow[d] & E(\gamma_n^q) \arrow[d] \\
& X \arrow[r] & G_n(\mathbb{R}^q),
\end{tikzcd}$$

and we would like to claim that f and f' are homotopic. But we know that the isomorphism class of the pullback is determined by the Gauss maps. In fact, if the Gauss maps of E and E' were linearly independent at each point, we could simply use the homotopy

$$\hat{h}(e,t) = t\hat{g}(e) + (1-t)\hat{g}'(e),$$

which would construct the desired homotopy between f and f'.

In order to achieve the hypothesis that the Gauss maps of E and E' have linearly independent images, we use two maps

$$\begin{aligned} \alpha\colon\ \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ e_q &\mapsto e_{2q} \\ \beta\colon\ \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ e_q &\mapsto e_{2q+1}, \end{aligned}$$

which are both clearly homotopic to the identity. Note that regardless of what $\hat{g}$ and $\hat{g}'$ are, the maps $\alpha\hat{g}$ and $\beta\hat{g}'$ are necessarily linearly independent. We can therefore use the above linear homotopy, and conclude that f and f' are homotopic. $\square$

LECTURE 4

Monday, July 24, 2006[1]

1. Pullback and bundles over $B \times I$

Suppose $f : X \to Y$ is a continuous map. This induces a map $f^* : \mathcal{E}_n(X) \to \mathcal{E}_n(Y)$, where $\mathcal{E}_n(X)$ denotes the set of isomorphism classes of n-vector bundles on X. Our goal is to finish the proof of the following theorem.

THEOREM 4.1. *Homotopic maps induce isomorphic pullbacks of vector bundles.*

The essential point is to complete the proof of Claim 2.3. We need some preliminary lemmas.

LEMMA 4.2. *Set $p : E \to U \times [a, c]$ with $a < b < c$. If $p|_{U\times[a,b]}$ and $p|_{U\times[b,c]}$ are trivial, then p is trivial.*

PROOF. The idea is to glue the two trivializations for the subbundles together to explicitly construct a trivialization for the whole bundle. $\square$

LEMMA 4.3. *There exists a finite open cover $\{U_1, ..., U_n\}$ of B such that $p|_{U_i\times I}$ is trivial for $1 \leq i \leq n$.*

PROOF. Use the compactness of B and an inductive argument using the previous lemma. $\square$

PROPOSITION 4.4. *Suppose $p : E \to B \times I$ is an n-plane bundle, where B is compact. Let $r : B \times I \to B \times I$ be defined by $r(b, t) = r(b, 1)$. Then there exists a map of bundles $g : E \to E$ such that the following diagram commutes.*

[1] These notes were taken and TEXed by Masoud Kamgarpour.

$$\begin{tikzcd}
E \arrow[dr, dotted, "\cong"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right] & & \\
& r^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
& B \times I \arrow[r, "r"'] & B \times I
\end{tikzcd}$$

Thus, $E \cong r^*E$.

PROOF. Observe that there exists $\lambda_i : B \to I$ such that $\lambda_i^{-1}(0,1] = U_i$. Let $\nu_i : B \to I$ be defined by $\nu_i(b) = \lambda_i(b)/max\{\lambda_1(b), ..., \lambda_m(b)\}$. Then we have $max_i\,\{v_i(b)\} = 1$. Let

$$r_i(b,t) := (b,\, max(\nu_i b, t)), \qquad g_i(\phi_i(b,t,v) := \phi_i(b, r_i(b,t), v)$$

Then g_i and r_i fit into the following commutative diagram.

$$\begin{tikzcd}
E \arrow[r, "g_i"] \arrow[d] & E \arrow[d, "p"] \\
B \times I \arrow[r, "r_i"'] & B \times I
\end{tikzcd}$$

Then $r_m \circ ... \circ r_1 = r$ and we can define $g_m \circ ... \circ g_1 = g$. □

PROOF. (of Theorem 4.1) Suppose f_0 and f_1 are maps $A \to B$, which are homotopic via the homotopy h. Write $f_0 = h \circ i_0$ and $f_1 = h \circ i_1 : A \to B$. Now by functoriality we have:

$$f_0^*E = (h \circ i_0)^*E = i_0^*h^*E \cong i_0^*r^*h^*E \cong i_1^*h^*E \cong f_1^*E$$

□

2. Stably Equivalent Bundles

Write $BO(n)$ as the union $\cup_q G_n(\mathbb{R}^n \oplus \mathbb{R}^q)$. Let $i_n : BO(n) \to BO(n+1)$ be the canonical map. (Geometrically this corresponds to adding ϵ, the trivial one dimensional bundle).

