DIFFERENTIAL FORMS ON NONCOMMUTATIVE SPACES

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ABSTRACT. This paper is intended as an introduction to noncommutative geometry for readers with some knowledge of abstract algebra and differential geometry. We show how to extend the theory of differential forms to the "noncommutative spaces" studied in noncommutative geometry. We formulate and prove the Hochschild-Kostant-Rosenberg theorem and an extension of this result involving the Connes differential.

CONTENTS

1. INTRODUCTION

Noncommutative geometry is a subject in which constructions from noncommutative algebra are interpreted spatially. Very often in mathematics we consider the collection of continuous functions from a space to the real or complex numbers. This collection of functions forms a commutative algebra under pointwise addition, multiplication, and scaling, and it is often convenient to study a space by studying its algebra of functions rather than the space itself. In noncommutative geometry, instead of starting with a space, we start with a noncommutative algebra and think of its elements as if they were functions on some "noncommutative space" even though an actual underlying space does not exist.

In this way, noncommutative geometry is a natural generalization of differential geometry, where one studies a space by studying the smooth functions defined on the space, and algebraic geometry, where one studies a space by studying algebraic functions from that space to the complex numbers. Noncommutative geometry also has applications in mathematical physics. The possible states of a physical system are represented in classical physics by points on a manifold, and observables are represented by smooth functions on this manifold of states. When we model a quantum mechanical system, we replace the commutative algebra of observables by a noncommutative algebra of operators on a Hilbert space.

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It turns out that many structures defined for spaces also make sense for noncommutative spaces. In this paper we show how to extend the theory differential forms to noncommutative geometry. We start with an associative but not necessarily commutative algebra A and use this algebra to define differential forms and exterior derivatives for the noncommutative space having A as its algebra of functions. The Hochschild-Kostant-Rosenberg (HKR) theorem says that this noncommutative theory reduces to the familiar theory of differential forms under certain conditions. For a more advanced discussion of these results and noncommutative geometry in general, see [1].

To appreciate the results of this paper, the reader should have some knowledge of abstract algebra (especially the theory of rings and modules) and of differential geometry (especially differential forms and de Rham cohomology). We do not assume any familiarity with homological algebra, and we introduce the necessary terminology from homological algebra as we need it. For more complete treatments of homological algebra, see [2] and [3].

The rest of this paper is organized as follows. In section 2, we introduce homological algebra and work up to the definition of the Tor and Ext functors. In section 3, we use Tor and Ext to define Hochschild homology and cohomology. To gain some intuition for these constructions, we perform some computations and make an analogy with differential geometry. In section 4, we prove some technical results from homological algebra. In section 5, we introduce Koszul complexes and use them to compute Tor. Finally, in section 6, we prove the HKR theorem and discuss the Connes differential.

2. Homological Algebra

Homological algebra is an abstract branch of mathematics that generalizes ideas from topology and differential geometry. One of the most important ideas in differential geometry is that the boundary of a boundary is zero and the exterior derivative of an exterior derivative is zero. More precisely, if d_n is the linear map that takes chains of n-simplices to their boundaries, then we have $d_n \circ d_{n+1} = 0$, and if d^n is the linear map that takes n-forms to their exterior derivatives, then we have $d^n \circ d^{n-1} = 0$. The notions of chain and cochain complexes generalize these ideas.

Definition 2.1. Let R be a ring. A *chain complex* C_* is a sequence of R-linear maps

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

of R-modules such that $d_n \circ d_{n+1} = 0$ for all n. That is, $\operatorname{im} d_{n+1} \subseteq \ker d_n$. Elements of $\ker d_n$ are called *cycles*, and the elements of $\operatorname{im} d_{n+1}$ are called *boundaries*. The quotient $H_n(C_*) = \ker d_n / \operatorname{im} d_{n+1}$ is called the *homology module* of C_* of *degree* n.

A *cochain complex* C^* is a sequence of R-linear maps

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} \cdots$$

of R-modules such that $d^n \circ d^{n-1} = 0$ for all n. Elements of $\ker d^n$ are called *cocycles*, and the elements of $\operatorname{im} d^{n-1}$ are called *coboundaries*. The quotient $H^n(C^*) = \ker d^n / \operatorname{im} d^{n-1}$ is called the *cohomology module* of C^* of *degree* n.

In differential geometry, cochain complexes are a tool for measuring the failure of closed differential forms to be exact. In this case the cohomology module $H^n(C^*)$ is the de Rham cohomology, and we have $H^n(C^*) = 0$ if and only if every closed n-form is exact. In general, we say that a sequence of maps

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

is *exact* at C_n if $\ker d_n = \operatorname{im} d_{n+1}$. We say the sequence is *exact* if it is exact at C_n for every n.

Now that we have defined chain and cochain complexes, we can define maps between them. A *morphism* $f : C_* \to C'_*$ of chain complexes is a set of R-linear maps $f_n : C_n \to C'_n$ such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \\ \cdots & \longrightarrow & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \longrightarrow & \cdots \end{array}$$

commutes. This means that any two composites from one fixed module to another are equal. A morphism between cochain complexes is defined similarly. The following definition is useful whenever we want to study a class of mathematical objects and the maps between them.

Definition 2.2. A *category* $\mathcal{C}$ consists of the following.

(1) A collection $\mathrm{Ob}(\mathcal{C})$, whose elements are called the *objects* of the category.
(2) For every pair X, Y of objects, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, whose elements are called the *morphisms* from X to Y in $\mathcal{C}$.
(3) For every triple X, Y, Z of objects, a binary operation

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

called *composition*, sending morphisms f and g to their *composite* $g \circ f$.
(4) For every object X, a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ called the *identity morphism* on X.

These data must satisfy the following axioms.

(1) Composition of morphisms is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

whenever either side is defined.
(2) An identity morphism is a two-sided unit for composition:

$$f \circ 1_X = f = 1_Y \circ f$$

for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

For a fixed ring R, the collection of all R-modules is a category whose morphisms are R-linear maps. Composition of morphisms in this category is just composition of functions, and the identity on an R-module M is the identity map $1_M : M \to M$, $m \mapsto m$. The collection of all chain complexes forms another category with the morphisms defined above. We define the composite of two morphisms f and g in the category of chain complexes by $(g \circ f)_n = g_n \circ f_n$. The identity morphism on a chain complex C_* is given by $(1_{C_*})_n = 1_{C_n}$. Similarly, the collection of all cochain complexes forms a category.

Suppose that $f : C_* \to C'_*$ is a morphism of chain complexes and that z_1 and z_2 are representatives for the same coset in $H_n(C_*)$. Then $z_1 - z_2 \in \operatorname{im} d_{n+1}$ so there exists x such that $z_1 - z_2 = d_{n+1}x$. Since the diagram above commutes, we have $f_n(z_1) - f_n(z_2) = f_n(z_1 - z_2) = f_n(d_{n+1}x) = d'_{n+1}(f_{n+1}(x)) \in \operatorname{im} d'_{n+1}$. It follows that $f_n(z_1) + \operatorname{im} d'_{n+1} = f_n(z_2) + \operatorname{im} d'_{n+1}$. This proves that $z + \operatorname{im} d_{n+1} \mapsto f(z) + \operatorname{im} d_{n+1}$ is a well defined linear map $H_n(C_*) \to H_n(C'_*)$. Thus we can assign to every chain complex C_* the nth homology module $H_n(C_*)$, and whenever we have a morphism $C_* \to C'_*$, we get a morphism $H_n(C_*) \to H_n(C'_*)$ of modules. In other words, we have a mapping between two categories.

