TATE'S THESIS ON ZETA FUNCTIONS ON NUMBER FIELDS

JOHN BINDER

ABSTRACT. In this paper, we examine John Tate's seminal work calculating functional equations for zeta functions over a number field k. Tate examines both 'local' properties of k, completed with respect to a given norm, and 'global' properties. The global theory examines the idele and adele groups of k as a way of encoding information from all of the completions of k into single structures, each with its own meaningful topology, measure, and character group. Finally, Tate uses techniques from Fourier analysis, both on the local fields and on the adele group to find functional equations for the zeta functions he defines.

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1. Number Fields: Basic Principles

We begin with a few words on assumed prerequisite knowledge. We assume the reader is familiar with basic point-set topology, group theory, and a little bit of field theory. We assume the reader is familiar with topological groups, and we will use the fact that a topological group has a Haar measure that is unique up to multiplication by a constant. We assume basic properties of characters on those groups (continuous homomorphisms from the group to $\mathbb{C}$), and how the the topology on the group itself relates to the property of the character group. We assume basic knowledge of Fourier analysis, especially the Fourier inversion formula.

Over the course of this work, there will be a number of facts from algebraic number theory that we ask the reader to take as fact. When this happens, we will provide references for further reading.

Throughout this paper, I have followed Tate's Thesis as given in Cassel's and Frohlich. Otherwise, I have used Lang's *Algebraic Number Theory* and Ramakrishnan's *Fourier Analysis on Number Fields* as the main reference works.

With those formalities out of the way, we start with a definition:

Definition 1.1. A finite extension $k/\mathbb{Q}$ is called a *number field.*

The most important subset of k will be the collection of integral elements. We define these elements below:

Definition 1.2. Let k be a number field. We say an element x is *integral* in k (or an *integer* in k) if x satisfies a monic polynomial with coefficients in $\mathbb{Z}$.

We prove an alternate, slightly more workable definition.

Proposition 1.3. *For a number field k, $x \in k$ is integral if and only if there is a finitely-generated $\mathbb{Z}$-module $M \subset k$ with $xM \subset M$.*

Proof. On the one hand, if x satisfies $x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, then the submodule M generated by $1, x, \ldots, x^{n-1}$ satisfies the requirements. On the other hand, if $M = \langle v_1, \ldots, v_m \rangle$, then for each $1 \leq i \leq m$, x satisfies $v_i x = a_{i1}v_1 + \ldots + a_{im}v_m$. It follows that x is an eigenvalue of $[a_{ij}]$, so that it satisfies the characteristic polynomial of that matrix. □

Corollary 1.4. *Given a number field k, the collection of integers is a subring of k.*

Proof. If $xM \subset M$ and $yN \subset N$, where both are finitely generated, then MN is finitely generated (multiplication is inherited from k) and is fixed by both $x + y$ and xy. □

Henceforth, we will use $\mathcal{O}$ to denote the ring of integers in k. We prove:

Proposition 1.5. *The ring $\mathcal{O}$ is an integral domain whose field of fractions is precisely k.*

Proof. Since $\mathcal{O} \subset k$, and k has no zero divisors, neither does $\mathcal{O}$. Since k is closed under division and $\mathcal{O}$ is contained in k, then so is its field of fractions. On the other hand, let $\alpha \in k$ satisfy the integral polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Then $a_n\alpha$ satisfies $y^n + a_{n-1}y^{n-1} + \ldots + a_1 a_n^{n-2} y + a_0 a_n^{n-1}$, so that α is the quotient of an integral element in k and a rational integer. □

Further properties of $\mathcal{O}$ (for instance, that it is Noetherian), can be found in the first chapter of Lang.

Throughout this work, we will examine the ideal structure in $\mathcal{O}$. In particular, each ideal is an $\mathcal{O}$-module. Therefore, it will be helpful to extend our conception of ideals to encompass other finitely generated $\mathcal{O}$ modules. We therefore introduce the concepts of inverse and fractional ideals.

Definition 1.6. Let $I \subset \mathcal{O}$ be an ideal. We define

$$I^{-1} = \{x \in k \mid \forall y \in \ I, xy \in \mathcal{O}\}$$

Definition 1.7. Let I and J be ideals in $\mathcal{O}$. Then we define a $I/J = IJ^{-1} = \{xy \mid x \in I, y \in J^{-1}\}$. Such a structure is called a fractional ideal.

We may multiply fractional ideals in the obvious manner: as subsets of k. The ring $\mathcal{O}$ itself is clearly the identity under this multiplication. Finally, it takes some work to show that $I \cdot I^{-1} = \mathcal{O}$, so that every fractional ideal has an inverse. We refer the reader to the first chapter of Lang for a proof. Once we accept this, we have that the set of fractional ideals in k (excluding k itself and the 0 ideal) form an abelian group under multiplication. This group is called the *ideal group* of k.

Without proof, we accept the fact that the ideal group of k is freely generated by the prime ideals in $\mathcal{O}$ (CITE). Specifically, every fractional ideal has a unique factorization as a product and quotient of prime ideals.

We now consider the principal ideals in $\mathcal{O}$ (i.e. the cyclic $\mathcal{O}$-modules). Since $\mathcal{O}$ contains 1, we know that the product of two cyclic modules is again cyclic (and generated by the product of their generators), and that the inverse of a cyclic module is cyclic and generated by the inverse of the generator. This leads us to define:

Definition 1.8. If Γ is the ideal group of k and H is the subgroup of principal ideals, then Γ/H is known as the *ideal class group* of k, and $|\Gamma : H|$ is called the *class number* of k.

Showing that the class number is finite is nontrivial. Lang completes the proof of this fact at the beginning of chapter V.

One ideal in particular will be of great help to us in the future. Consider the trace map from k to $\mathbb{Q}$. Then:

Definition 1.9. We define the *inverse different* of k as the set

$$\mathcal{D}^{-1} = \{x \in k \mid \forall y \in \mathcal{O},\ \mathrm{Tr}(xy) \in \mathbb{Z}\}$$

Proposition 1.10. *The set $\mathcal{D}^{-1}$ is a proper $\mathcal{O}$-submodule of k containing $\mathcal{O}$.*

Proof. Given a, $b \in \mathcal{D}^{-1}$, $\mathrm{Tr}((a+b)x) = \mathrm{Tr}(ax+bx) = \mathrm{Tr}(ax)+\mathrm{Tr}(bx) \in O_p$ for all $x \in \mathcal{O}$. Similarly, if $y \in \mathcal{O}$, then $\mathrm{Tr}(axy) \in \mathbb{Z} \Rightarrow ax \in \mathcal{D}^{-1}$, so $\mathcal{D}^{-1}$ is an $\mathcal{O}$-module. To see that it is proper, note that if $x \in \mathbb{Q}_p$, then $\mathrm{Tr}(x) = nx$ ($n = [k : \mathbb{Q}]$), so simply pick x where $nx \notin \mathbb{Z}$.

To see $\mathcal{D}^{-1} \supset \mathcal{O}$, it suffices to show that $\mathrm{Tr}(x) \in \mathbb{Z}$ for all $x \in \mathcal{O}$. Recall that $\mathrm{Tr}(x)$ is an integral multiple of the second coefficient in the minimal monic polynomial m_x of x. Let g be the result of 'clearing the denominators' in m_x. By hypothesis, x satisfies a monic polynomial f with integral coefficients. Then $g \mid f$. By Gauss' lemma, their quotient has integral coefficients. This must mean that g is monic, so that $g = m_x$, so that m_x has integral coefficients, completing the proof. □

2. Norms and Completions on Number Fields

All mathematical facts unproved in this section may be found in the beginning chapters of Lang.

We wish to ascribe a metric to k and then complete k with respect to that metric. Just as all norms on $\mathbb{Q}$ are either the standard archimedean norm or the p-adic norm for a prime p, there are two types of norms on k: *archimedean* norms and *$\wp$-adic* norms (which Tate calls *discrete* norms).

We define both types of norms here:

First, the **archimidean** norms. Note that, as a purely algebraic structure, k is isomorphic to some algebraic subfield of $\mathbb{C}$. To ascribe an archimedean norm to k, we simply pick one of these isomorphic subfields of $\mathbb{C}$ and use the induced Euclidean norm. Furthermore, this norm depends on the embedding we choose even though the algebraic structures remain unchanged. Therefore, a number field k may have a number of archimedean norms, and the completion of the field with respect to that norm is either $\mathbb{R}$ or $\mathbb{C}$. In particular, if $k = \mathbb{Q}(\alpha)$, then k will have one real completion for each real Galois conjugate of α and one complex completion for each complex conjugate *pair* of Galois conjugates. (¡–confusing?)

Second, the **$\wp$-adic** or **discrete** norms. These will require some heavier mathematical lifting. We begin with a definition:

Definition 2.1. Let $\wp$ be a prime ideal in $\mathcal{O}$. We define $\mathcal{N}\wp$, called the *norm* of $\wp$, as the number of cosets of $\wp$ in $\mathcal{O}$.

We can extend the norm to all fractional ideals as a homomorphism on the ideal group.

We localize $\mathcal{O}$ with respect to $\wp$ by considering the fraction ring $\mathcal{O}_\wp = \{\frac{a}{b} \mid a \in \mathcal{O}, b \in \mathcal{O} \setminus \wp\}$. Then $\wp\mathcal{O}_\wp$ is the unique maximal ideal in $\mathcal{O}_\wp$, and it is generated by a single element π (we refer the reader to Lang's sections 1.6 and 1.7 for a discussion of Dedekind rings, discrete valuation rings, and a proof of this fact). We define the $\wp$-adic norm on k as follows: if $x = \pi^v \cdot \frac{a}{b}$ where $a, b \in \mathcal{O} \setminus \wp$, then $|x|_\wp = (\mathcal{N}\wp)^{-v}$. This is entirely analogous to the p-adic norm on $\mathbb{Q}$.