DEFINITION 4.5. Two vector bundles E and D on X are said to be **stably equivalent** if there exists an isomorphism $D \oplus \epsilon^m \cong E \oplus \epsilon^n$ for some non-negative integers m and n. Let $\mathcal{E}_{st}(X)$ be the set of stable equivalence classes of vector bundles.

Note that if we want to consider based maps, then we add a disjoint base point to X, to make it a based space X_+. In this case, the homotopies and the maps we are considering will also be based. Next let $BO = \cup_n BO(n)$. We have:

THEOREM 4.6. *If X is compact, $[X_+, BO] \cong \mathcal{E}_{st}(X)$.*

PROOF. Because X is compact, the image of any map from X to BO lands in some $BO(n)$. Now use $[X_+, BO(n)] \cong \mathcal{E}_n(X)$. $\square$

PROPOSITION 4.7. *Let X be a compact space. Then for any bundle E, there exists a bundle D such that $E \oplus D$ is trivial.*

PROOF. The idea is to use a Gauss map to construct an orthogonal complement. (See the section on the sum of tangent bundle and normal bundle) $\square$

COROLLARY 4.8. *Every element $\zeta - \nu$ of $K(X)$ can be written of the form $\alpha - q.\epsilon$ for some integer q and some $\alpha \in \mathcal{E}_{st}(X)$.*

COROLLARY 4.9. *Let X be connected and compact. Then, $\mathcal{E}_{st}(X)$ is naturally isomorphic to $\tilde{K}(X)$.*

PROOF. Let $\zeta - q$ be an element of $\tilde{K}(X)$. Then $dim\zeta = q$. Define a map $\tilde{K}(X) \to \mathcal{E}_{st}(X)$ by $\zeta - dim\zeta \mapsto [\zeta]$. It is to see that this is an isomorphism. $\square$

COROLLARY 4.10. *Under the same assumptions on X we have:*

(1) $[X_+, BO \times \mathbb{Z}] \cong KO(X)$.
(2) $[X_+, BU \times \mathbb{Z}] \cong KU(X)$.

It is not hard to show that $\tilde{K}(S^2) \cong \mathbb{Z}$. This boils down to the fact that $\pi_2(BU) = \pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$. A surprising and fundamental fact is the following theorem:

THEOREM 4.11. **(Bott Periodicity)** *The canonical map*

$$\otimes : K(X) \otimes K(S^2) \to K(X \times S^2)$$

is an isomorphism.

LECTURE 5

Thursday, July 23, 2009

(These notes fill in the proof of Proposition 4.7 and the deduction of Corollary 4.8 in the notes from 2006.)

PROPOSITION 5.1 (4.7). *Let X be a compact space. Then for any bundle E over X, there exists a bundle D such that $E \oplus D$ is trivial.*

PROOF. The idea is first to show that for all bundles $E \xrightarrow{\eta} X$ and all $t \in \mathbb{N}$, there exists a surjective map of bundles

$$\begin{tikzcd} X \times \mathbb{R}^t \arrow[r, two heads] \arrow[dr, "\pi_1"'] & E \arrow[d, "\eta"] \\ & X \end{tikzcd}$$

and then to give bundles metrics which restrict to metrics on fibers, inducing splittings of vector bundles – for example, $X \times \mathbb{R}^t \xrightarrow{\pi_1} X$ above – to yield the desired bundle D.

Given a bundle $E \xrightarrow{\xi} X$, define the **sections** ΓE of E by

$$\Gamma E := \{s : X \to E \,|\, \xi s = 1_X\}$$

Since the fibers of ξ are real vector spaces, we then observe that given two sections $s, t : X \to E$ and any $r \in \mathbb{R}$, we have a well-defined notions of sum $s + t$, given by $(s + t)(x) := s(x) + t(x)$, and multiplication by r, $(rs)(x) := r(s(x))$. Thus ΓE is a real vector space as well. Also note that we have an evaluation

$$\begin{aligned} \epsilon : X \times \Gamma E &\to E \\ (x, s) &\longmapsto s(x) \end{aligned}$$

We need to find an appropriate finite-dimensional subspace of ΓE:

CLAIM 5.2. There exists a finite dimensional subspace $V \leq \Gamma E$ such that the restriction $\epsilon \restriction_{X \times V} : X \times V \to E$ is epi.