For another example of a mapping between two categories, let N be any R-module. Then the operation $\operatorname{Hom}_R(-, N)$ which sends each R-module M to the module $\operatorname{Hom}_R(M, N)$ of all morphisms $M \to N$ and sends each R-linear map $f : M \to M'$ to the map $\operatorname{Hom}_R(M', N) \to \operatorname{Hom}_R(M, N)$, $g \mapsto g \circ f$ is a mapping from the category of R-modules to itself.

Definition 2.3. A *covariant functor* $F : \mathcal{C} \to \mathcal{C}'$ from a category $\mathcal{C}$ to a category $\mathcal{C}'$ consists of the following.

(1) A map sending each object X of $\mathcal{C}$ to an object $F(X)$ of $\mathcal{C}'$.
(2) A map sending each morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{C}'$.

These data must satisfy the following axioms.

(1) F preserves composition:
$$F(g \circ f) = F(g) \circ F(f)$$
whenever the composite morphism $g \circ f$ is defined.
(2) F preserves identities:
$$F(1_X) = 1_{F(X)}$$
for every object X of $\mathcal{C}$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{C}'$ is defined in the same way, except that F sends each morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{C}'$, satisfying $F(g \circ f) = F(f) \circ F(g)$.

Thus homology is a covariant functor from the category of chain complexes to the category of R-modules, and one can similarly show that cohomology is a covariant functor from the category of cochain complexes to the category of R-modules. The operation $\operatorname{Hom}_R(-, N)$ described above is an example of a contravariant functor.

For the remainder of this section, we discuss Tor and Ext, two of the most important functors in homological algebra. These functors are useful for studying chain and cochain complexes, and they will be used to define the Hochschild homology and cohomology. Before defining Tor and Ext, we state some general definitions involving R-modules. An R-module P is said to be *projective* if for any surjective morphism $f : A \to B$ and any morphism $\beta : P \to B$ there exists a morphism

$\alpha : P \to A$ such that

$$\begin{array}{ccc} & & P \\ & \overset{\alpha}{\swarrow} & \downarrow \beta \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes. It is easy to show for example that if a module is free then it is projective. A *projective resolution* $P_* \to M$ of M is a chain complex P_* of projective modules together with a map $P_0 \to M$ so that

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is an exact sequence.

The operation $N \otimes_R -$ which sends each module M to the tensor product $N \otimes_R M$ and sends each linear map $f : M \to M'$ to the linear map $N \otimes_R M \to N \otimes_R M'$ defined by $n \otimes_R m \mapsto n \otimes_R f(m)$ is a functor from R-modules to abelian groups. Given a projective resolution $P_* \to M$, we can apply this functor to P_* to get the induced sequence

$$\cdots \longrightarrow N \otimes_R P_2 \longrightarrow N \otimes_R P_1 \longrightarrow N \otimes_R P_0 \longrightarrow 0$$

which is in fact a chain complex. We have seen that $\mathrm{Hom}_R(-, N)$ is a contravariant functor from the category of R-modules to itself. If we apply this functor to the projective resolution above, we get the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(P_0, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow \mathrm{Hom}_R(P_2, N) \longrightarrow \cdots$$

which is in fact a cochain complex.

Definition 2.4. Let M and N be R-modules. For any projective resolution of M, we define $\mathrm{Tor}_n^R(N, M)$ to be the nth homology module of the chain complex induced by $N \otimes_R -$. We define $\mathrm{Ext}_R^n(M, N)$ to be the nth cohomology module of the cochain complex induced by $\mathrm{Hom}_R(-, N)$.

Of course we must prove that these definitions do not depend on the choice of projective resolutions. In proving this result, it will be useful to have a criterion to tell when two different morphisms of chain complexes induce the same map on homology or cohomology modules. Thus, given morphisms $\{f_n\}$ and $\{g_n\}$ from a chain complex C_* to a chain complex C'_*, we define a *chain homotopy* to be a collection of maps $s_n : C_n \to C'_{n+1}$ with the property that $f_n - g_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$.

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \\ & \overset{s_n}{\swarrow} & \downarrow f_n - g_n & \overset{s_{n-1}}{\swarrow} & \\ C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} \end{array}$$

If such a collection exists, we say that $\{f_n\}$ and $\{g_n\}$ are *chain homotopic*. In this case, if z is any cycle in C_n then $(f_n - g_n)(z) = d'_{n+1}(s_n(z))$ is a boundary, so $f_n - g_n$ is the zero map on homology and hence $f_n = g_n$ on homology. Similarly, we can define cochain homotopies and prove that cochain homotopic maps induce equivalent maps on cohomology.

Lemma 2.5. *Let $f : M \to M'$ be a morphism of modules, and take projective resolutions $P_* \to M$ and $P'_* \to M'$. Then for each $n \geq 0$ there exists a morphism $f_n : P_n \to P'_n$ such that the following diagram commutes.*

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \xrightarrow[d'_2]{} & P'_1 & \xrightarrow[d'_1]{} & P'_0 & \xrightarrow[d'_0]{} & M' & \longrightarrow & 0
\end{array}
$$

These morphisms f_n are said to lift *the morphism f. If $g_n : P_n \to P'_n$ are any other morphisms lifting f then $\{f_n\}$ and $\{g_n\}$ are chain homotopic.*

Proof. We first define the maps f_n inductively. Since the bottom row of the above diagram is an exact sequence, we know that $\operatorname{im} d'_0 = M'$. Thus d'_0 is surjective and, by definition of the projective module P_0, there exists $f_0 : P_0 \to P'_0$ such that

$$
\begin{array}{ccc}
 & & P_0 \\
 & \swarrow{\scriptstyle f_0} & \downarrow{\scriptstyle f \circ d_0} \\
P'_0 & \xrightarrow[d'_0]{} & M'
\end{array}
$$

commutes. Now suppose that we have defined maps f_i for $i \leq n$ so that

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & P'_{n+1} & \xrightarrow[d'_{n+1}]{} & P'_n & \xrightarrow[d'_n]{} & P'_{n-1} & \longrightarrow & \cdots & \longrightarrow & M' & \longrightarrow & 0
\end{array}
$$

commutes. If $z \in P_{n+1}$ then by commutativity of the diagram we have

$$d'_n(f_n(d_{n+1}z)) = f_{n-1}(d_n(d_{n+1}z)) = f_{n-1}(0) = 0$$

and hence $\operatorname{im} f_n \circ d_{n+1} \subseteq \ker d'_n$. The solid horizontal arrow in the diagram

$$
\begin{array}{ccc}
 & & P_{n+1} \\
 & \swarrow{\scriptstyle f_{n+1}} & \downarrow{\scriptstyle f_n \circ d_{n+1}} \\
P'_{n+1} & \xrightarrow[d'_{n+1}]{} & \ker d'_n
\end{array}
$$

is surjective by exactness of the projective resolution of M', so there is a morphism $f_{n+1} : P_{n+1} \to P'_{n+1}$ making the diagram commute. Continuing inductively, we obtain a morphism f_n for every n.