If $k_\wp$ is a field with a $\wp$-adic norm, the ideal structure is considerably simplified by the fact that the ring of integers $\mathcal{O}_\wp$ consists precisely of those elements of norm ≤ 1. In particular, we have

Proposition 2.2. *Let $\wp$ be a prime ideal in $\mathcal{O}$ and let k be complete with respect to the $\wp$-adic norm. Then all proper $\mathcal{O}_\wp$ modules of k are of the form $\wp^k$ for some $k \in \mathbb{Z}$. In particular, all ideals in $\mathcal{O}$ are of the form $\wp^k$ for $k \in \mathbb{N}$.*

Proof. It suffices to show that if M is an $\mathcal{O}$, $x \in M$, and $|y| \leq |x|$, then $y \in M$. Fortunately, M is closed under multiplication by $\mathcal{O}$, and $\left|\dfrac{y}{x}\right| \leq 1 \Rightarrow \dfrac{y}{x} \in \mathcal{O} \Rightarrow y \in M$. The second statement follows because all ideals are submodules. □

We say a prime ideal $\wp \subset \mathcal{O}$ *lies above* the prime $p \in \mathbb{Z}$ if $\wp \cap \mathbb{Z} = (p)$. Then the $\wp$-adic absolute value, restricted to $\mathbb{Q}$, is a constant power of the p-adic absolute value, so that the two absolute values induce the same topology. It follows that completing $\mathbb{Q}$ with respect to the $\wp$-adic absolute value yields $\mathbb{Q}_p$, so that $k_\wp \supset \mathbb{Q}_p$. In particular, this allows us to define a trace map from $k_\wp$ to $\mathbb{Q}_\wp$, and a corresponding definition of a local inverse different:

Definition 2.3. Let $\wp$ lie over p. Then the *local inverse different* of $k_\wp$ is defined as

$$\mathcal{D}_\wp^{-1} = \{x \in k \mid \forall y \in \mathcal{O}_\wp,\ \mathrm{Tr}(xy) \in O_p\}$$

Where O_p is the p-adic ring of integers.

Following the proof of proposition 1.10, we have that the local inverse different is a proper $\mathcal{O}_\wp$-submodule of k that contains the local ring of integers. With proposition 2.1 in hand, we have that the local inverse different must be equal to $\wp^k$ for some non-positive integer k.

Finally, we present one more definition and extraordinary fact:

Definition 2.4. A prime $\wp$ is called *ramified* if it satisfies the three equivalent conditions:

- The prime $\wp$ divides the global different $\mathcal{D}$ in the ideal group.
- If $\wp$ lies above p, then $\wp^2 \cap \mathbb{Z} \neq (p)$.
- The local different $\mathcal{D}_\wp$ is **strictly** contained in the local ring of integers $\mathcal{O}_\wp$.

A prime not satisfying these conditions is called *unramified.*

Proving that these three properties are equivalent takes a bit of work. For a full discussion of ramification, we invite the reader to see Cassels and Frohlich, pages 18-22. Lang includes a further discussion relating the ramified primes to the global different.

The remarkable fact to take away from this 'profinition' is that, for all but finitely many primes $\wp$, we have $\mathcal{D}_\wp = \mathcal{O}_\wp$ (since only finitely many primes divide the global different). This fact will be of great use to us when we examine the global theory.

3. The Local Theory

Throughout this section, we work locally in a field k completed with respect to a given norm (we will examine means of simultaneously examining information from all norms simultaneously when we consider the global theory). In general, we denote a given $\wp$-adic completion by $k_\wp$ with *localized* ring of integers $\mathcal{O}_\wp$ and local inverse different $\mathcal{D}_\wp^{-1}$. In this section, however, we drop the pesky subscript and simply use k, $\mathcal{O}$, and $\mathcal{D}^{-1}$ respectively.

We wish to do Fourier analysis on k. Therefore, we must first examine the characters of the additive group of k (henceforth denoted k^+).

3.1. **The Additive Characters.** We begin with a definition:

Definition 3.1. A character on a topological group G is a continuous homomorphism from G to S^1.

The characters on a locally compact group G form a group under multiplication. We call this group $\widehat{G}$, and ascribe to it the compact-open topology. We assume basic knowledge of how the topology on G relates to the topology on $\widehat{G}$. We also assume the Pontryagin duality, which says $\widehat{\widehat{G}} \cong G$, where the characters on $\widehat{G}$ are induced by evaluation of a member of $\widehat{G}$ at an element of G. We refer the reader to the third chapter of Ramakrishnan for further explanation.

In this section, we wish to examine the character group $\widehat{k^+}$ of k^+.

Proposition 3.2. *The character group $\widehat{k^+}$ is nontrivial.*

Proof. We will construct a special character χ for each completion.

If k is real, define $\Lambda(x) = -x$. If k is complex, define $\Lambda(x) = -2\mathrm{Re}(x) = -\mathrm{Tr}(x)$. In either case, Λ is nontrivial, continuous, and additive.

If k is $\wp$-adic, with $\wp$ lying above p, then k is a finite extension of $\mathbb{Q}_p$, so there is a continuous, additive trace map from k to $\mathbb{Q}_p$. Once in $\mathbb{Q}_p$, we have a map $\lambda : \mathbb{Q}_p \to \mathbb{R}$ given by the tail of a number's p-adic expansion. That is, if $x = \sum_{j=v}^{\infty} a_j p^j$, then we set $\lambda(x) = \sum_{j=v}^{-1} a_j p^j$. It is easy to check that this map is both continuous and additive modulo 1. We therefore set $\Lambda(x) = \lambda(\mathrm{Tr}(x))$ in the $\wp$-adic case. The map Λ is again nontrivial since λ's kernel is precisely the p-adic integers, and it is both continuous and additive.

In all cases, we set $\chi(x) = e^{-2\pi i \Lambda(x)}$, giving us our desired nontrivial character. □

From here, we have

Theorem 3.3. *The topological groups $\widehat{k^+}$ and k^+ are isomorphic.*

Proof. We will show the map $\eta \mapsto \chi(\eta x)$ is an isomorphism from k^+ to its character group. This function is clearly a character. We have $\chi((\eta_1+\eta_2)x) = \chi(\eta_1 x)\cdot\chi(\eta_2 x)$. Since this holds for all x, this map is a homomorphism. For $\eta \neq 0$, $\eta \cdot k = k$, and since χ is nontrivial, so is $\chi(\eta x)$, so that the map is injective.

To prove surjectivity, we first show that the characters of the form $x \mapsto \chi(\eta x)$ are dense in the character group. Let H be the subgroup consisting of these characters, and let I be its topological closure in the character group. Then $\widehat{k^+}/I$ is itself a topological group. Assume $I \neq \widehat{k^+}$, so that $\widehat{k^+}/I$ is not the trivial group. Then, there is a nontrivial character on $\widehat{k^+}$ that is trivial on I. By the Pontryagin duality, this character consists of evaluation at a certain $y \in k^+$. Since this character is trivial on I, then we must have $\chi(y \cdot x)$ trivial. This is possible only if $y = 0$. But evaluation at 0 always yields 1, so that the only trivial character on I is trivial on all of $\widehat{k^+}$. It follows that I is the entirety of the character group.

Finally, we wish to prove the bicontinuity of this map, proving that H is locally compact, and therefore closed, in the character group, completing the proof. Let $B = \{x \in k \mid |x| \leq M\}$. As $\eta \to 0$, then $\eta B \to 0$, so that $\chi(\eta B) \to 1$ in $\mathbb{C}$. Since this we may use this to map any such B into any open neighborhood of 1 in $\mathbb{C}$, then $\chi(\eta x)$ approaches the identity in the character group under the compact-open topology. On the other hand, let x_0 be a fixed element of k with $\chi(x_0) \neq 1$. As $\chi(\eta \cdot x)$ approaches the identity character, eventually we must have $\chi(\eta B)$ closer to 1 than $\chi(x_0)$. This implies that $x_0 \notin \eta B$, which is only possible if $|\eta$ is sufficiently small. Therefore, the map is bicontinuous, so that k^+ is algebraically and topologically isomorphic to its character group. □

In the case that our norm is $\wp$-adic, the local inverse different plays a particularly important role:

Proposition 3.4. *For all $\eta \in \mathcal{O}$, the character $x \mapsto \chi(\eta x)$ is trivial on $\mathcal{D}^{-1}$.*

Proof. Let $\wp$ lie above p, and note that $\mathrm{Tr}(y) \in O_p \Rightarrow \chi(y) = 1$, so that if $x \in \mathcal{D}^{-1}$, $\eta \in \mathcal{O}$, then $\chi(\eta x) = 1$. □

In future calculations, we would like to talk about the size of $\mathcal{D}^{-1}$ relative to $\mathcal{O}$. Therefore, we define $\mathcal{D}$ as the inverse ideal of $\mathcal{D}^{-1}$; then the quantity $\mathcal{N}\mathcal{D}$, or the number of cosets of $\mathcal{D}$ in $\mathcal{O}$, is well-defined, and the number of distinct additive translates of $\mathcal{O}$ inside $\mathcal{D}^{-1}$ will be $\mathcal{N}\mathcal{D}$. Since $\mathcal{D}$ is a power of $\wp$, we also have that $\mathcal{N}\mathcal{D}$ will be some power of $\mathcal{N}\wp$.

3.2. **Additive Haar Measure and Absolute Value.** A Haar measure on a locally compact group G is a measure on G for which compact sets have finite measure, open sets have positive measure, and the measure is invariant under multiplication by a group element. The existence of a Haar measure, and its uniqueness up to a multiplicative constant, is assumed. We refer the interested reader to the first chapter of Ramakrishnan for further explanation.

Let μ be a Haar measure on k. By examining this Haar measure, we can define the absolute value of an element $x \in k$ by the amount it 'stretches' a set. That is, we can define $|x|$ such that $\mu(xM) = |x|\mu(M)$. Such a quantity is well-defined, since $\mu_x(M) := \mu(xM)$ is itself an additive Haar measure, and the Haar measure is unique up to a constant. Moreover, this quantity is multiplicative, since $\mu(xyM) = |x|\mu(yM) = |x||y|\mu(M)$.

From this perspective, it is clear that the 'natural' absolute value depends on the completion in which we work. When k is a real field, we define $|x|$ as the standard real absolute value. When k is complex, we use the *square* of the standard absolute value.

For $\wp$-adic k, we defined $|\pi^v a| = (\mathcal{N}\wp)^{-v}$, where a is a unit in $\mathcal{O}$ (we denote the group of units in $\mathcal{O}$ by $\mathcal{U}$). We will show here that this absolute value is 'natural' from the above perspective. If $|x| = (\mathcal{N}\wp)^{-v}$, then x is in the annulus $\wp^v \setminus \wp^{v+1}$, so that $x\mathcal{O} = \wp^v$. If $v \geq 0$, then there are exactly $\mathcal{N}\wp^v$ cosets of $\wp^v$ in $\mathcal{O}$, so that $\frac{\mu(x\mathcal{O})}{\mu(\mathcal{O})} = \mathcal{N}\wp^{-v} = |x|$. If $v < 0$, it suffices to examine x^{-1}.