PROOF. Since we have assumed X is compact, let $\{U_i\}_{i=1}^m$ be an open cover of X such that we have local trivializations

$$\begin{tikzcd} U_i \times \mathbb{R}^n \arrow[rr, "\phi_1"] \arrow[dr, "\pi_1"'] & & \xi^{-1}(U_i) \arrow[dl, "\xi\restriction U_i"] \\ & U_i & \end{tikzcd}$$

If we additionally assume that X is normal, we can use Urysohn's lemma to obtain functions $\lambda_i : X \to I$ with $\lambda_i^{-1}(0,1] \subseteq U_i$. Without loss of generality – by rescaling – we can assume that the λ_i form a **partition of unity**; namely, that they satisfy $\sum_i \lambda_i(x) = 1$ for all $x \in X$.

Now, for each U_i, we can pick out a finite dimensional subspace V_i of $\Gamma(\xi^{-1}(U_i))$ as follows. We have a local trivialization $\phi_i : U_i \times \mathbb{R}^n \to \xi^{-1}(U_i)$ which we can rewrite as $\phi_i(x, v) = s(v)(x)$, since for each fixed $v \in \mathbb{R}^n$, $\phi_i(-, v)$ is a section of ξ by definition. Taking the standard basis $\{e_i\}_{i=1}^m$ for $\mathbb{R}^n$, the collection of sections $\{\phi_i(-, e_i)\}_{i=1}^m$ are what we use to generate the finite dimensional subspace V_i. Now we need to patch these together to get a map of the desired form on all of X. To do this, for each i define $\theta_i : U_i \to \Gamma E$ by

$$\theta_i(s)(x) := \begin{cases} \lambda_i(x)s(x) & x \in U_i \\ 0 & x \notin U_i \end{cases}$$

and then define $\theta : \prod_i V_i \to \Gamma E$ by

$$\theta(s_1, \ldots, s_m)(x) = \sum_i \theta_i(s)(x).$$

Observe that θ is epi at each point by construction. □

Thus we have the desired epi; we now need to show that every epi splits. To prove this, we need to define the notion of a metric on a bundle. Observe that the collection of all symmetric bilinear forms on a real vector space itself forms a real vector space; this yields a functor from the collection of real vector spaces to itself, which we can apply fiberwise.[1] Explicitly, let $Sym(E)_x$ denote the collection of symmetric bilinear forms on the fiber E_x. The collection $Sym(E) = \uplus_{x \in X} Sym(E)_x$ forms a bundle over X.

DEFINITION 5.3. A **metric** on a bundle E is a section $d : X \to Sym(E)$ such that for all $x \in X$, $d(x)$ is positive definite.

[1]For the complex case, simply replace symmetric bilinear forms with Hermitian forms and the analogous statement holds.

We construct such a section d by defining metrics on the sets of the local trivialization and patching them together. First, let $d_i^{U_i}$ be defined over $\xi^{-1}(U_i)$ via $U_i \times \mathbb{R}^n$, by letting $d_i^{U_i}$ be the standard inner product on $\mathbb{R}^n$. Then we have a metric on $\xi^{-1}(U_i)$ via

$$d_i(x) := \begin{cases} \lambda_i(x) d_i^{U_i}(x) & x \in U_i \\ 0 & x \notin U_i \end{cases}$$

Patching these together, we obtain the desired section: $d(x) := \sum_i d_i(x)$. This completes the proof of the proposition. $\square$

Now deducing the desired corollary is a simple matter.

Corollary 5.4 (4.8). *Every element $\zeta - \nu$ of $K(X)$ can be written in the form $\zeta' - [\varepsilon_r]$ for some integer r and some $\zeta' \in K(X)$.*

Proof. By Proposition 4.7, we know that for $E \xrightarrow{\nu} X$, there exists a bundle $E \xrightarrow{\nu^\perp} X$ such that $\nu \oplus \nu^\perp \cong \varepsilon_r$ for some r. Then we can write:

$$\begin{aligned} \zeta - \nu &= (\zeta - \nu) + \nu^\perp - \nu^\perp \\ &= (\zeta \oplus \nu) - (\nu \oplus \nu^\perp) \\ &= (\zeta \oplus \nu) - \varepsilon_r \end{aligned}$$

which proves the claim. $\square$

LECTURE 6

Wednesday, July 26, 2006[1]

Recall that last time we defined the important notion of stable vector bundles on a topological space X, and showed that for compact spaces X, we have a canonical isomorphism:

$$\mathcal{E}_{st}(X) = [X_+, BO].$$

Similarly for complex vector bundles on X we have the canonical isomorphism

$$\tilde{K}(X) = [X_+, BU].$$

Next observe that the (exterior) tensor product, gives us a map $K(X) \otimes K(Y) \to K(X \times Y)$. Bott Periodicity for complex vector bundles states that this map is an isomorphism for $Y = S^2$. Thus, we see that it's fundamental to understand the K-theory of spheres. This will lead us to the 'Hopf invariant one' problem.