Next we use induction to define the morphisms s_n. Put $s_{-1} = 0$, $f_{-1} = g_{-1} = f$, and $h_n = f_n - g_n$. Since $h_{-1} = f - f = 0$ we have $d'_0 \circ h_0 = h_{-1} \circ d_0 = 0$ and hence $\operatorname{im} h_0 \subseteq \ker d'_0 = \operatorname{im} d'_1$. Since P_0 is projective, there exists a morphism $s_0 : P_0 \to P'_1$ such that

$$
\begin{array}{ccc}
 & & P_0 \\
 & \swarrow{\scriptstyle s_0} & \downarrow{\scriptstyle h_0} \\
P'_1 & \xrightarrow[d'_1]{} & P'_0
\end{array}
$$

commutes, and this morphism s_0 satisfies $h_0 = d'_1 \circ s_0 = d'_1 \circ s_0 + s_{-1} \circ d_0$, as desired. To apply induction, suppose we have defined s_i for $i \leq n$ satisfying $d'_{i+1} \circ s_i = h_i - s_{i-1} \circ d_i$. Since

$$\begin{aligned} d'_{n+1} \circ (h_{n+1} - s_n \circ d_{n+1}) &= d'_{n+1} \circ h_{n+1} - d'_{n+1} \circ s_n \circ d_{n+1} \\ &= d'_{n+1} \circ h_{n+1} - (h_n - s_{n-1} \circ d_n) \circ d_{n+1} \\ &= d'_{n+1} \circ h_{n+1} - h_n \circ d_{n+1} + s_{n-1} \circ d_n \circ d_{n+1} \\ &= 0 \end{aligned}$$

we see that $\operatorname{im}(h_{n+1} - s_n \circ d_{n+1}) \subseteq \ker d'_{n+1} = \operatorname{im} d'_{n+2}$. Since P_{n+1} is projective, there exists a morphism s_{n+1} such that

$$\begin{array}{ccc} & & P_{n+1} \\ & \overset{s_{n+1}}{\swarrow} & \downarrow{\scriptstyle h_{n+1} - s_n \circ d_{n+1}} \\ P'_{n+2} & \xrightarrow[d'_{n+2}]{} & \operatorname{im} d'_{n+2} \end{array}$$

commutes. This completes the proof by induction. □

Theorem 2.6. *The groups* $\operatorname{Tor}_n^R(N, M)$ *and* $\operatorname{Ext}_R^n(M, N)$ *depend only on* M*,* N*,* n *and not on the choice of projective resolution* $P_* \to M$*.*

Proof. Suppose $P_* \to M$ and $P'_* \to M$ are two projective resolutions of M. By the lemma, there is a collection of morphisms $f_n : P_n \to P'_n$ lifting 1_M and a collection of morphisms $g_n : P'_n \to P_n$ lifting 1_M. Since $N \otimes_R -$ is a functor, we get a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & N \otimes_R P_1 & \longrightarrow & N \otimes_R P_0 & \longrightarrow & N \otimes_R M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle 1_{N \otimes_R M}} & & \\ \cdots & \longrightarrow & N \otimes_R P'_1 & \longrightarrow & N \otimes_R P'_0 & \longrightarrow & N \otimes_R M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle 1_{N \otimes_R M}} & & \\ \cdots & \longrightarrow & N \otimes_R P_1 & \longrightarrow & N \otimes_R P_0 & \longrightarrow & N \otimes_R M & \longrightarrow & 0 \end{array}$$

where φ_n is the map induced by f_n and ψ_n is the map induced by g_n. Thus the maps $\psi_n \circ \varphi_n$ are a lift of $1_{N \otimes_R M}$. But the collection of identity maps $N \otimes_R P_n \to N \otimes_R P_n$ is another lift of $1_{N \otimes_R M}$, so we see that $\{1_{N \otimes_R P_n}\}$ is chain homotopic to $\{\psi_n \circ \varphi_n\}$. Hence $\psi_n \circ \varphi_n$ is the identity map on $\operatorname{Tor}_n^R(N, M)$. Reversing the roles of $P_* \to M$ and $P'_* \to M$ in this argument, we see also that $\varphi_n \circ \psi_n$ is the identity on $\operatorname{Tor}_n^R(N, M)$. Hence φ_n and ψ_n are isomorphisms between the modules $\operatorname{Tor}_n^R(N, M)$ obtained from $P_* \to M$ and $P'_* \to M$.

The proof for $\operatorname{Ext}_R^n(M, N)$ is similar, with the arrows in the diagram reversed since $\operatorname{Hom}_R(-, N)$ is a contravariant functor. □

If $f : M \to M'$ is a morphism of R-modules, then the morphism $N \otimes_R M \to N \otimes_R M'$ induced by f induces another morphism $\operatorname{Tor}_n^R(N, M) \to \operatorname{Tor}_n^R(N, M')$ of abelian groups so that $\operatorname{Tor}_n^R(N, -)$ is a covariant functor. Similarly, $\operatorname{Ext}_R^n(-, N)$ is a contravariant functor.

3. Hochschild Homology and Cohomology

Now that we have introduced some terminology from homological algebra, we can see how to define 1-forms in noncommutative geometry. Given any smooth manifold M, we write $\Omega^0(M)$ for the algebra of smooth functions on M and $\Omega^1(M)$ for the module of 1-forms. We would like to have a completely algebraic analogue of $\Omega^1(M)$. Thus, for any commutative algebra A over a field $\Bbbk$, we define the module $\Omega^1_{A|\Bbbk}$ of *Kähler differentials* to be the A-module generated by the symbols da with $a \in A$ satisfying

$$d(a+b) = da + db$$
$$d(\lambda a) = \lambda da$$
$$d(ab) = a(db) + b(da)$$

for all a, $b \in A$ and $\lambda \in \Bbbk$. If we view A as an algebraic analogue of $\Omega^0(M)$, then this module of $\Omega^1_{A|\Bbbk}$ of Kähler differentials is an algebraic analogue of $\Omega^1(M)$. Our goal in this section is to define an analogue of the module of Kähler differentials when A is not commutative.

Recall that the *opposite* of an algebra A is the algebra denoted A^{op} with the same underlying vector space as A and multiplication defined by $a \cdot^{\mathrm{op}} b = b \cdot a$ for a, $b \in A$. The *enveloping algebra* of A is the algebra $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ where $\otimes$ denotes tensor product over the field $\Bbbk$. An A-bimodule is an abelian group M which is both a left and right A-module and satisfies $a(mb) = (am)b$ for all a, $b \in A$. Note that any A-bimodule is canonically an A^{e}-module where the action of A^{e} on M is given by $(a \otimes b)m = amb$. The multiplication $\cdot^{\mathrm{op}}$ is needed here to ensure that this is an action. Conversely, if M is an A^{e}-module, then M is an A-bimodule with left and right actions defined by $am = (a \otimes 1)m$ and $ma = (1 \otimes a)m$, respectively.

Definition 3.1. If M is a bimodule, the *Hochschild homology* of M is the $\Bbbk$-vector space $H_n(M, A) = \mathrm{Tor}_n^{A^{\mathrm{e}}}(M, A)$.