3.3. **Self-Dual Additive Haar Measures.** The 'big idea' behind Tate's thesis is that we can find functional equations for zeta functions by applying Fourier analysis. Recall the definition of a Fourier transform over a self-dual group:

Definition 3.5. Let $f \in L^1(G)$ (that is, $|f|$ is integrable). Then we define

$$\hat{f}(y) = \int_G f(x)\chi(xy)\,dx$$

Moreover, for functions f such that $\hat{f}$ is also L^1, we have the following fact from Fourier analysis, proved in section 3.3 of Ramakrishnan):

$$\hat{\hat{f}}(x) = n \cdot f(-x) \text{ for some constant } n \tag{3.6}$$

For future calculations, it will be important to define a Haar measure dx such that the constant in the above equation is 1. To this end, we set

- dx = standard Lesbegue measure for real k
- dx = twice standard Lesbegue measure for complex k
- dx such that $\mathcal{O}$ gets measure $\mathcal{N}\mathcal{D}^{-1/2}$ for $\wp$-adic k

Proposition 3.7. *If we define dx as above, we have $\hat{\hat{f}}(x) = f(-x)$.*

Proof. By equation 3.6, it is sufficient to check only one function. For real k, we use $f(x) = e^{-\pi x^2}$. For complex k, we use $f(x) = e^{-\pi|x|}$. For $\wp$-adic k, we use the characteristic function of $\mathcal{O}$.

To save space, we will only carry out computations for the latter case. We have

$$\hat{f}(y) = \int_{\mathcal{O}} \chi(xy)\, dx$$

If we define $\chi_y(x) = \chi(xy)$, then χ_y is trivial on $\mathcal{O}$ if and only if $y \in \mathcal{D}^{-1}$. Since χ_y is a character and $\mathcal{O}$ is an additive subgroup, then this integral will be 0 if $x \notin \mathcal{D}^{-1}$, and it will be $\mu(\mathcal{O}) = \mathcal{N}\mathcal{D}^{-1/2}$ if $x \in \mathcal{D}^{-1}$.

Then

$$\hat{\hat{f}}(x) = \int_{\mathcal{D}^{-1}} \mathcal{N}\mathcal{D}^{-1/2}\chi(xy)\, dy$$

As before, this character is trivial on the domain if and only if $x \in \mathcal{O}$, in which case the integral comes to $\mathcal{N}\mathcal{D}^{-1/2}\mu(\mathcal{D}^{-1}) = \mathcal{N}\mathcal{D}^{-1/2}((\mu(\mathcal{O})(\mathcal{N}\mathcal{D})) = 1$. Otherwise, if the character is nontrivial, then the integral is 0 since $\mathcal{D}^{-1}$ is an additive subgroup, so that $\hat{\hat{f}}(x) = f(-x)$, and the measure is self-dual. □

Note in particular that $\mathcal{D} = \mathcal{O}$ for all but finitely many primes, so that $\mu(\mathcal{O}) = 1$ in all but finitely many $\wp$-adic extensions. This will be of use in the global theory.

3.4. Multiplicative Haar Measure.

We will characterize the multiplicative Haar measure with respect to the absolute values we defined in section 3.2.

Proposition 3.8. *The measure* $\mu^*(M) = \int_M \frac{dx}{|x|}$ *is a multiplicative Haar measure.*

Proof. Because $\mu(xM) = |x|\mu(M)$, we have $\int f(x)\, dx = |a| \int f(ax)\, dx$. Specifically, for any f, we have

$$\int f(x)\frac{dx}{|x|} = |a| \int f(ax)\frac{dx}{|ax|} = \int f(ax)\frac{dx}{|x|}$$

Therefore, the measure $d^*x = \frac{dx}{|x|}$ is invariant under multiplicative translations. Taking f to be the characteristic function of M proves the proposition. □

For archimedean k, the measure $d^*x = \frac{dx}{|x|}$ will suffice. However, when k is $\wp$-adic, then the subgroup of elements in absolute value 1 in k, henceforth denoted $\mathcal{U}$, is simultaneously compact and open. We wish for $\mathcal{U}$ to have the same *multiplicative* measure as $\mathcal{O}$ has *additive* measure, so that in particular we have $\mu^*(\mathcal{U}) = 1$ for all but finitely many completions. Therefore, in the $\wp$-adic case, we set

$$d^*x = \frac{\mathcal{N}\wp}{\mathcal{N}\wp - 1}\frac{dx}{|x|}$$

3.5. Multiplicative Characters.

In this section, we discuss the the characters on the multiplicative group k^* of k. The major difference between theory of the multiplicative characters and the additive characters is that we will now allow functions from k^* to all of $\mathbb{C}$, instead of restricting our range to the circle of elements with absolute value 1. We begin with some notation:

Definition 3.9. We call a character c *unitary* if $|c(x)| = 1$ for all $x \in k^*$. Otherwise, we call it a *quasi-character*.

Definition 3.10. We call a quasi-character c *unramified* if it is trivial on $\mathcal{U}$. Otherwise, we call it *ramified*.

Since $\mathcal{U}$ is multiplicative in all cases, and the elements of absolute value 1 are the maximal compact subgroup of $\mathbb{C}$, then all characters on $\mathcal{U}$ are unitary.

We begin by classifying the unramified quasi-characters.

Proposition 3.11. *The unramified quasi-characters are those characters of the form $x \to |x|^s$, where s is determined uniquely if k is archimedean or determined up to an integer multiple of $\frac{2\pi i}{\log(\mathcal{N}\wp)}$ if k is $\wp$-adic.*

Proof. If c is trivial on $\mathcal{U}$, then it must be constant on all sets with the same absolute value, so that $c(x)$ is actually a quasi-character on the group of values attained by the absolute value function. In the archimedean case, the value group is $\mathbb{R}^+_\times$. We leave it as an exercise that the quasi-characters on $\mathbb{R}^+_\times$ are those specified.

On the other hand, in the $\wp$-adic case, the value group is restricted to integer powers of $\mathcal{N}\wp$. Therefore, for any $x \in k^*$, $s \in \mathbb{C}$, we have

$$\begin{aligned} |x|^{s+2\pi i/\log(\mathcal{N}\wp)} &= |x|^s \cdot |x|^{2\pi i/\log(\mathcal{N}\wp)} = |x|^s \cdot (\mathcal{N}\wp)^{v\cdot 2\pi i/\log(\mathcal{N}\wp)} \\ &= |x|^s \cdot e^{\log(\mathcal{N}\wp)\cdot v\cdot 2\pi i/\log(\mathcal{N}\wp)} = |x|^s \end{aligned}$$

since $v \in \mathbb{Z}$. □

From a geometric standpoint, n the archimedean case, the space of unramified quasi-characters looks like a copy of the complex plane. On the other hand, since the quasi-characters in the $\wp$-adic case are only defined modulo a real multiple of i, then the space of quasi-characters in this space is an infinite cylinder.

Armed with this knowledge, we wish to examine the space of all quasi-characters. To this end, we call two quasi-characters equivalent if their quotient is unramified. Such a relation is clearly an equivalence relation. To define an equivalence class, we need simply to understand how a quasi-character acts on $\mathcal{U}$; once this is accomplished, we can map $x \in k^*$ down to $\tilde{x} \in \mathcal{U}$ using $\tilde{x} = x/|x|$ the real case, $\tilde{x} = x/\sqrt{|x|}$ in the complex case, and $\tilde{x} = x/\pi^v$ where π is the generator of $\wp$ from before and v is the $\wp$-adic valuation of x, in the $\wp$-adic case. Then $\tilde{x} = x$ on $\mathcal{U}$, and so given any quasi-character c whose restriction to $\mathcal{U}$ is $\tilde{c}$, then the quasi-character $x \mapsto c(x)/\tilde{c}(\tilde{x})$ is unramified and hence is of the form $|x|^s$, with s determined as above. It follows that the space of quasi-characters is a collection of the above surfaces, with each surface indexed by a character $\tilde{c}$ on $\mathcal{U}$.

Specifically, this tells us that, for any quasi-character c, that $|c(x)| = ||x|^s| = |x|^\sigma$, for $\sigma = \mathrm{Re}(s)$. We call σ the *exponent* of c.

Finally, we examine the characters $\tilde{c}$.

In the real case, $\mathcal{U} = \{\pm 1\}$, so that there are precisely two characters on $\mathcal{U}$: the identity character and the trivial character. Therefore, the space of quasi-character in the real case is a pair of complex planes.

In the complex case, $\mathcal{U}$ is the circle of absolute value 1. The characters on this subgroup are of the form $x \mapsto e^{2\pi i n x}$, $n \in \mathbb{Z}$. Therefore, the space of quasi-characters here is a countable set of complex planes indexed by the integers.

In the $\wp$-adic case, we accept as fact that the subsets $1 + \wp^n$ are multiplicative subgroups of $\mathcal{U}$ for $n \geq 1$. Pick a ball around 1 in $\mathbb{C}$ not containing any nontrivial multiplicative subgroups: then for sufficiently large n, $\tilde{c}(1+\wp^n)$ must be inside this ball, and therefore must be 1. Let n be minimal so that $\tilde{c}(1+\wp^n) = 1$; then $\wp^n$ is called the conductor of $\tilde{c}$. The collection of characters on $\mathcal{U}$ with conductor $\wp^n$ is actually the collection of characters on the quotient group $\mathcal{U}/(1+\wp^n)$; since this group is finite, there are finitely many such characters. Since there are countably many possibilities for the conductor, then there are only countably many characters on $\mathcal{U}$. It follows that the space of quasi-characters in the $\wp$-adic case is a countable collection of cylinders.

3.6. **Local Zeta Functions.** We begin with a definition:

Definition 3.12. For a sufficiently nice function $f : k^* \to \mathbb{C}$, and a quasi-character c of exponent > 0, we define

$$\zeta(f, c) = \int_{k^*} f(x)c(x)\, d^*x$$

The term 'sufficiently nice' differs by author. Tate uses the space of functions f where both f and $\hat{f}$ are continuous and in $L^1(k^+)$, and $f(x)|x|^\sigma$, $\hat{f}(x)|x|^\sigma \in L^1(k^*)$. Sally, on the other hand, restricts the analysis to a space of test functions, such as the Schwartz space on $\mathbb{R}$ or $\mathbb{C}$, or the space of locally constant, compactly supported functions on $\wp$-adic fields. The latter approach has the advantage that those function spaces are easy to work with, invariant under the Fourier transform, and dense in L^p spaces for $1 \leq p < \infty$. For the analysis present this paper, either definition will suffice, so we will use Sally's simpler definition. Call this space $\mathcal{S}$.

Recall that each equivalence class of quasi-characters is a surface in the shape of either the whole complex plane, or a quotient group of the complex plane. In this context, it is reasonable to apply ideas such as analytic continuation (in the sense of complex analysis) to functions on the character group. At least, if we have a function on the character space, we may speak of analytic continuation from one subset of an equivalence class to a larger subset.

We would like to apply these ideas to the local zeta functions. First, we need for $\zeta(f, c)$ to be holomorphic in the domain of characters with exponent > 0. However, because of the restrictions we placed on f, the integral $\int f(x)c(x)\, d^*x$ is absolutely convergent, so that the function it defines is holomorphic in the character group.