1. Definition of smash products

Let $X \vee Y = X \times \{*\} \cup \{*\} \times Y$. Define $X \wedge Y = X \times Y / X \vee Y$. $X \wedge Y$ is known as the **smash product** of the spaces X and Y. For example, $S^1 \wedge S^1 = S^2$. In fact, it is easy to show that $S^m \wedge S^n = S^{m+n}$. (Do this). This trivial result is at the foundation of the stable homotopy theory of spheres.

1.1. Cones and Suspension. Let $f : X \to Y$ be a continuous basepoint preservng map between based spaces. We define the cofiber of f, denoted by Cf, to be the topological space $Y \cup_f CX$, where CX is the cone of X, defined by

$$CX = X \times I/(X \times \{*\} \cup \{*\} \times I).$$

We thus get a sequence

$$X \to Y \to Cf \to \Sigma X \to \Sigma Y \to ...$$

This is thought of as an 'exact sequence' of topological spaces. Recall that a sequence of abelian groups is exact if the image of each map is equal to the kernel of the next.

[1] These notes were taken and TEXed by Masoud Kamgarpour.

EXAMPLE 6.1. The sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

is an example of a non-split short exact sequence in the category of abelian groups.

Next consider the contravariant functor $Z \to [-, Z]$ applied to the above exact sequence of topological spaces. Then we get an exact sequence of based-sets

$$[X, Z] \leftarrow [Y, Z] \leftarrow [Cf, Z] \leftarrow [\Sigma X, Z] \leftarrow ....$$

(See A Concise Course in Algebraic Topology for details).

EXERCISE 6.2. Suppose we have a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "i"] \arrow[d, "\beta"] & Cf \arrow[r] \arrow[d, dotted, "\gamma"'] & \Sigma X \arrow[d] \\
X \arrow[r, "f'"'] & Y' \arrow[r] & Cf' \arrow[r] & \Sigma X'
\end{tikzcd}$$

Such that the left hand square homotopy commutes. Show that there exists a map $\gamma : Cf \to Cf'$ which makes the two right squares homotopy commutative. (However, γ is NOT unique, why?)

Note that in the above exact sequence we can replace $[\Sigma X, Z]$ by the isomorphic based set $[Ci, Z]$, where i is the inclusion $Y \to Cf$ because there is a collapsing out of a cone homotopy equivalence $Ci \to \Sigma X$. See Concise. Furthermore, one can show that $[\Sigma X, Z]$ is a group. Applying this construction twice, one gets that $[\Sigma^2 X, Z]$ is an **abelian** group. (Exercise! The proofs of these facts are parallels of the proof that the homotopy groups are actually groups, and that higher homotopy groups are abelian.)

Next note that the map (cofibration)

$$X \vee Y \to X \times Y \to X \wedge Y$$

gives us an isomorphism:

$$K(X \times Y) \cong K(X \wedge Y) \oplus K(X) \oplus K(Y).$$

Using the above isomorphism, it is easy to show that

$$\tilde{K}(S^q) = \begin{cases} \mathbb{Z} & q \text{ even} \\ 0 & q \text{ odd} \end{cases}$$

Our goal is to prove Hopf invariant one, which is one of the fundamental results in mathematics. Suppose n is an even integer and let $f : S^{2n-1} \to S^n$ be a map. Denote by $X = Cf$ the cofiber of f. We attach to this datum a number $h(f)$ known as the Hopf invariant of f. The exact sequence of spaces

$$S^{2n-1} \longrightarrow S^n \longrightarrow X = Cf \longrightarrow S^{2n} \longrightarrow S^{n-1}$$

gives rise to an exact sequence of abelian groups. Here the zeros come from the fact that $\tilde{K}(S^q) = 0$ if q is odd.

$$\tilde{K}(S^{2n-1}) = 0 \leftarrow \tilde{K}(S^n) \leftarrow \tilde{K}(S^{2n}) \leftarrow \tilde{K}(S^{n+1}) = 0.$$

Here $\tilde{K}(S^n)$ is a copy of $\mathbb{Z}$ and we call its generator i_n. Similarly we have a generator i_{2n} for $\tilde{K}(S^{2n})$. Since the diagonal map $\Delta\colon S^n \to S^n \wedge S^n = S^{2n}$ represents an element of $\pi_{2n}(S^n) = 0$, it must be null homotopic. This implies that $i_n^2 = 0$ and similarly $i_{2n}^2 = 0$.