To show how this relates to differential forms, we perform a simple computation. By the results of the previous section, we can compute Hochschild homology by choosing any projective resolution of A and computing the homology of the complex induced by $M \otimes_{A^{\mathrm{e}}} -$. It is convenient to use the *bar resolution* $B_*(A) \to A$ given by

$$\cdots \longrightarrow A^{\otimes 4} \xrightarrow{b'_2} A^{\otimes 3} \xrightarrow{b'_1} A^{\otimes 2} \longrightarrow A \longrightarrow 0$$

where $B_n(A) = A^{\otimes n+2}$ is the tensor product of $n+2$ copies of A, the map $A^{\otimes 2} \to A$ is multiplication, and b'_n is given by

$$b'_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

It is tedious but straightforward to check that this is a chain complex. We claim that the bar resolution is a projective resolution of A. Indeed, if we define $s_n : A^{\otimes n+2} \to A^{\otimes n+3}$ by the formula

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1},$$

we see that

$$\begin{aligned}
&b'_{n+1}(s_n(a_0 \otimes \cdots \otimes a_{n+1})) + s_{n-1}(b'_n(a_0 \otimes \cdots \otimes a_{n+1})) \\
&\quad = b'_{n+1}(1 \otimes a_0 \otimes \cdots \otimes a_{n+1}) + s_{n-1}\left(\sum_{i=0}^{n}(-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}\right) \\
&\quad = 1a_0 \otimes \cdots \otimes a_{n+1} + \sum_{i=0}^{n}(-1)^{i+1} 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\
&\qquad + \sum_{i=0}^{n}(-1)^i 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\
&\quad = a_0 \otimes \cdots \otimes a_{n+1}.
\end{aligned}$$

Hence $b'_{n+1} \circ s_n + s_{n-1} \circ b'_n = 1_{B_n(A)}$ and the identity on the bar resolution is chain homotopic to the zero map. It follows that the homology modules of $B_*(A)$ are all trivial and $B_*(A)$ is an exact chain complex.

Next we must show that the each term in the complex is a projective A^{e}-module where the A^{e}-module structure is given by $(a \otimes b)a_0 \otimes \cdots \otimes a_{n+1} = aa_0 \otimes \cdots \otimes a_{n+1}b$. There is an isomorphism $A^{\otimes n+2} \cong A^{\mathrm{e}} \otimes A^{\otimes n}$ of A^{e}-modules, and since $\Bbbk$ is a field, the tensor power $A^{\otimes n}$ is free as a $\Bbbk$-module. Therefore

$$A^{\otimes n+2} \cong A^{\mathrm{e}} \otimes \bigoplus_{i \in I} \Bbbk \cong \bigoplus_{i \in I} A^{\mathrm{e}} \otimes \Bbbk \cong \bigoplus_{i \in I} A^{\mathrm{e}}$$

for some indexing set I. This proves that $A^{\otimes n+2}$ is free and hence projective as an A^{e}-module.

We now see that the bar resolution is a projective resolution of A. It remains to compute Hochschild homology. Any bimodule M is canonically a right A^{e}-module where the action of A^{e} on the right is given by $m(a \otimes b) = bma$. Tensoring the chain complex $B_*(A)$ with M, we obtain

$$\cdots \longrightarrow M \otimes_{A^{\mathrm{e}}} A^{\otimes 4} \longrightarrow M \otimes_{A^{\mathrm{e}}} A^{\otimes 3} \longrightarrow M \otimes_{A^{\mathrm{e}}} A^{\otimes 2} \longrightarrow 0.$$

Define linear maps $\varphi : M \otimes A^{\otimes n} \to M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2}$ and $\psi : M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2} \to M \otimes A^{\otimes n}$ by the formulas

$$\varphi(m \otimes (a_1 \otimes \cdots \otimes a_n)) = m \otimes (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$$

and

$$\psi(m \otimes (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1})) = a_{n+1} m a_0 \otimes (a_1 \otimes \cdots \otimes a_n)$$

where we have omitted all subscripts in the symbol for tensor product. It is immediate that $\psi \circ \varphi = 1_{M \otimes A^{\otimes n}}$. We also have

$$\begin{aligned}
\varphi(\psi(m \otimes (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}))) &= \varphi(a_{n+1} m a_0 \otimes (a_1 \otimes \cdots \otimes a_n)) \\
&= a_{n+1} m a_0 \otimes (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \\
&= m(a_0 \otimes a_{n+1}) \otimes (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \\
&= m \otimes (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}),
\end{aligned}$$

so $\varphi \circ \psi = 1_{M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2}}$. This shows that φ is an isomorphism $M \otimes A^{\otimes n} \cong M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2}$, and the complex whose homology we want to compute becomes

$$\cdots \longrightarrow M \otimes A^{\otimes 2} \xrightarrow{b_2} M \otimes A \xrightarrow{b_1} M \longrightarrow 0.$$

The maps b'_n in the bar resolution satisfy

$$\begin{aligned}
&(1_M \otimes b'_n)(m \otimes a_0 \otimes \cdots \otimes a_{n+1}) \\
&\quad = \sum_{i=0}^{n} (-1)^i m \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\
&\quad = (a_{n+1} m a_0 a_1) \otimes 1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1 \\
&\qquad + \sum_{i=1}^{n-1} (-1)^i (a_{n+1} m a_0) \otimes 1 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes 1 \\
&\qquad + (-1)^n (a_n a_{n+1} m a_0) \otimes 1 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes 1,
\end{aligned}$$

so the maps $b_n = \psi \circ (1_M \otimes b'_n) \circ \varphi : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ are given by

$$\begin{aligned}
b_n(m \otimes a_1 \otimes \cdots \otimes a_n) = m a_1 \otimes a_2 \otimes \cdots \otimes a_n \\
+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
+ (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}.
\end{aligned}$$

Computing the first Hochschild homology is now easy. The kernel of b_1 is generated by $m \otimes a \in M \otimes A$ such that $ma - am = 0$, and the image of b_2 is generated by expressions of the form $ma \otimes b - m \otimes ab + bm \otimes a$ with $m \in M$ and a, $b \in A$. The homology $H_1(M, A)$ is the quotient $\ker b_1 / \operatorname{im} b_2$.

In the special case where $M = A$ is commutative, the kernel of b_1 is all of $A \otimes A$, so the Hochschild homology $H_1(A, A)$ is the quotient of $A \otimes A$ by the relation

$$ab \otimes c - a \otimes bc + ca \otimes b = 0.$$

Define a $\Bbbk$-linear map $\varphi : H_1(A, A) \to \Omega^1_{A|\Bbbk}$ taking the coset of $a \otimes b$ to the element adb. By the relations defining $\Omega^1_{A|\Bbbk}$, we have

$$\begin{aligned}
\varphi(ab \otimes c - a \otimes bc + ca \otimes b) &= ab(dc) - ad(bc) + ca(db) \\
&= ab(dc) - ab(dc) - ac(db) + ca(db) \\
&= 0,
\end{aligned}$$

so this map φ is well defined. The $\Bbbk$-linear map $\psi : \Omega^1_{A|\Bbbk} \to H_1(A, A)$ given by $\psi(adb) = a \otimes b$ satisfies $b_1(\psi(adb)) = b_1(a \otimes b) = ab - ba = 0$ by commutativity of A, and

$$\psi(a(db) + b(da) - d(ab)) = 1a \otimes b + b1 \otimes a - 1 \otimes ab \in \operatorname{im} b_2,$$

so ψ is a well defined map into $H_1(A, A) = \ker b_1 / \operatorname{im} b_2$. It is clearly an inverse of φ. This proves that, in the special case where $M = A$ is commutative, $H_1(A, A)$ is isomorphic to the module $\Omega^1_{A|\Bbbk}$ of Kähler differentials.