Finally, we give a definition and prove the *Main Lemma* of the local theory:

Definition 3.13. For a quasi-character c, define $\hat{c}(x) = |x|c(^{-1}(x)$. We should remark that $\hat{c}$is **not** the Fourier Transform of c as a function on k^+, but rather a sort of inverse character.

We note that, if c has exponent σ, then $\hat{c}$ has exponent $1 - \sigma$. Then we hwve

Theorem 3.14. *Then for any f, $g \in \mathcal{S}$, and c with exponent between 0 and 1 we have*

$$\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \zeta(g, c)\zeta(\hat{f}, \hat{c})$$

Specifically, we have $\zeta(f,\, c) = \rho(c)\zeta(\hat{f},\, \hat{c})$, where ρ is independent of f.

Proof. We have

$$\zeta(f,\, c)\zeta(\hat{g},\, \hat{c}) = \int f(x)c(x)\, d^*x \int \hat{g}(y)c^{-1}(y)|y|\, d^*y$$

We use the substitution $y \mapsto xy$ in the second integral, leaving the integral unchanged, so that

$$\zeta(f,\, c)\zeta(\hat{g},\, \hat{c}) = \int f(x)c(x)\, d^*x \int \hat{g}(xy)c^{-1}(xy)|xy|\, d^*y$$
$$= \int \int f(x)\hat{g}(xy)c(y^{-1})|xy|\, d^*x\, d^*y$$

$$= \int \int f(x) \left(\int g(z)\chi(xyz)\, dz \right) c(y^{-1})|xy|\, d^*x\, d^*y$$

Since $dz = K|z|\, d^*z$, for a constant K depending only on k, the above quantity simplifies to

$$= K \int \left(\int \int (f(x)g(z)\chi(xyz)|xz|\, d^*x\, d^*z \right) |y|\, d^*y$$

$$= K \int \left(\int \int (f(x)g(z)\chi(xyz)|z| d^*x\, d^*z \right) |y|\, d^*y$$

$$= K \int \int \hat{f}(yz)g(z)|yz|c(y^{-1})\, d^*x\, d^*z$$

Which is known to equal $\zeta(g,\, c)\zeta(\hat{f},\, \hat{c})$ by the above calculations. Since this holds for all appropriate functions f, g, then there must be some function ρ depending only on c such that $\zeta(f,\, c) = \rho(c)\zeta(\hat{f},\, \hat{c})$. □

This lemma is important because it gives us a means of analytically continuing our local zeta function, defined only on quasi-characters of exponent $\sigma > 0$, to the space of all quasi-characters. Since $\hat{(c)}$ is again a quasi-character with exponent $1 - \sigma$, and the above formula holds on all quasi-characters in the open region $0 < \sigma < 1$, then the formula gives an analytic continuation, provided we find ρ. This is the goal of the next section.

3.7. **Explicit Calculations for $\rho(c)$.** Tate goes through explicit detail in calculating the functional equations for local zeta functions under real, complex, and $\wp$-adic norms. As above, we will not display all calculations. Rather, we will restrict our attention to the $\wp$-adic cases, where the calculations are most informative. The interested reader may examine the real and complex calculations in Tate's section 2.5.

Recall that the additive Haar measure, dx, is normalized to give the integer ring $\mathcal{O}$ measure of $(\mathcal{N}\mathcal{D})^{-1/2}$, and that $d^*x = \frac{\mathcal{N}\wp}{\mathcal{N}\wp - 1}\frac{dx}{|x|}$ gives the multiplicative subgroup U the same measure.

Recall that two multiplicative characters are equivalent if their quotient is ramified. Let c' be any *unitary* character with conductor $\wp^n$. Then c' is equivalent to a unitary character c with the same conductor, and with $c(\pi) = 1$.

We wish to calculate

$$\rho(c) = \frac{\zeta(f,\, c)}{\zeta(\hat{f},\, \hat{c})}$$

Since $\rho(c)$ is independent of f, we have the luxury of hand-picking an f that is sufficiently easy to work with. We set:

$$f_n(x) = \begin{cases} \chi(-x) & \text{if } x \in \mathcal{D}^{-1}\wp^n, \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.15. *We have*

$$\hat{f}_n(x) = \begin{cases} (\mathcal{N}\mathcal{D})^{-1/2}(\mathcal{N}\wp)^{-n} & \textit{if } x \in \wp^n + 1, \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We have

$$\hat{f}_n(x) = \int f_n(y)\chi(xy)\,dy = \int_{\mathcal{D}^{-1}\wp^{-n}} \chi(y(x-1))\,dy$$

This integral will vanish if $\chi(y(x-1))$ is nontrivial on $\mathcal{D}^{-1}\wp^{-n}$, which occurs if if $(x-1)\wp^{-n} \not\subset \mathcal{O}$ or $x \notin \wp^n + 1$. Otherwise, the character in question is trivial on the additive subgroup, so that the integral in question is simply the additive measure of the set for $x \in \wp^n$. □

For the purpose of our calculations, it will be useful to partition k^* into the annuli $A_v = \{|x| = (\mathcal{N}\wp)^{-v}\}$, so that A_v for large v are close to 0. Note that all such annuli are multiplicative translates of one another, so that they all have the same measure under d^*x: $\mathcal{N}\mathcal{D}^{-1/2}$.

We now wish to calculate zeta functions on each equivalence class, using the function f_n on all equivalence classes of quasi-characters with conductor $\wp^n$. We begin with the equivalence class of unramified quasi-conductors: those of the form $|\cdot|^s$. In this case, f_0 is simply the characteristic function of $\mathcal{D}^{-1}$.

Let $\mathcal{D} = \wp^d$, so that $\mathcal{D}^{-1} = \wp^{-d} = \cup_{v=-d}^{\infty} A_v$. Then we have

$$\begin{aligned}
\zeta(f_0, |\cdot|^s) &= \sum_{v=-d}^{\infty} \int_{A_v} |x|^s\,d^*x \\
&= \sum_{v=-d}^{\infty} (\mathcal{N}\wp)^{-vs}(\mathcal{N}\mathcal{D})^{-1/2} \\
&= \frac{(\mathcal{N}\wp)^{ds}}{1-(\mathcal{N}\wp)^{-s}}(\mathcal{N}\mathcal{D})^{-1/2} \\
&= \frac{(\mathcal{N}\mathcal{D})^{s-1/2}}{1-(\mathcal{N}\wp)^{-s}}
\end{aligned}$$

Recall that $\hat{f}_0$ is $(\mathcal{N}\mathcal{D})^{1/2}$ times the characteristic function of $\mathcal{O}$, and since $\widehat{|\cdot|^s} = |\cdot|^{1-s}$, we have

$$\begin{aligned}
\zeta(\hat{f}_0, \widehat{|\cdot|^s}) &= \mathcal{N}\mathcal{D}^{1/2} \int_{\mathcal{O}} |x|^{1-s}\,d^*x \\
&= \sum_{v=0}^{\infty} (\mathcal{N}\wp)^{(s-1)v} = \frac{1}{1-(\mathcal{N}\wp)^{s-1}}
\end{aligned}$$

since Re(s) ¡ 1. Therefore, for the unramified quasi-characters $|\cdot|^s$, we have

$$\rho(|\cdot|^s) = (\mathcal{N}\mathcal{D})^{s-1/2}\frac{1-(\mathcal{N}\wp)^{s-1}}{1-(\mathcal{N}\wp)^{-s}}$$

We now treat the ramified case: let c be a quasi-character of with conductor $\wp^n$, $n \geq 1$, and $c(\pi) = 1$. Then

$$\zeta(f_n, c|\cdot|^s) = \int_{\mathcal{D}^{-1}\wp^{-n}} \chi(-x)c(x)|x|^s\,d^*x = \sum_{v=-d-n}^{\infty} (\mathcal{N}\wp)^{-vs} \int_{A_v} \chi_0(-x)c(x)\,d^*x$$

We will show that $\int_{A_v} \chi(-x)c(x)\,d^*x = 0$ for all $v > -d-n$.

Case 1: $v \geq -d$. In this case, $A_v \subset \mathcal{D}^{-1}$, so that χ is trivial on this annulus, and the integral is $\int_{A_v} c(x)\,d^*x$. Using the substitution $x \mapsto \pi^v x$, that $c(\pi) = 1$, and that c is nontrivial on the subgroup $\mathcal{U} = A_0$, this integral becomes

$$= \int_{\mathcal{U}} c(\pi^v x)\,d^*x = \int_{\mathcal{U}} c(x)\,d^*x = 0$$

Case 2: $-d-n < v < -d$. To handle this case, we write A_v as a disjoint union of balls $x_0 + \mathcal{D}^{-1} = x_0(1+\wp^{-d-v})$. We know χ will be constant on all such subsets (and $= \chi(x_0)$) and then we have

$$\int_{x_0+\mathcal{D}^{-1}} \chi(-x)c(x)\,d^*x = \chi(-x_0)\int_{x_0+\mathcal{D}^{-1}} c(x)\,d^*x$$

The integral above is 0, because if we apply the substitution $x \mapsto x_0 \cdot x$, we have

$$\int_{x_0+\mathcal{D}^{-1}} c(x)\,d^*x = \int_{1+\wp^{-d-v}} c(x_0\cdot x)\,d^*x = c(x_0)\int_{1+\wp^{-d-v}} c(x)\,d^*x$$

which is equal to 0 because $1+\wp^{-d-v}$ is a subgroup on which c is nontrivial.

We have now shown that

$$\zeta(f_n, c|\cdot|^s) = (\mathcal{N}\wp)^{(d+n)s}\int_{A_{-d-n}} \chi(-x)c(x)\,d^*x$$

where the integral evaluates to some complex constant z_c. After we evaluate $\zeta(\hat{f}_n, \widehat{c|\cdot|^s})$, we will be able to show that z_c is nonzero. However, since $\hat{f}_n$ is $(\mathcal{N}\mathcal{D})^{-1/2}(\mathcal{N}\wp)^{-n}$ times the characteristic function of $1+\wp^n$, a subgroup upon which c is constant, we have

$$\zeta(\hat{f}_n, \widehat{c|\cdot|^s}) = (\mathcal{N}\mathcal{D})^{-1/2}(\mathcal{N}\wp)^{-n}\int_{1+\wp^n} c^{-1}(x)|x|^{1-s} = (\mathcal{N}\mathcal{D})^{-1/2}(\mathcal{N}\wp)^{-n}\int_{1+\wp^n} d^*x$$

which comes to $(\mathcal{N}\mathcal{D})^{-1/2}$. Therefore, for ramified c, we have

$$\rho(c|\cdot|^s) = (\mathcal{N}\mathcal{D})^{s-1/2}(\mathcal{N}\wp^n)^s z_c = \mathcal{N}(\mathcal{D}\wp^n)^{s-1/2}(\mathcal{N}\wp)^{-n/2}z_c$$

The reason we use this convoluted form is to use the following lemma to show that ρ is nonzero.