The exact sequence above must split, and we can write

$$\tilde{K}(X) = \mathbb{Z}a \oplus \mathbb{Z}b,$$

where $a \mapsto i_n$ and $b \mapsto i_{2n}$. As $i_n^2 = i_{2n}^2 = b^2 = 0$, we see that a^2 has to be an integer multiple of b. This multiple is known as the **Hopf invariant** of f, denoted $h(f)$. Our goal is to sketch the proofs of two theorems. The first one is easy. The second one is very deep.

THEOREM 6.3. *Given a map $\phi : S^{n-1} \times S^{n-1} \to S^{n-1}$ of bi-degree (p, q). Then there exists a map $f = H(\phi) : S^{2n-1} \to S^n$ such that $h(f) = \pm pq$.*

THEOREM 6.4. **Hopf Invariant One** *If $f : S^{2n-1} \to S^n$ has $h(f) = \pm 1$; then, $n = 2, 4,$ or 8.*

LECTURE 7

Friday, July 28, 2006[1]

Aside: The following theorem is very useful.

THEOREM 7.1. *F is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Essentially surjective means that every object of the target category is isomorphic to FX for some object X of the source category. Note however that this result depends on the Axiom of Choice, since we must choose such objects X to prove the result.

1. Division Algebras over $\mathbb{R}$

Let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be such that $\phi(x, y) = 0$ implies that $x = 0$ or $y = 0$. Assume further that there exists $e \in \mathbb{R}^n$ such that $\phi(e, y) = y = \phi(y, e)$. We have the following surprising theorem.

THEOREM 7.2. *If ϕ is as above, then $n = 1, 2, 4, or 8$.*

The only known proofs for this purely algebraic result are topological. We now sketch a proof of this theorem based on Hopf Invariant One theorem.

DEFINITION 7.3. Let X be a topological space. X is said to be an H-space if there is a continuous multiplication map $\mu : X \times X \to X$ and an 'identity' element $e \in X$, such that the two maps $X \to X$ given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity by homotopies based at e.

Exercise: The multiplication ϕ induces the structure of an H-space on S^{n-1}.

Since S^0 is not very interesting (we can view it as the group $\mathbb{Z}/2\mathbb{Z}$), we assume that $n > 1$ henceforward.

[1]These notes were taken and TEXed by Masoud Kamgarpour.

Thus, we are reduced to determining which spheres are H-spaces. Let (p, q) be the bi-degree defined by the following diagram:

$$\begin{tikzcd} S^{n-1} \times S^{n-1} \arrow[r, "\phi"] & S^{n-1} \\ S^{n-1} \arrow[u, "i_2", "i_1"'] \arrow[ur, "q", "p"'] & \end{tikzcd}$$

Where i_1 and i_2 are the two inclusions, and p and q are the maps (or more precisely, the degree of the maps) induced. We have the following theorem:

Theorem 7.4. *If $\phi : S^{n-1} \times S^{n-1} \to S^{n-1}$ is given, and n is odd, then $p = 0$ or $q = 0$.*

Proof. ϕ induces a map on K-theory,

$$\phi^* : K(S^{n-1}) \to K(S^{n-1} \times S^{n-1}) = K(S^{n-1}) \otimes K(S^{n-1}).$$

One can check that

$$\phi^*(k) = p(i \otimes 1) + q(1 \otimes j) + r(i \otimes j)$$

where i, j, k are generators of the appropriate K-groups. Now last time we saw that $i^2 = j^2 = 0$; This shows at once that $(\phi^*(k))^2 = 2pq(i \otimes j)$. On the other hand, as $k^2 = 0$, we see that $\phi^*(k^2) = (\phi^*)^2(k) = 0$. It follows that $p = 0$ or $q = 0$. $\square$

2. Towards Hopf Invariant One

From now on we assume n is even and $n > 2$.