We shall see later that the Hochschild homology provides a good notion of differential forms when the algebra A is not commutative. In the meantime, let us define Hochschild cohomology $H^*(A, M)$ and compute $H^1(A, M)$.

Definition 3.2. The *Hochschild cohomology* of M is the vector space $H^n(A, M) = \operatorname{Ext}^n_{A^e}(A, M)$.

As in the previous computation, we look at the sequence induced by the bar resolution. Applying the functor $\mathrm{Hom}_{A^e}(-, M)$ to the bar resolution, we obtain

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A^{\otimes 2}, M) \longrightarrow \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \longrightarrow \cdots.$$

Since $A^{\otimes 2}$ is free of rank one as an A^e-module, we see there is an isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes 2}, M) \cong M$, namely the map that evaluates morphisms on $1 \otimes 1$. Moreover, there is an isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, M)$ which assigns to each morphism $\phi : A^{\otimes n+2} \to M$ the morphism defined by $a_1 \otimes \cdots \otimes a_n \mapsto \phi(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$. Thus the complex whose cohomology we need to compute reduces to

$$0 \longrightarrow M \xrightarrow{b^0} \mathrm{Hom}_{\Bbbk}(A, M) \xrightarrow{b^1} \mathrm{Hom}_{\Bbbk}(A^{\otimes 2}, M) \xrightarrow{b^2} \cdots.$$

If $\phi : A^{\otimes n+2} \to M$ is any morphism of A^e-modules, then

$$\begin{aligned} \phi \circ b'_{n+1}(1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1) = \phi(a_1 \otimes \cdots \otimes a_{n+1} \otimes 1) \\ + \sum_{i=1}^{n} (-1)^i \phi(1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes 1) \\ + (-1)^{n+1} \phi(1 \otimes a_1 \otimes \cdots \otimes a_{n+1}). \end{aligned}$$

If $f : A^{\otimes n} \to M$ is an $\Bbbk$-linear map, then by surjectivity of the isomorphism described above there exists an A^e-linear map $\phi : A^{\otimes n+2} \to M$ with the property that $\phi(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = f(a_1 \otimes \cdots \otimes a_n)$. Under this isomorphism the differential $b^n : \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, M) \to \mathrm{Hom}_{\Bbbk}(A^{\otimes n+1}, M)$ is therefore given by

$$\begin{aligned} b^n f(a_1 \otimes \cdots \otimes a_{n+1}) &= \phi \circ b'_{n+1}(1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1) \\ &= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &\quad + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}. \end{aligned}$$

Using this formula for b^n we can easily compute the Hochschild cohomology. The kernel of b^1 consists of all maps f such that $af(b) - f(ab) + f(a)b = 0$, or $f(ab) = af(b) + f(a)b$, so being a cocycle is the same as being a derivation. The image of b^0 consists of maps f for which there exists some m such that $f(a) = am - ma$. These maps f are called *inner derivations*, and the quotient $H^1(M, A) = \ker b^1 / \operatorname{im} b^0$ is called the module of *outer derivations*.

In the special case where $M = A$ is commutative, the image of b^0 is trivial and the Hochschild cohomology $H^1(A, M)$ is simply the vector space of derivations. In fact, if A is not commutative, we can view the cohomology $H^1(A, A)$ as the space of vector fields on a noncommutative space.

4. Long Exact Sequences

In the last section, we showed that if A is a commutative algebra over a field $\Bbbk$ then the Hochschild homology $H_1(A, A)$ is isomorphic to the module $\Omega^1_{A|\Bbbk}$ of Kähler differentials. We can therefore view $H_1(A, A)$ as the module of 1-forms on a noncommutative space. In section 6, we shall prove the HKR theorem which implies that we can view $H_n(A, A)$ as the module of n-forms on a noncommutative space.

To prove this result, we need we need to develop further techniques from homological algebra.

Lemma 4.1. *Consider a commutative diagram*

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \end{array}$$

in the category of R-modules. If the rows of this diagram are exact, then there is an exact sequence of maps

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma.$$

Proof. Since this result plays a technical role in our discussion, we sketch the proof and leave the details to the reader. Recall that the cokernel of a map $h : X \to Y$ is the quotient $\operatorname{coker} h = Y/\operatorname{im} h$. We get the maps between kernels by restricting the horizontal maps in the given diagram, and we get the maps between cokernels by the universal property of a quotient. Given an element z of $\ker\gamma$ there exists $y \in B$ such that $g(y) = z$ by surjectivity of g. By commutativity of the diagram, we have $g'(\beta(y)) = \gamma(g(y)) = \gamma(z) = 0$, so $\beta(y)$ is contained in the kernel of g' and hence in the image of f'. Therefore we can find $x \in A'$ such that $f'(x) = \beta(y)$. The map $\delta : \ker\gamma \to \operatorname{coker}\alpha$, $z \mapsto x + \operatorname{im}\alpha$ is well defined and the resulting sequence is exact by inspection. □

By a *short exact sequence* of chain complexes we mean an exact sequence

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

of morphisms of chain complexes.

Theorem 4.2. *If* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *is a short exact sequence of chain complexes then there are maps $\partial_n : H_n(C_*) \to H_{n-1}(A_*)$ so that*

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A_*) \longrightarrow H_n(B_*) \longrightarrow H_n(C_*) \xrightarrow{\partial_n} H_{n-1}(A_*) \longrightarrow H_{n-1}(B_*) \longrightarrow H_{n-1}(C_*) \xrightarrow{\partial_{n-1}} \cdots$$

is an exact sequence called the long exact sequence *in homology.*

Proof. Let us denote the set of cycles in C_n by $Z_n(C_*)$. For each n we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow[f_{n-1}]{} & B_{n-1} & \xrightarrow[g_{n-1}]{} & C_{n-1} & \longrightarrow & 0 \end{array}$$

with exact rows, so the lemma implies that there is an exact sequence of maps

$$0 \to Z_n(A_*) \to Z_n(B_*) \to Z_n(C_*) \xrightarrow{\delta} \frac{A_{n-1}}{d_n A_n} \to \frac{B_{n-1}}{d_n B_n} \to \frac{C_{n-1}}{d_n C_n} \to 0$$

for every n. It follows that the rows of

$$\begin{array}{ccccccc} & & A_n/d_{n+1}A_{n+1} & \longrightarrow & B_n/d_{n+1}B_{n+1} & \longrightarrow & C_n/d_{n+1}C_{n+1} & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & Z_{n-1}(A_*) & \longrightarrow & Z_{n-1}(B_*) & \longrightarrow & Z_{n-1}(C_*) & \end{array}$$

are exact. Applying the lemma a second time yields an exact sequence

$$H_n(A_*) \longrightarrow H_n(B_*) \longrightarrow H_n(C_*) \xrightarrow{\partial_n} H_{n-1}(A_*) \longrightarrow H_{n-1}(B_*) \longrightarrow H_{n-1}(C_*),$$

and the desired long exact sequence is obtained by piecing these ones together. □

Explicitly, the map ∂_n can be described as follows. Given a coset $z + \operatorname{im} d_{n+1}$ in $H_n(C_*)$, let y be a cycle in B_n such that $g_n(y) = z$. Then $d_n y \in B_{n-1}$ actually belongs to $Z_{n-1}(A_*)$ and represents $\partial_n z \in H_{n-1}(A_*)$.