Lemma 3.16. *For any quasi-character c of exponent $1/2$, we have $|\rho(c)| = 1$.*

Proof. If c has exponent $1/2$, $|c(x)||\bar{c}(x)| = |c(x)|^2 = |x| = |\bar{c}(x)||\hat{c}(x)|$, so that $|\bar{c}(x)| = |\hat{c}(x)|$.

On the one hand, we have

$$\zeta(f,c) = \rho(c)\zeta(\hat{f},\hat{c}) = \rho(c)\rho(\hat{c})\zeta(\hat{\hat{f}},\hat{\hat{c}})$$

Since $\hat{\hat{f}}(x) = f(-x)$ and $\hat{\hat{c}} = c$, we use the substitution $x \mapsto -x$ to find $\zeta(\hat{\hat{f}},\hat{\hat{c}}) = c(-1)\zeta(f,c)$. Therefore, $\rho(c)\rho(\hat{c}) = c(-1)$.

On the other hand, using the substitution $x \mapsto -x$ in the formula for $\hat{f}$ yields $\bar{\hat{f}}(x) = \hat{\bar{f}}(-x)$. Since $\bar{\hat{c}} = \hat{\bar{c}}$, we have $\zeta(\hat{\bar{f}},\hat{\bar{c}}) = c(-1)\zeta(\bar{\hat{f}},\bar{\hat{c}})$ (using the same substitution as above in the integral defining ζ). This yields

$$\overline{\zeta(f,c)} = \zeta(\bar{f},\bar{c}) = \rho(\bar{c})\zeta(\hat{\bar{f}},\hat{\bar{c}}) = \rho(\bar{c})c(-1)\zeta(\bar{\hat{f}},\bar{\hat{c}}) = \rho(\bar{c})c(-1)\overline{\zeta(\hat{f},\hat{c})}$$

Since $\overline{\zeta(f,c)} = \overline{\rho(c)}\overline{\zeta(\hat{f},\hat{c})}$, we have $\rho(\bar{c}) = c(-1)\overline{\rho(c)}$.

Combining these formulas yields $\rho(c)\overline{\rho(c)} = 1$. ∎

In particular, the above lemma gives us that $z_c \neq 0$. Since $c|\cdot|^{1/2}$ has exponent $1/2$, then $1 = |\rho(c|\cdot|^{1/2})| = (\mathcal{N}\wp)^{-n/2} z_c$.

The $\wp$-adic calculations are the most interesting and informative ones, which is why they have been included. We will state the values of ρ for the archimedean calculations without proof.

For real completions, there are two equivalence classes, corresponding to the two characters on $\{\pm 1\}$. For the unramified class, we have

$$\rho(|\cdot|^s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)$$

whereas for the other equivalence class, where $c(-1) = -1$, we have

$$\rho(c|\cdot|^s) = -i2^{1-s}\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)$$

For the complex completions, the equivalence classes are indexed by the characters c_n ($n \in \mathbb{Z}$ on S^1, with $c_n : z \mapsto z^n$. We have

$$\rho(c_n|\cdot|^s) = (-i)^{|n|}\frac{(2\pi)^{1-s}\Gamma\left(s + |n|/2\right)}{(2\pi)^s\Gamma((1-s) + |n|/2)}$$

4. Towards a Global Theory: Restricted Direct Product

It is our goal to look at 'global' properties of our number field k, independent of the norm and completion we ascribe to it. To do this, we use the 'restricted direct product' of locally compact groups in order to examine all of the completions of k simultaneously.

Henceforth, 'almost all' will mean 'all but finitely many.'

Definition 4.1. Let $G_\wp$ be a collection of locally compact groups, almost all of which have a subgroup $H_\wp$ which is simultaneously open and compact. Then we define their *restricted direct product* with respect to the subgroups $H_\wp$ as

$$G = \prod_\wp{}' G_\wp = \{x = (\ldots, x_\wp, \ldots) \mid x_\wp \in G_\wp \text{ and } x_\wp \in H_\wp \text{ for almost all } \wp\}$$

where the group operation is defined componentwise.

We will examine the topology, characters, and measure on G.

4.1. **Topology.**

Let S be a finite set, and let G_S be those elements of G with $x_\wp \in H_\wp$ for all $\wp \notin S$. Then G_S is isomorphic to $\prod_{\wp \in S} G_\wp \times \prod_{\wp \notin S} H_\wp$. The former is locally compact, and the latter is compact by Tychonoff's theorem, so that G_S is locally compact. Moreover, each G_S is open in G and G is the union of the G_S, so that G is itself locally compact.

Within each G_S, we ascribe the product topology. This induces a topology on all of G.

It will be important to understand the open sets in G. To do so, it is sufficient to find a neighborhood basis at 1.

Proposition 4.2. *The collection of 'boxes' $N = \prod_\wp N_\wp$, where $1 \in N_\wp$ and $N_\wp$ open for all $\wp$ and $N_\wp = H_\wp$ for almost all $\wp$ is a neighborhood basis of* 1 *in* G.

Proof. Because the open sets in G_S as a basis for the topology on G, we need only check that given a set S, then the set of such boxes for which $\{\wp \mid N_\wp \neq H_\wp\} \subset S$ is a neighborhood basis in that G_S. Since we ascribe the product topology to each G_S, this is true by definition, so every neighborhood of 1 contains such a box. On the other hand, since $N_\wp = H_\wp$ for almost all $\wp$ by hypothesis, then any such box is a subset of G_S, where S is the set of set of indices in which $N_\wp \neq H_\wp$. Because G_S has the product topology and each $N_\wp$ is open, then each such box is open in G_S and therefore in G. □

It will also be necessary to examine the compact sets. We accomplish this task with another proposition.

Proposition 4.3. *A subset of G is relatively compact if and only if it is contained in some box $\prod_\wp B_\wp$, where $B_\wp$ is compact for all $\wp$ and $B_\wp = H_\wp$ for almost all $\wp$.*

Proof. On the one hand, any such box lies in some G_S, and is compact in the G_S in which it lies. On the other hand, any compact subset of G is in some G_S because the G_S form an open cover of G, and $G_{S_1} \cup G_{S_2} \subset G_{S_1 \cup S_2}$. Then the projection of any compact set onto any component is again compact, and any compact set is contained in the direct product of its projections. □

4.2. **Characters.** To simplify our analysis of the characters of G, it will be convenient to define yet another subgroup. As above, let S be a finite set of indices, and let $G^S \subset G_S$ such that $x_\wp = 1$ for all $\wp \in S$. Specifically, G^S is isomorphic to the direct product $\prod_{\wp \notin S} H_\wp$, and we have $G_S = G^S \times \prod_{p \in S} G_\wp$.

Let c be a quasi-character on G, and let $c_\wp$ be the restriction of c to the subgroup $G_\wp$.

Lemma 4.4. *If c is quasi-character on G, then $c_\wp$ is trivial on $H_\wp$ for almost all $\wp$, and for any $x \in G$, we have*

$$c(x) = \prod_\wp c_\wp(x_\wp)$$

Proof. Let U be a neighborhood of 1 in $\mathbb{C}^\times$, sufficiently small as to contain no other subgroups of $\mathbb{C}^\times$. Because c is continuous, there is a box $N = \prod N_\wp$ such that $c(N) \subset U$. Since $N_\wp = H_\wp$ for almost all $\wp$, then $N \supset G^S$ for some S, and then since G^S is a subgroup, we must have $c(G^S) = 1$. Specifically, this gives us that $c_\wp(H_\wp) = 1$ for all $\wp \notin S$. The second half of the theorem follows immediately if for any $x \in G$ we write $x = x^S \cdot \prod_{\wp \in S} x_\wp$, with $x^S \in G^S$. □

Having characterized the quasi-characters on G, we would like for all functions of that type to actually be characters. Again, this is taken care of by a simple lemma.

Lemma 4.5. *Let $c_\wp$ be a character on $G_\wp$ that is trivial on $H_\wp$ for almost all $\wp$. Then $c(x) = \prod_\wp c_\wp(x_\wp)$ is a character on G.*

Proof. For a given x, we have $x_\wp \in H_\wp$ for almost all $\wp$, so that $c_\wp(x_\wp) = 1$ for almost all $\wp$. It follows that the product is well-defined, and it is clearly multiplicative.

We must show that this product is continuous. As usual, it is sufficient to prove continuity at 1. Let S be a finite set containing those $\wp$ where $c_\wp$ is nontrivial on $H_\wp$. Let U be a neighborhood of 1 in $\mathbb{C}$, and let $\mathbb{A}$ be such that $\mathbb{A}^{|S|} \subset U$. Since each $c_\wp$ is continuous, we may find an $N_\wp$ with $c_\wp(N_\wp) \subset \mathbb{A}$. If we set $N = \prod_{\wp \in S} N_\wp \times \prod_{\wp \notin S} H_\wp$, then $c(N) \subset U$, and c is continuous. □

We now restrict our attention from the quasi-characters on G to the unitary characters on G. Note that c is unitary if and only if $c_\wp$ is for all $\wp$. Let $\widehat{G}$ be the character group of G, and let $H_\wp^* \subset \widehat{G_\wp}$ be the subgroup of characters that are trivial on $H_\wp$, so that $H_\wp^* \cong \widehat{G_\wp/H_\wp}$ and $\widehat{H_\wp} \cong \widehat{G_\wp}/H_\wp^*$. Specifically, this gives us that $H_\wp^*$ is simultaneously open and compact. First, $H_\wp$ is compact, so that $\widehat{H_\wp}$ is discrete. Since $\widehat{H_\wp} \cong \widehat{G_\wp}/H_\wp^*$, then $H_\wp^*$ is open. Second, $H_\wp$ is open, so that $G_\wp/H_\wp$ is discrete. Since $H_\wp^* \cong \widehat{G_\wp/H_\wp}$, then we have $H_\wp^*$ compact. The relations between the topologies of groups and those of their dual groups are proven in proposition 3.2 of Ramakrishnan.