Theorem 7.5. *Let $\phi : S^{n-1} \times S^{n-1} \to S^{n-1}$ be given with bi-degree (p, q). There exists a map $f = H(\phi) : S^{2n-1} \to S^n$ such that $h(f) = \pm pq$.*

Proof. To construct f it's enough to construct a map on the homotopic spaces:

$$f : CS^{n-1} \times S^{n-1} \cup S^{n-1} \times CS^{n-1} \longrightarrow \Sigma S^{n-1}$$

For this, we write

$$f([x, t], y) = [\phi(x, y), (1 + t)/2], \qquad f(x, [y, t]) = [\phi(x, y), (1 - t)/2]$$

The essential point is that we have the following commutative diagram, where $X = Cf$ and α and β are the canonical maps in our exact sequence of spaces

$$S^{2n-1} \xrightarrow{f} S^n \xrightarrow{\alpha} X \xrightarrow{\beta} S^{2n}.$$

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[dd, "\beta"'] & X \wedge X \\
& S^n \wedge S^n \arrow[u, "\alpha\wedge\alpha"'] \\
S^{2n} \arrow[r, "\equiv"] & S^n \wedge S^{2n} \arrow[u, "p\wedge q"']
\end{tikzcd}$$

It just remains to chase the diagram

$$\begin{tikzcd}
a^2 & a \otimes a \arrow[l] \arrow[d] \\
& i_n \otimes i_n \arrow[d, leftrightarrow] \\
pqi_{2n} \arrow[uu] & pq(i_n \otimes i_n) \arrow[l]
\end{tikzcd}$$

But by the definition of the Hopf invariant, $a^2 = h(f)i_{2n}$ and therefore $h(f) = pq$. □

2.1. Adams Operations on K-theory. There exist natural ring homomorphisms $\psi^k : K(X) \to K(X)$ for k a positive integer, satisfying:

(1) $\psi^1 = id$.
(2) $\psi^p(x) \equiv x^p \pmod p$ for a prime p.
(3) $\psi^k\psi^l = \psi^{kl} = \psi^l\psi^k$.
(4) If ζ is a line bundle, then $\psi^k(\zeta) = \zeta^k$.
(5) $\psi^k(x) = n^k x$ if $x \in \tilde{K}(S^{2n})$.

Given these operations, let us see how we can prove Hopf invariant one:

THEOREM 7.6. *If $h(f) = \pm 1$, $n > 1$, then $n = 2$, 4, or 8.*

PROOF. Let $n = 2m$. Then

$$\phi^k(a) = k^m a + \mu_k b,$$

$$\phi^k(b) = k^{2m} b,$$

and

$$\psi^2(a) \equiv a^2 \pmod 2.$$

As $a^2 = h(f)b$, we see that

$$\mu_2 \equiv h(f) \pmod 2.$$

Now

$$\psi^2\psi^k(a) = \psi^2(k^m a + \mu_k b) = 2^m k^m a + k^m \mu_2 b + 2^{2m}\mu_k b.$$

On the other hand, we have:

$$\psi^k\psi^2(a) = \psi^k(2^m a + \mu_2 b) = 2^m k^m a + 2^m \mu_k b + b^{2m}\mu_2 b.$$

Since these are equal, it follows that

$$k^m\mu_2 + 2^{2m}\mu_k = 2\mu_k + k^{2m}\mu_2$$

which in turn implies that

$$k^m(k^m - 1)\mu_2 = 2^m(2^m - 1)\mu_k.$$

We have assumed that μ_2 is odd; thus, $2^m | k^m - 1$ for all odd k, or equivalently

$$k^m \equiv 1 \quad (mod\, 2^m).$$

Remembering that $n = 2m$, the rest is easy number theory. (We could even restrict our number theory to considering the case $k = 3$, which is what Adams and Atiyah did in their original proof). Once can first check that if $m > 1$, m has to be even. [It is relevant that the group of units in $\mathbb{Z}/2^m\mathbb{Z}$ has even order, r_m say. We can write $r_m = 2^q k$, k odd. I didn't recover the argument at the time]. Let $k = 1 + 2^{m/2}$. Then,

$$k^m \equiv 1 + m2^{m/2} \quad (mod\, 2^m).$$

It follows that

$$2^m | m2^{m/2},$$

hence $m \geq 2^{m/2}$. But this is true if and only if $m = 2$ or 4. $\square$