We conclude this section by defining a construction on chain complexes. Just as we can take the tensor product of two modules, there is a notion of tensor product $(C \otimes_R C')_*$ of chain complexes C_* and C'_*. The module in degree n is $(C \otimes_R C')_n = \bigoplus_{p+q=n} C_p \otimes_R C_q$ and the differential map is defined by the formula

$$d_n(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$$

where $|x|$ denotes the degree of x.

5. Koszul Complexes

Let R be a commutative ring. Given an m-tuple $x = (x_1, \dots, x_m)$ of elements of R, we define the *Koszul complex* $K_*(x)$ to be the chain complex

$$0 \longrightarrow \Lambda^m R^m \xrightarrow{d_m} \Lambda^{m-1} R^m \xrightarrow{d_{m-1}} \cdots \longrightarrow R^m \xrightarrow{d_1} R \xrightarrow{d_0} 0$$

with the exterior power $\Lambda^n R^m$ in degree n. The map d_1 takes $v \in R^m$ to the inner product $x \cdot v$, and the other maps are given by

$$d_{n+1}(v_0 \wedge \cdots \wedge v_n) = \sum_{i=0}^{n} (-1)^i d_1(v_i) v_0 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_n$$

where the $\widehat{\ }$ on top of v_i means that this vector is omitted from the product. It is straightforward to check that $K_*(x)$ is indeed a chain complex. Note that the modules $K_n(x)$ in this chain complex are trivial for $n > m$ by anticommutativity of the wedge product.

Let R be a commutative algebra over $\Bbbk$. We call the m-tuple $(x_1, \dots, x_m)$ a *regular sequence* if multiplication by x_i in $R/(x_1 R + \cdots + x_{i-1} R)$ is injective for each i.

Lemma 5.1. *If I is an ideal of R which is generated by a regular sequence, then $K_*(x) \to R/I$ is a projective resolution of R/I.*

Proof. Choose a basis $\{e_1, \dots, e_m\}$ for R^m. Then the elements $e_{i_1} \wedge \cdots \wedge e_{i_n}$ form a basis for $\Lambda^n R^m$ and hence $\Lambda^n R^m \cong R^{\oplus \binom{m}{n}}$. This proves that every module in the Koszul complex is free and hence projective. To finish the proof, we argue by induction on m that

$$0 \longrightarrow \Lambda^m R^m \xrightarrow{d_m} \Lambda^{m-1} R^m \xrightarrow{d_{m-1}} \cdots \longrightarrow R^m \xrightarrow{d_1} R \longrightarrow R/I \longrightarrow 0$$

is an exact sequence. For $m = 1$ the ideal I is generated by a single element $x_1 \in R$, and this sequence is

$$0 \longrightarrow R \xrightarrow{d_1} R \longrightarrow R/I \longrightarrow 0.$$

The projection $R \to R/I$ is surjective, so the kernel of the map $R/I \to 0$ equals the image of $R \to R/I$. The map d_1 is multiplication by x_1, and this element x_1 generates I, so the kernel of $R \to R/I$ equals the image of d_1. Since multiplication by x_1 is assumed to be injective, the kernel of d_1 equals the image of $0 \to R$. This proves exactness in the case $m = 1$.

Suppose now that the sequence is exact for $m = k - 1$ and let us prove that it is exact for $m = k$. If $x = (x_1, \dots, x_k)$ is the regular sequence generating I then we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{1_R} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle d_1} & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow[1_R]{} & R & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the map d_1 is multiplication by the ring element x_k. It follows by inspection that this is a short exact sequence

$$0 \longrightarrow \mathcal{K}_0 \longrightarrow K(x_k) \longrightarrow \mathcal{K}_1 \longrightarrow 0$$

of chain complexes where $\mathcal{K}_0$ is the sequence whose only nonzero term is R in degree 0, and $\mathcal{K}_1$ is the sequence whose only nonzero term is R in degree 1. Tensoring each term of this short exact sequence by the Koszul complex $L = K(x_1, \dots, x_{k-1})$ induces another short exact sequence whose middle term is $L \otimes_R K(x_k) = K(x)$, as the reader can check using our definition of the tensor product. The associated long exact sequence in homology turns out to be

$$\cdots \longrightarrow K_0 \otimes_R H_{n+1}(L) \longrightarrow H_{n+1}(K(x)) \longrightarrow K_1 \otimes_R H_n(L) \xrightarrow{\partial_n}$$
$$\longrightarrow K_0 \otimes_R H_n(L) \longrightarrow H_n(K(x)) \longrightarrow K_1 \otimes_R H_{n-1}(L) \longrightarrow \cdots$$

where $K_0 = K_1 = R$. Using our description of the connecting morphism, one can show that ∂_n is multiplication by x_k. Therefore we have an exact sequence

$$0 \to \operatorname{coker}(H_n(L) \stackrel{x_k}{\to} H_n(L)) \to H_n(K(x)) \to \ker(H_{n-1}(L) \stackrel{x_k}{\to} H_{n-1}(L)) \to 0$$

for every n. Our inductive hypothesis says that the complex L has homology $H_n(L) = 0$ for indices $n > 0$ and $H_0(L) = R/(x_1R + \cdots + x_{k-1}R)$. It follows that $H_n(K(x)) = 0$ for $n > 1$. For $n = 1$ the module $H_1(K(x))$ is the kernel of multiplication by x_k in $R/(x_1R+\cdots+x_{k-1}R)$. Since this multiplication is injective, we get $H_1(K(x)) = 0$. Finally, for $n = 0$ the module $H_0(K(x))$ is the cokernel of multiplication by x_k, and this cokernel equals R/I. This completes the proof by induction. □

A *graded ring* R is a ring with a direct sum decomposition $R = \bigoplus_{i=0}^{\infty} R_i$ into additive subgroups R_i so that if $r \in R_i$ and $s \in R_j$ then $rs \in R_{i+j}$. An algebra A over a ring R is called a *graded algebra* if it is graded as a ring. For example, the collection $\Omega(M)$ of differential forms on a manifold M is an algebra over the ring of smooth functions. We think of $\Omega(M)$ as a direct sum

$$\Omega(M) = \bigoplus_{n=0}^{\infty} \Omega^n(M)$$

where $\Omega^n(M)$ is the vector space of n-forms on M. Note that this sum has only finitely many nonzero terms since there are no nonzero forms of degree greater than the dimension of the manifold. The wedge product operation makes $\Omega(M)$ into a graded algebra. Similarly, if we put $\Omega^n_{A|\Bbbk} = \Lambda^n \Omega^1_{A|\Bbbk}$ then $\Omega^*_{A|\Bbbk}$ is a graded algebra over A. The HKR theorem states that for certain nice algebras A there is an isomorphism $H_*(A, A) \cong \Omega^*_{A|\Bbbk}$ of graded algebras.

The following lemma is an essential ingredient in the proof of the HKR theorem.