We conclude with a theorem characterizing $\widehat{G}$:

Theorem 4.6. *The character group $\widehat{G}$ is isomorphic, both algebraically and topologically, to the restricted direct product of the character groups $\widehat{G_\wp}$ with respect to the compact, open subgroups $H_\wp^*$.*

Proof. Letting $c = (\ldots, c_\wp, \ldots)$, we use the map $c(x) \mapsto \prod_\wp c_\wp(x_\wp)$. The previous two lemmas tell us that this map is an algebraic isomorphism, and we must show that it is also a topological isomorphism. Let B be a compact 'box' as described in proposition 4.3, with $1 \in B_\wp$ for all $\wp$ and $B_\wp = H_\wp$ for almost all $\wp$. Then c is 'close' to the identity character in the compact-open topology on $\widehat{G}$ if and only if $c(B) = c(\prod_\wp B_\wp) = \prod_\wp c_\wp(B_\wp)$ is close to 1 in $\mathbb{C}$. By letting $x_\wp = 1$ at all places except for one, we see that this occurs if and only if $c_\wp(B_\wp)$ is close to 1 for all $\wp$. Specifically, since $H_\wp$ is a subgroup, then we can be 'close enough' to 1 in $\mathbb{C}$ if and only if $c_\wp(H_\wp) = 1$ at these primes. Specifically, we have $c_\wp \in H_\wp^*$ for almost all $\wp$ and $c_\wp(B_\wp)$, and $c_\wp$ close to 1 in the compact-open topology on $\widehat{G_\wp}$ for all other places. Specifically, c is close to 1 in $\widehat{G}$ if and only if $(\ldots, c_\wp, \ldots)$ is close to 1 in the direct product of the $\widehat{G_\wp}$'s (following our characterization of the neighborhood basis of 1 in proposition 4.2), so the map is bicontinuous. □

4.3. **Measure.** We wish to create a measure on the restricted direct product that is, in some sense, the product of the measures on the existing groups. Let $\mu_\wp$ be measures on the $G_\wp$ (represented by $dx_\wp$ in an integral), with $\mu_\wp(H_\wp) = 1$ for almost all $\wp$. To do this, it is sufficient to fix a measure on each G_S. We consider the G_S as the finite direct product $\prod_S G_\wp \times G^S$, and set $dx_S = \prod_S dx_\wp \times dx^S$, where dx^S is the measure on the compact group G^S such that $\mu^S(G^S) = \prod_{\wp \notin S} \mu_\wp(H_\wp)$ which is actually a finite product. It is easy to check that this measure is independent of the set of indices we choose, so that it defines a measure on all of G.

Let f be an L^1 function on G. Then $\int f(x)\,dx$ is the limit of the integrals of f over compact subsets $B \subset G$. From proposition 4.3, we know that all compact subsets of G are contained in a G^S, so that we have, in some sense,

$$\int_G f(x)\,dx = \lim_S \int_{G_S} f(x)\,dx$$

where the limit is taken as S gets larger. Rigorously, given any neighborhood U of $\int f(x)$ in $\mathbb{C}$, there is a set $S(U)$ such that for all larger sets S we have $\int_{G_S} f(x)\,dx \in U$.

This gives us an easy means of calculating integrals of certain functions:

Proposition 4.7. *For each $\wp$, let $f_\wp \in L^1(G_\wp)$ be a continuous function, and let $f_\wp(H_\wp) = 1$ for almost all $\wp$. Define $f(x) = \prod_\wp f_\wp(x_\wp)$. Then*

$$\int_G f(x)\,dx = \prod_\wp \left(\int_{G_\wp} f_\wp(x_\wp)\,dx_\wp \right)$$

Proof. Let S be any set of indices containing those places where either $f_\wp(H_\wp) \neq 1$ or $\mu_\wp(H_\wp) \neq 1$. Then we have

$$\int_{G_S} f(x)\,dx = \int_{G_S} f(x)\,dx_S = \int_{G_S} \left(\prod_{\wp \in S} f_\wp(x_\wp) \right) \left(\prod_{\wp \in S} dx_\wp \cdot dx^S \right)$$

Since $f = 1$ on G^S, this is equal to

$$= \mu(G^S) \cdot \prod_{\wp \in S} \left(\int_{G_\wp} f_\wp(x_\wp)\,dx_\wp \right) = \prod_{\wp \in S} \left(\int_{G_\wp} f_\wp(x_\wp)\,dx_\wp \right)$$

since S contains all indices for which $H_\wp$ has a measure other than 1, so that $\mu(G^S) = 1$.

This statement holds for all 'sufficiently large' S. Taking limits in the manner described above completes the proof. □

In particular, the proposition above allows us to easily study Fourier analysis on these restricted direct products and to find a dual measure to the measure we have described on G. In this section, we restrict our attention to *unitary* characters on G. For each $\wp$, let $dc_\wp$ be the dual measure to $dx_\wp$ (denoted by $\widehat{\mu_\wp}$ outside of an integral).

Let $f_\wp$ be the characteristic function of $H_\wp$, so that $\hat{f}_\wp$ is $\mu_\wp(H_\wp)$ times the characteristic function of $H_\wp^*$ (since the integral defining the Fourier transform is nonzero if and only if the character in question is trivial on $H_\wp$). In particular, because μ and $\hat{\mu}$ are dual, we have $\mu_\wp(H_\wp)\widehat{\mu_\wp}(H_\wp^*) = 1$, so that $\widehat{\mu_\wp}(H_\wp^*) = 1$ for almost all $\wp$. In particular, this gives us that the measure dc on $\widehat{G}$, defined as the restricted direct product measure over the $\widehat{G_\wp}$'s with the measures $dc_\wp$, is well-defined.

Finally, we have:

Proposition 4.8. *The measure dc as defined above is dual to the measure dx.*

Proof. From abstract Fourier analysis, we know that $\hat{\hat{f}}(-x)$ differs from $f(x)$ by a multiplicative constant, and so it suffices to check one function. Let $f_\wp$ be the characteristic function of $H_\wp$, and let $f(x) = \prod_\wp f_\wp(x_\wp)$. Then, from proposition 4.7, we have

$$\hat{f}(c) = \int f(x)\overline{c(x)}\,dx = \prod_\wp \int f_\wp(x_\wp)\overline{c_\wp(x_\wp)} = \prod_\wp \widehat{f_\wp}(c_\wp)$$

Applying the above calculation to $\hat{f}(c)$, we have that $\hat{\hat{f}}(x) = \prod_{\wp} \widehat{\widehat{f_\wp}}(x)$. Since we chose dx and dc to be dual, we know that $\widehat{\widehat{f_\wp}}(x_\wp) = f_\wp(-x_\wp)$, whence the theorem follows. □

5. The Global Theory

For the rest of the paper, $\wp$ represents any norm on k, whether $\wp$-adic for an actual prime ideal or an archimedean norm. By abuse of notation, we call an archimedean norm a 'prime at infinity.'

Having discussed the properties of local completions of k, we wish to study the global properties of k. To do this, we consider a restricted direct product of the additive and multiplicative subgroups of k, as follows.

Definition 5.1. The *adele group*, $\mathbb{A}$, of k is the restricted direct product of the additive groups of the completions $k_\wp$, relative to the subgroups $\mathcal{O}_\wp$ of integral elements. We shall denote the generic adele by $x = (\ldots, x_\wp, \ldots)$.

We should note that Tate uses the term 'valuation vector' instead of newer term 'adele', and uses V to denote the group they form.

Definition 5.2. The *idele group*, $\mathbb{I}I$, of k is the restricted direct product of the multiplicative groups of the completions $k_\wp$, relative to the subgroups $\mathcal{U}_\wp$ of multiplicative units. We shall denote the generic idele by $a = (\ldots, a_\wp, \ldots)$.

Remark 5.3. Recall that, to define a restricted direct product, we needed only for a compact, open subgroup $H_\wp$ to exist in *almost all* places. When considering the adele and idele groups, the subgroups $\mathcal{O}_\wp$ and $\mathcal{U}_\wp$ are defined only for non-archimedean places. Moreover, recall that we have $\mu(\mathcal{O}_\wp) = \mu^*(\mathcal{U}_\wp) = 1$ for almost all $\wp$, and that all additive measures on $k_\wp$ were chosen to be self-dual.

5.1. The Additive Theory: The Adele Group.

We begin with a remarkable fact: since $k_\wp^+$ is naturally its own character group, and $\chi_\wp(\eta x)$ is trivial on $\mathcal{O}_\wp$ for $\eta \in \mathcal{D}_\wp^{-1}$, then $\widehat{\mathbb{A}}$ is the restricted direct product of the additive groups $k_\wp^+$ with respect to the subgroups $\mathcal{D}_\wp^{-1}$. However, since $\mathcal{D}_\wp^{-1} = \mathcal{O}_\wp$ in almost all places, then the character group $\widehat{\mathbb{A}}$ is isomorphic to $\mathbb{A}$ as a topological group.

To get at the underlying mathematics a little bit more, we set $\chi_0(x) = \prod_\wp \chi_\wp(x_\wp)$, where χ_0 is the basic global character on $\mathbb{A}$. Moreover, we define multiplication on the adele group in the obvious way:

$$xy = (\ldots, x_\wp, \ldots)(\ldots, y_\wp, \ldots) = (\ldots, x_\wp y_\wp, \ldots)$$

In particular, this offers us the natural way to define the isomorphism between $\mathbb{A}$ and its character group:

Theorem 5.4. *The adele group $\mathbb{A}$ is isomorphic to its character group if identify an element $y \in \mathbb{A}$ with the character $x \mapsto \chi_0(xy)$.*

Furthermore, recall that we picked each local measure $dx_\wp$ to be self-dual, so the same will be true of our restricted direct product measure dx on $\mathbb{A}$. Formally, we have

Theorem 5.5. *Let $f \in L^1(\mathbb{A})$. We define the Fourier transform*

$$\hat{f}(y) = \int f(x)\chi_0(xy)\, dx$$

Then $\hat{\hat{f}}(x) = f(-x)$

It will be necessary to understand how k is embedded in its adele group $\mathbb{A}$. Given $\xi \in k$, we identify ξ with the adele $(\xi, \xi, \xi, \ldots)$ (we know such an element is actually an adele because $\xi = a/b$, where $a, b \in \mathcal{O}$. Then $\xi \notin \mathcal{O}_\wp \Rightarrow \wp \mid (b)$, which only occurs for finitely many primes). Then k rests as a subgroup in $\mathbb{A}$.

To identify the properties of k as a subgroup, we would like to find an appropriate representation for the quotient space $\mathbb{A}/k$. That is, we want to find a set D such that we have $\mathbb{A} = \cup_{\xi \in k}(\xi + D)$, and the union is disjoint. Such a set D will be called a 'fundamental domain' for $\mathbb{A}$ mod k.

Let S_∞ be the set of archimedean (infinite) primes, and consider the subgroup $\mathbb{A}_{S_\infty}$ (i.e, those elements such that $x_\wp \in \mathcal{O}_\wp$ for all discrete primes $\wp$). We have

Lemma 5.6. *With* $\mathbb{A}_{S_\infty}$ *as above, we have* $k \cap \mathbb{A}_{S_\infty} = \mathcal{O}$ *and* $k + \mathbb{A}_{S_\infty} = \mathbb{A}$.

Proof. The first statement follows because an element of k is integral at each prime if and only if it is actually in the ring of integers of k.

To prove the second statement, let $(\ldots, x_\wp, \ldots)$ be an adele. Then $x_\wp \in \mathcal{O}_\wp$ at almost all places, and it is our goal to add an element of k to this adele to move it into $\mathcal{O}_\wp$ at all finite places. We will correct it one place at a time. Pick $\wp$ such that $x_\wp \notin \mathcal{O}_\wp$. Our goal is to find a $y_\wp \in k$ so that first, $y_\wp + x_\wp \in \mathcal{O}_\wp$, and second, $y_\wp \in \mathcal{O}_q$ for all $q \neq \wp$. First, pick $y'_\wp \in k \cap (-x_\wp + \mathcal{O}_\wp$: this will serve as our 'first approximation.' We have finite set S of discrete places such that, for $q \in S$ $y'_\wp \notin \mathcal{O}_q$. Let $N_q = -\text{ord}_q(y'_\wp)$, and let $N = -\text{ord}_\wp(y'_\wp)$. By the Chinese Remainder Theorem (discussed in Lang, section 1.4), there is an $a \in \mathcal{O}$ such that, first, $a \in q^{N_q}$ for all $q \in S$, and second, $a \in 1 + \wp^N$.

Let $y_\wp = a \cdot y'_\wp$, and we must now show that the necessary conditions are satisfied. For $\wp$, we note that $y_\wp \in (-x_\wp + \mathcal{O}_\wp)(1 + \wp^{\text{ord}_\wp(x_\wp)}) = -x_\wp + \mathcal{O}_\wp$, so the first condition is satisfied. For $q \in S$, we picked a to be sufficiently close to 0 in k_q so that $y_\wp \in \mathcal{O}_q$. And, for $q \notin S$, since a, $y'_\wp \in \mathcal{O}_q$, then so is $y_\wp$. This completes the proof. □

From here, it should be clear that $\mathbb{A}_{S_\infty}$ should play a role in helping us to find our fundamental domain D. Let $\mathbb{A}^\infty$ be the finite cartesian product of the archimedean completions of k, and let x^∞ be the projection of x into $\mathbb{A}^\infty$. Let $k = \mathbb{Q}(\alpha)$. If α has r_1 real Galois conjugates and r_2 pairs of complex Galois conjugates, then there will be precisely r_1 real completions of k and r_2 complex completions, since each root of the generating equation must be mapped to another root by any automorphism. Therefore, $\mathbb{A}^\infty$ is will have dimension $r_1 + 2r_2$ over $\mathbb{R}$, which is the degree $[k : \mathbb{Q}]$.

Let $\{\omega_1, \ldots, \omega_n\}$ be a basis for the ring of integers $\mathcal{O}$ over $\mathbb{Z}$, and let ω_i^∞ be the embedding of these ω_i in the n-space $\mathbb{A}^\infty$. Consider the 'box'

$$D^\infty \subset \mathbb{A}^\infty = \{\sum_{i=1}^{n} t_i \omega_i^\infty \mid 0 \leq t_1 < 1\}$$

Finally, let

$$D = D^\infty \times \prod_{\wp \notin S_\infty} H_\wp \subset \mathbb{A}_{S_\infty}$$

Then

Proposition 5.7. *The set D, as given above, is a fundamental domain for $\mathbb{A}$ mod k. That is, $D + k = \mathbb{A}$, and each element of $\mathbb{A}$ can be written uniquely as a sum of an element of D and and element of k.*

Proof. First, from the second part of lemma 5.6, we know that any element of $\mathbb{A}$ can be brought into $\mathbb{A}_{S_\infty}$ by subtracting an element of k. Since $k \cap \mathbb{A}_{S_\infty} = \mathcal{O}$, then this element must be unique modulo $\mathcal{O}$. Once we are in $\mathbb{A}_{S_\infty}$, we must adjust the infinite component of our adele so that it is in D^∞, by the way we picked D^∞, there is a unique element of $\mathcal{O}$ by which we may translate in order to do so. □

Tate also goes through considerable detail in showing that D has volume 1 by explicitly evaluating the determinant of the matrix formed by the ω_i^∞'s. We omit this part of the proof because a purely algebraic proof, based solely on Fourier analysis, will arise in the next section. The reader interested in the geometric proof should see Lemma 4.1.4 and Theorem 4.1.3 in Tate. Lang provides good further explanation in chapter V, section 2.

With facts about D in hand, we may now talk about how k lies in $\mathbb{A}$. Since D has an interior, then k must be a discrete subgroup. On the other hand, since D is relatively compact, then $\mathbb{A}$ mod k must be compact.

Finally, we wish to see how the character group behaves on $\mathbb{A}$ mod k. Specifically, we want to find a 'dual group' for k: that is, a group $\tilde{k} \subset \mathbb{A}$ such that $\chi_0(xy) = 1$ for all $y \in \tilde{k}$, $x \in k$.

Lemma 5.8. $\chi_0(\xi) = 1$ *for all* $\xi \in k$.

Proof. Recall that χ_0 was the product of local characters, each of whom depended on the quantity $\lambda_p(\mathrm{Tr}_\wp(\xi))$, where $\wp$ lies above p. Therefore, it is sufficient to show $\sum_\wp \lambda_p(\mathrm{Tr}_\wp(\xi))$ is integral. We have

$$\sum_\wp \lambda_p(\mathrm{Tr}_\wp(\xi) = \sum_p \lambda_p \left(\sum_{\wp | p} \mathrm{Tr}_\wp(\xi) \right)$$

Here, we import a fact from the classical theory, which states that:

$$\sum_{\wp|p} \mathrm{Tr}_\wp(\xi) = Tr(\xi), \text{ so that } \sum_\wp \lambda_p(\mathrm{Tr}_\wp(\xi)) = \sum_p \lambda_p(\mathrm{Tr}(\xi))$$

(this fact is proven in chapter 12, section 3 of Lang's *Algebra*). Then $\mathrm{Tr}(\xi) = r$ is rational, so it suffices to show that $\sum_p \lambda(r)$ is integral for all prime q. Clearly, for $p \neq q$ $\lambda_p(r)$ is integral with respect to q since $\lambda_p(r)$ is a sum of fractions with denominators equal to powers of p. Therefore, we must only check λ_q and the infinite prime. But for the infinite prime, $\lambda(r) = -r$, and $\lambda_q(r) - r$ is integral with respect to q by definition. □

Finally, we have

Theorem 5.9. *The dual group* $\tilde{k} = k$

Proof. We defined $\tilde{k}$ as the dual group to $\mathbb{A}$ mod k, so $\tilde{k}$ is discrete. We consider the group $\tilde{k}$ mod k. This is contained in the compact group $\mathbb{A}$ mod k and is therefore compact, which means it must be finite.

On the other hand, it is easy to check that $\tilde{k}$ is a vector space over k. Because the index of k in $\tilde{k}$ is finite, and k is itself infinite, then $\tilde{k}$ cannot have dimension > 1 over k. It follows that $\tilde{k} = k$. □

This fact will be of great importance when we apply Fourier analysis to the adele group in two sections.

5.2. **The Multiplicative Theory: The Idele Group.** In this section, we examine the global multiplicative theory via the idele group, $\mathbb{I}$. We first note that there is a continuous homomorphism from the idele group into the ideal group (with the discrete topology), given by:

$$\varphi(a) = \prod_{\wp \notin S_\infty} \wp^{\mathrm{ord}_\wp(a_\wp)}$$

with kernel $\mathbb{I}_{S_\infty}$ (those ideles whose coordinates are units at all non-archimedean places).

From our section on general restricted direct products, we know that the quasi-characters on $\mathbb{I}$ are $c(a) = \prod_\wp c_\wp(a_\wp)$, where $c_\wp$ is unramified (trivial on $\mathcal{U}_\wp$) at almost all places. We choose a measure $d^*a = \prod_\wp d^*a_\wp$.

As with the adele group, we embed k^* in $\mathbb{I}$ by $\alpha \mapsto (\alpha, \alpha, \alpha, \ldots)$. Then $\varphi(\alpha)$ is the principal ideal $\alpha\mathcal{O}$. Moreover, we note that αD is a fundamental domain for k in $\mathbb{A}$, since D is. Therefore, we may write $\alpha D = \cup(\alpha D \cap \xi + D)$ and $D = \cup(D \cap -\xi + \alpha D)$, where ξ ranges over elements of k, and both unions are disjoint. Each piece of αD is a translate by ξ of the corresponding piece of D, so two corresponding pieces have the same measure. It follows that $\mu(\alpha D) = \mu(D)$, so that $|\alpha| = 1$ for all nonzero $\alpha \in k$.

We call J the subgroup of $\mathbb{I}$ of elements with absolute value 1. As with the additive theory, we wish to find a fundamental domain for J with respect to k^*, since this will tell us important information about how k^* lies in J topologically. As above, the archimedean primes will play a crucial role in this analysis. Consider the subgroup $J_{S_\infty} \leq J$ of ideles whose coordinates are units at all discrete places and which have absolute value 1. Pick one archimedean prime $\wp_0$ and let $S'_\infty = S_\infty \setminus \{\wp_0\}$, and let $|S'_\infty| = r$. We then have a continuous, surjective homomorphism $l : J_{S_\infty} \to \mathbb{R}^r$, given by $l(b) = (\log|b_{\wp_1}|, \ldots, \log|b_{\wp_r}|)$.

As above, we have $J_{S_\infty} \cap k^* = \mathcal{U}$, since those elements of k^* that are units in all discrete completions are the units in $\mathcal{O}$. If we restrict l to k^*, the kernel of l is precisely the roots of unity in k, since these are the only elements that have absolute value 1 in all embeddings of k into $\mathbb{C}$ (This theorem is due to Kronecker, and has been proved more recently by Joel Spencer and Gerhard Greither). In particular, if $\epsilon_1, \ldots, \epsilon_r$ are a minimum generating set for $\mathcal{U}$ modulo the roots of unity, then $l(\epsilon_1), \ldots, l(\epsilon_r)$ is a basis for $l(U)$ over $\mathbb{Z}$ in $\mathbb{R}^r$.

Let P be the parallelotope spanned by the $l(\epsilon_i)$ (i.e. $P = \{\sum t_i l(\epsilon_i) \mid 0 \leq t_i < 1\}$). Define $l^{-1}(P)$ as the subset of J_{S_∞} on which $l(b) \in P$. Let E_0 be the subset of $l^{-1}(P)$ such that $0 \leq \arg b_{\wp_0} < \frac{2\pi}{w}$, where w is the number of roots of unity in k.

Let h be the class number of k, and let $b_1, \ldots, b_h$ be ideles in J such that $\varphi(b_1), \ldots, \varphi(b_h)$ are elements of distinct ideal classes. Finally, we set $E = \cup_{i=1}^h E_0 b_i$, and claim:

Proposition 5.10. *The set E is a fundamental domain for the k in J, in the sense that*

$$J = \cup_{\alpha \in k^*} \alpha E_0$$

and the union is disjoint.