Lemma 5.2. *Let R be a commutative ring and I an ideal of R which is generated by a regular sequence in R. Then there is an isomorphism $\mathrm{Tor}^R_*(R/I, R/I) \cong \Lambda^*_{R/I}(I/I^2)$ of graded algebras.*

Proof. Let x be a regular sequence generating I. Since the Koszul complex $K_*(x)$ is a projective resolution of R/I, we may use it to compute $\mathrm{Tor}^R_*(R/I, R/I)$. Tensoring this complex with R/I gives a complex

$$0 \longrightarrow (R/I) \otimes_R \Lambda^m R^m \xrightarrow{1\otimes_R d_m} \cdots \longrightarrow (R/I) \otimes_R R^m \xrightarrow{1\otimes_R d_1} (R/I) \otimes_R R \longrightarrow 0.$$

Our assumption that x generates I implies in particular that every element of x is contained in I. It then follows from the definition of the maps d_n that $1 \otimes d_n = 0$. Therefore the nth homology group of this sequence is

$$\begin{aligned}\mathrm{Tor}^R_n(R/I, R/I) &\cong (R/I) \otimes_R \Lambda^n_R(R^m)\\ &\cong (R/I) \otimes_R R^{\oplus\binom{m}{n}}\\ &\cong ((R/I) \otimes_R R)^{\oplus\binom{m}{n}}\\ &\cong (R/I)^{\oplus\binom{m}{n}}\\ &\cong \Lambda^n_R(R/I)^m.\end{aligned}$$

The sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is exact, and we obtain a short exact sequence of chain complexes by choosing a projective resolution for each term in this sequence. By examining the associated long exact sequence in homology, one finds that $\mathrm{Tor}_1^R(R/I, R/I)$ is the kernel of $R/I \otimes I \to R/I \otimes R$. Now

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^2 & \longrightarrow & I & \longrightarrow & R/I \otimes I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I \otimes R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

is a short exact sequence of chain complexes. Denote the columns from left to right by A_*, B_*, and C_*. By Theorem 4.2 we have an exact sequence

$$H_1(B_*) \longrightarrow H_1(C_*) \longrightarrow H_0(A_*) \longrightarrow H_0(B_*).$$

Now $H_1(B_*) \cong 0$, $H_1(C_*) \cong \mathrm{Tor}_1^R(R/I, R/I)$, $H_0(A_*) \cong I/I^2$, and $H_0(B_*) \cong R/I$. The map $I/I^2 \to R/I$ is the zero map, so we have $\mathrm{Tor}_1^R(R/I, R/I) \cong I/I^2$ by exactness. It follows that $\mathrm{Tor}_n^R(R/I, R/I)$ is an exterior power of $I/I^2 \cong \mathrm{Tor}_1^R(R/I, R/I) \cong (R/I)^m$.

It remains to check that the canonical product on Tor is identified under the isomorphism with the exterior algebra product. This follows from the fact that the wedge product $K(x) \otimes_R K(x) \to K(x)$ is a morphism of complexes lifting $1_{R/I}$. □

6. Noncommutative Differential Forms

In this section we formulate and prove the HKR theorem. This theorem suggests that we can view Hochschild homology classes as differential forms on a noncommutative space. The proof of the HKR theorem involves "localizing" rings at a maximal ideal. This powerful technique comes from commutative algebra and is analogous to the familiar idea of localizing at a point when studying the behavior of a function on the real line.

The construction of the localization of a commutative ring R is similar to the construction of the rational numbers from the integers. Let S be a subset of R which contains 1 and is closed under multiplication. There is an equivalence relation $\sim$ on the set $R \times S$ where $(r_1, s_1) \sim (r_2, s_2)$ if there exists $s \in S$ such that $ss_2r_1 = ss_1r_2$. Let R_S be the set of equivalence classes and denote the equivalence class of the pair (r, s) by the formal fraction r/s. We can define addition and multiplication on R_S by

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2r_1 + s_1r_2}{s_1s_2} \qquad \text{and} \qquad \frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}.$$

These operations are well defined, and they make R_S into a ring which we call the *localization* of R at S.

Similarly, if M is an R-module there is an equivalence relation on $M \times S$, also denoted $\sim$, where $(m_1, s_1) \sim (m_2, s_2)$ if there exists $s \in S$ such that $ss_2m_1 = ss_1m_2$. Let M_S be the set of equivalence classes and denote the class of (m, s) by m/s. The operations of addition and scalar multiplication defined by

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2} \qquad \text{and} \qquad \frac{r}{s_1}\frac{m}{s_2} = \frac{rm}{s_1 s_2}$$

are well defined and make M_S into an R_S-module called the *localization* of M at S. An important example for our purposes is where M is an R-module and $\mathfrak{m}$ is a maximal ideal of M. Since $\mathfrak{m}$ is a prime ideal, it follows that we can localize R at $R - \mathfrak{m}$. By a common abuse of language, we call this the localization at $\mathfrak{m}$ and denote it by $R_{\mathfrak{m}}$.

If $f : M \to M'$ is a morphism of R-modules then the assignment $m/s \mapsto f(m)/s$ is a well defined R_S-linear map $M_S \to M'_S$ which we denote by f_S. Localization at S is therefore a functor from the category of R-modules to the category of R_S-modules. It is straightforward to show that localization commutes with direct sums, quotients, kernels, and projectives. The following result is an algebraic analogue of the fact from differential geometry that a morphism of vector bundles over a manifold is an isomorphism if it restricts to an isomorphism on every fiber.

Theorem 6.1. *A morphism $f : M \to M'$ of R-modules is an isomorphism if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Suppose that $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m} \subseteq R$. Let $x \in \ker f$ and suppose $x \neq 0$. We can choose a maximal ideal $\mathfrak{m} \subseteq R$ containing the annihilator $\operatorname{Ann} x$ of x in R. If x represents the zero class in $(\ker f)_{\mathfrak{m}} = \ker f_{\mathfrak{m}}$, then there exists $s \notin \mathfrak{m}$ such that $sx = 0$. Then we have $s \in \operatorname{Ann} x$, a contradiction. Therefore we must have $x \neq 0$ in $\ker f_{\mathfrak{m}}$. But this contradicts the fact that $f_{\mathfrak{m}}$ is injective. This proves that f is itself injective.

A similar argument with $\ker f$ replaced by $\operatorname{coker} f$ shows that f is surjective. □

To state the HKR theorem, we need to impose a condition on the algebra A which we are thinking of as an algebra of functions. The following notion comes from algebraic geometry where it is related to smooth manifolds.

Definition 6.2. Let A be an algebra over a field $\Bbbk$ with multiplication map $\mu : A \otimes A \to A$. We say that A is *smooth* if it is commutative and the kernel of the localized map $\mu_{\mathfrak{m}} : (A \otimes A)_{\mu^{-1}(\mathfrak{m})} \to A_{\mathfrak{m}}$ is generated by a regular sequence for every maximal ideal $\mathfrak{m}$ of A.

Let us now define a map appearing in the statement of the HKR theorem. Let $\varepsilon_n : M \otimes \Lambda^n A \to M \otimes A^{\otimes n}$ be given by

$$\varepsilon_n(a_0 \otimes a_1 \wedge \cdots \wedge a_n) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

where the sum is taken over all permutations of $\{1, \ldots, n\}$ and $\operatorname{sgn} \sigma$ denotes the sign of a permutation σ. Define $\delta_n : M \otimes \Lambda^n A \to M \otimes \Lambda^{n-1} A$ by the formula

$$\begin{aligned} \delta_n(a_0 \otimes a_1 \wedge \cdots \wedge a_n) &= \sum_{i=1}^{n} (-1)^i [a_0, a_i] \otimes a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_n \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j-1} a_0 \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge \widehat{a_j} \wedge \cdots \wedge a_n. \end{aligned}$$

An argument by induction (see [2]) shows that the diagram

$$\begin{array}{ccc} M\otimes\Lambda^n A & \xrightarrow{\varepsilon_n} & M\otimes A^{\otimes n} \\ \downarrow{\scriptstyle\delta_n} & & \downarrow{\scriptstyle b_n} \\ M\otimes\Lambda^{n-1}A & \xrightarrow[\varepsilon_{n-1}]{} & M\otimes A^{\otimes n-1} \end{array}$$

commutes for every n. In particular, if A is commutative then the commutators $[a_i,a_j]$ all vanish, so $\delta_n=0$ and we have $b_n\circ\varepsilon_n=0$. Therefore we can take homology and get a well defined map $M\otimes\Lambda^n A\rightarrow H_n(M,A)$ which we also denote by ε_n. There is a natural map $M\otimes\Lambda^n A\rightarrow M\otimes\Omega^n_{A|\Bbbk}$ given by $m\otimes a_1\wedge\cdots\wedge a_n\mapsto m\otimes da_1\wedge\cdots\wedge da_n$. The kernel of this map is is generated by elements of the form

$$ma_1\otimes a_2\wedge a_3\wedge\cdots\wedge a_{n+1}+ma_2\otimes a_1\wedge a_3\wedge\cdots\wedge a_{n+1} \\ -m\otimes a_1a_2\wedge a_3\wedge\cdots\wedge a_{n+1}.$$

To show that the domain of the map ε_n is actually $M\otimes\Omega^*_{A|\Bbbk}$ we must therefore show that

$$\varepsilon_n(ma_1\otimes a_2\wedge a_3\wedge\cdots\wedge a_{n+1})+\varepsilon_n(ma_2\otimes a_1\wedge a_3\wedge\cdots\wedge a_{n+1}) \\ -\varepsilon_n(m\otimes a_1a_2\wedge a_3\wedge\cdots\wedge a_{n+1})$$

is a boundary. This is true because this element equals

$$b\left(\sum_\sigma(\operatorname{sgn}\sigma)m\otimes a_{\sigma^{-1}(1)}\otimes\cdots\otimes a_{\sigma^{-1}(n)}\right)$$

where the sum runs over all $\sigma\in S_{n+1}$ with $\sigma(1)<\sigma(2)$. Taking $M=A$ we get a map $\Omega^n_{A|\Bbbk}\rightarrow H_n(A,A)$ which we call the *antisymmetrization map* and denote again by ε_n.

Theorem 6.3 (Hochschild-Kostant-Rosenberg). *For any smooth algebra A over $\Bbbk$, the antisymmetrization map is an isomorphism*

$$\Omega^*_{A|\Bbbk}\cong H_*(A,A)$$

of graded algebras.

Proof. Since A is commutative we have $A\otimes A^{\mathrm{op}}\cong A\otimes A$ and hence $H_n(A,A)\cong\operatorname{Tor}_n^{A\otimes A}(A,A)$. By the previous theorem, it suffices to show that the map

$$(\Omega^n_{A|\Bbbk})_{\mathfrak{m}}\rightarrow(\operatorname{Tor}_n^{A\otimes A}(A,A))_{\mathfrak{m}}$$

obtained by localizing ε_n is an isomorphism of $A_{\mathfrak{m}}$-modules for every maximal ideal $\mathfrak{m}\subseteq A$. One can show that there are isomorphisms

$$(\operatorname{Tor}_n^{A\otimes A}(A,A))_{\mathfrak{m}}\cong\operatorname{Tor}_n^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_{\mathfrak{m}},A_{\mathfrak{m}}).$$

and $(\Omega^n_{A|\Bbbk})_{\mathfrak{m}}\cong\Omega^n_{A_{\mathfrak{m}}|\Bbbk}$ of $A_{\mathfrak{m}}$-modules for every n. It therefore suffices to show that the map

$$\Omega^n_{A_{\mathfrak{m}}|\Bbbk}\rightarrow\operatorname{Tor}_n^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_{\mathfrak{m}},A_{\mathfrak{m}})$$

is an isomorphism of $A_{\mathfrak{m}}$-modules. Let I be the kernel of the localized map $\mu_{\mathfrak{m}}$. Define $R=(A\otimes A)_{\mu^{-1}(\mathfrak{m})}$. Then there are isomorphisms $A_{\mathfrak{m}}\cong R/I$ and $I/I^2\cong\Omega^1_{A_{\mathfrak{m}}|\Bbbk}$, so we need to prove that the map

$$\Lambda^n_{R/I}(I/I^2)\rightarrow\operatorname{Tor}_n^R(R/I,R/I)$$

is an isomorphism. Indeed, this is just the isomorphism from Lemma 5.2. □

This result implies that Hochschild homology is a generalization of the usual graded algebra of differential forms. Given a noncommutative algebra A, we can think of $H_n(A, A)$ as the module of n-forms on the noncommutative space described by A. We now extend this result to get a chain complex of noncommutative differential forms.

There is a natural linear action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on $A^{\otimes n+1}$ defined by

$$t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

where $t_n = t$ is the generator of $\mathbb{Z}/(n+1)\mathbb{Z}$. We define two more operators N and s by $N = 1 + t + t^2 + \cdots + t^n$ and $s(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n$.

Definition 6.4. The *Connes differential* is the map $B = (1 - t)sN$.

One can check (see [2] for details) that $(1-t)b' = b(1-t)$ and $b'N = Nb$ where b' is the boundary map in the bar resolution and we have omitted all subscripts for clarity. Then

$$\begin{aligned} bB + Bb &= b(1-t)sN + (1-t)sNb \\ &= (1-t)(b's + sb')N \\ &= (1-t)N \\ &= 0 \end{aligned}$$

by an earlier computation, so B is a morphism from the complex $C_*(A)$ with $A^{\otimes n+1}$ in degree n to the complex $C_*(A)[1]$ with $A^{\otimes n+2}$ in degree n. It is a convention to multiply the differential by -1 when shifting a complex in this manner. It follows that the Connes differential induces a well defined map on Hochschild homology.

Theorem 6.5. *The diagram*

$$\begin{array}{ccc} \Omega^n_{A|\Bbbk} & \xrightarrow{\varepsilon_n} & H_n(A,A) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle B} \\ \Omega^{n+1}_{A|\Bbbk} & \xrightarrow[\varepsilon_n]{} & H_{n+1}(A,A) \end{array}$$

commutes.

Proof. A straightforward computation shows that

$$\begin{aligned} B(\varepsilon_n(a_0 da_1 \wedge \cdots \wedge da_n)) &= \varepsilon_{n+1}(1 \otimes a_0 \wedge \cdots \wedge a_n) \\ &= \varepsilon_{n+1}(d(a_0 da_1 \wedge \cdots \wedge da_n)). \end{aligned}$$

The result follows by taking homology. □

We have thus specified a cochain complex which reduces to the de Rham complex when A is a smooth algebra. This result illustrates how structures defined on spaces can be defined for noncommutative spaces.

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