Proof. Note that, since multiplication by k does not change the ideal class of $\varphi(b)$, then αb may lie in only one of $b_h E_0$'s. We may therefore assume b corresponds to the principle ideal class.

Since $b_\wp$ is a unit at all but finitely many places, we can multiply by an element of k^*, unique up to units, that brings $b_\wp$ into the units at all places, and so bringing b into $\mathbb{I}_{S_\infty}$. Now, if we examine our embedding of J^∞ into $\mathbb{R}^r$, we see may multiply by a unit, unique modulo the roots of unity, that will bring the image of the J^∞ component (under l) into our parallelotope. Finally, we need to adjust the $\wp_0$ component, at this point using only the roots of unity. But since we wish to only change the argument so that it is between 0 and $2\pi/w$, there is a unique root of unity by which we may multiply to accomplish this. Thus, there is a unique $\alpha \in k$ such that $\alpha b \in E$. □

The volume of E will appear in the functional equation of the global zeta functions we find, but is otherwise unimportant. We omit the calculations of this volume (the interested reader should see pages 336-8 of Cassels and Frohlich), and, following Tate, denote the volume by κ.

Having found a fundamental domain, we now see exactly how k^* rests inside the idele group. First, because E has an interior, then k^* is discrete inside J and therefore inside $\mathbb{I}$. Second, because E is relatively compact, then J mod k^* is compact.

These topological properties give us further information about the quasi-characters on $\mathbb{I}$. For our analysis, as in the additive theory, we will be interested in those quasi-characters that are trivial on k^*. Since J mod k^* is compact, then all such characters will be unitary on J.

To find all of the quasi characters on $\mathbb{I}$, it is helpful to express it as a direct product $J \times T$, where T represents the possible absolute values taken on the idele group. We construct T as follows: consider the archimedean prime $\wp_0$, and let $T = \{(t, 1, 1, \ldots) \mid t \in \mathbb{R}^+\}$, whose absolute value is t if $\wp_0$ is real and t^2 if $\wp_0$ is complex. It is easy to see that $T \cap J = \{1\}$ and $TJ = \mathbb{I}$, so that $\mathbb{I} = T \times J$.

We ascribe the multiplicative measure $\frac{dt}{t}$ to T, and then to J we ascribe the unique measure d^*b so that $d^*a = d^*b \cdot \frac{dt}{t}$.

Therefore, to examine the quasi-characters on $\mathbb{I}$, we let $\tilde{a}$ be the projection of a onto J, and let $\tilde{c}$ be a character on J. Then, since $T \cong \mathbb{R}^+_\times$, all quasi-characters on T are of the form $t \mapsto t^s$ for some $s \in \mathbb{C}$. Therefore, all quasi-characters on $\mathbb{I}$ are of the form $a \mapsto c(a) = \tilde{c}(\tilde{a}) \cdot |a|^s$.

As in the local theory, we call $\sigma = \mathrm{Re}(s)$ the *exponent* of c. We call a character *unramified* if it is trivial on J, and we call two characters *equivalent* if their quotient is unramified. In particular, each equivalence class of quasi-characters is of the form $\{\tilde{c}(\tilde{a})|a|^s \mid s \in \mathbb{C}\}$ and is therefore isomorphic to $\mathbb{C}^+$.

Again, as in the local theory, it is now clear what we mean when we speak of a holomorphic function of quasi-characters, or of analytic continuation in the domain of quasi-characters (of course, we must have analytic continuation separately in each equivalence class).

5.3. **Fourier Transforms on $\mathbb{A}$ and the Riemann-Roch Theorem.** We call a function φ on $\mathbb{A}$ periodic if, for $\xi \in k$, we have $\varphi(x + \xi) = \varphi(x)$. In particular, a periodic function on $\mathbb{A}$ a function on the quotient group $\mathbb{A}$ mod k, and is uniquely determined by its action on D.

The quotient group $\mathbb{A}/k$ is itself a topological group. We ascribe to it that Haar measure induced by the existing measure on D. Specifically, we ascribe that measure which gives the whole group a measure of 1.

By theorem 5.9, we know that k is the dual group of D. We therefore have the following definition of Fourier transforms for periodic functions:

Definition 5.11. Given $\xi \in k$ and a continuous function φ on $\mathbb{A}$ mod k, we write

$$\hat{\varphi}(\xi) = \int_D \varphi(x)\chi_0(\xi x)\,dx$$

Because the dual group of $\mathbb{A}$ is different from the dual group of $\mathbb{A}/k$, we use different definitions for Fourier transforms, even though a function on $\mathbb{A}/k$ induces a periodic function on $\mathbb{A}$. We shall have a proposition relating the two quantities momentarily.

First, however, we check that the Fourier inversion formula holds:

Lemma 5.12. *For a continuous, periodic function $\varphi(x)$ for which $\sum_{\xi\in k} |\hat{\varphi}(\xi)| < \infty$, we have $\hat{\hat{\varphi}}(x) = \varphi(-x)$. That is:*

$$\varphi(x) = \sum_{\xi\in k} \hat{\varphi}(\xi)\chi_0(-\xi x)$$

Proof. The second condition assures that the Fourier transform is actually in $l^1(k)$. Since we know that the Fourier inversion formula holds up to a constant, it suffices to check only one function.

Let $\varphi(x)$ take the value of 1 uniformly. Since χ_0 is periodic and nontrivial on $\mathbb{A}$, it is nontrivial on D, so that $\chi_0(\xi x)$ is nontrivial for nonzero ξ on the group $\mathbb{A}$ mod k. Therefore, $\hat{\varphi}(\xi)$ is 0 for $\xi \neq 0$ and $\mu(D)$ for $\xi = 0$. Then $\hat{\hat{\varphi}} = \mu(D)$ uniformly. We shall see shortly that $\mu(D) = 1$. □

From here, we have:

Lemma 5.13. *Let $f(x)$ be a continuous, L^1 function on $\mathbb{A}$, such that the sum $\sum_{\eta\in k} f(x+\eta)$ is absolutely, uniformly convergent on D. Then, for the periodic function $\varphi(x) = \sum_{\eta\in k} f(x+\eta)$, we have, for all $\xi \in k$, $\hat{\varphi}(\xi) = \hat{f}(\xi)$ (where $\hat{\varphi}$ is as in definition 5.8, and $\hat{f}$ is as in theorem 5.5).*

Proof.

$$\hat{\varphi}(\xi) = \int_D \varphi(x)\chi_0(\xi x)\,dx = \int_D \left(\sum_{\eta\in k} f(x+\eta)\chi_0(\xi x)\right)dx$$

We may interchange the sum and the integral since the sum converges uniformly and D has finite measure (indeed, it is contained in a compact set), leaving

$$= \sum_{\eta\in k}\left(\int_D f(x+\eta)\chi_0(\xi x)\,dx\right) = \sum_{\eta\in k}\left(\int_{D+\eta} f(x)\chi_0(\xi(x-\eta))\,dx\right)$$

$$= \sum_{\eta\in k}\left(\int_{D+\eta} f(x)\chi_0(\xi x)\,dx\right)$$

Since $\chi_0(\xi\eta) = 1$. Finally, we have

$$= \int_{\mathbb{A}} f(x)\chi_0(\xi x)\,dx = \hat{f}(\xi)$$

□

If we combine the last two lemmas and put $x = 0$ in lemma 5.9, we have an analogue of the Poisson summation formula:

Theorem 5.14. *Let $f(x)$ be continuous and in $L^1(\mathbb{A})$, with $\hat{f} \in l^1(k)$. Then*

$$\sum_{\xi\in k} \hat{f}(\xi) = \sum_{\xi\in k} f(\xi)$$

First, this gives us an easy way to show that $\mu(D) = 1$. Had we not assumed this, from the Poisson formula we would have $\mu(D)\sum f(\xi) = \sum \hat{f}(x)$. Applying the Poisson formula twice would yield $\mu(D)^2 = 1 \Rightarrow \mu(D) = 1$.

Finally, we wish to see how the Poisson formula acts when we 'stretch' $\mathbb{A}$ via multiplication by an element. We would like this action to be an automorphism. Clearly, multiplication by any element is an additive homomorphism. Let $a = (\ldots, a_\wp, \ldots)$ be such an element. Then this multiplication is invertible if there is a $b = (\ldots, b_\wp, \ldots) \in \mathbb{A}$ where $a_\wp b_\wp = 1$. First and foremost, we need $b_\wp \neq 0$ at all places. Secondly, we need $b_\wp \in \mathcal{O}\wp$ for almost all $\wp$, so that for almost all $\wp$, we have both $a_\wp$ and $a_\wp^{-1} \in \mathcal{O}$. Indeed, this occurs if and only if $a_\wp \in \mathcal{U}$ at almost all places, or if a is an idele.

For an idele a, define $|a|$ such that $\mu(aM) = |a|\mu(M)$ (such a constant exists because $\mu(a\cdot)$ is a Haar measure). We then have

Proposition 5.15. *For an idele a, we have $|a| = \prod_\wp |a_\wp|$ (really a finite product since $|a_\wp| = 1$ almost everywhere).*

Proof. Pick a compact 'box' of the form $N = \prod_\wp N_\wp$. Then, using proposition 4.7, we have

$$\int_N dx = \prod_\wp \int_{N_\wp} dx$$

and

$$\int_{aN} dx = \prod_\wp \int_{a_\wp N_\wp} dx = \prod_\wp |a_\wp| \int_{N_\wp} dx$$

whence the proposition follows. □

We conclude with the 'crown jewel' of this section, the Riemann-Roch theorem:

Theorem 5.16. *Let $f(x) \in L^1(\mathbb{A})$ be continuous, with $\sum_{\xi\in k} f(a(x+\xi))$ uniformly, absolutely convergent on D for all ideles a, and $\hat{f}(ax) \in l^1(k)$ for all ideles a. Then we have*

$$\frac{1}{|a|}\sum_{\xi\in k} \hat{f}(\xi/a) = \sum_{\xi\in k} f(\xi a)$$

Proof. Let $g(x) = f(ax)$. Using the substitution $\eta \mapsto a\eta$ in the formula for $\hat{f}$, we have $\hat{g}(x) = \hat{f}(ax)/|a|$. This gives us that g satisfies the conditions of the Poisson formula above, leaving

$$\sum_{\xi \in k} \hat{g}(\xi) = \sum_{\xi \in k} g(\xi), \text{ so that } \frac{1}{|a|} \sum_{\xi \in k} \hat{f}(\xi/a) = \sum_{\xi \in k} f(\xi a)$$

□

5.4. **Zeta Functions, Analytic Continuation, and the Functional Equation.** We finally come to the payoff: defining, and finding functional equations for, global zeta functions. Let $f : \mathbb{A} \to \mathbb{C}$ satisfy the following conditions:

- f and $\hat{f}$ are both continuous and in $L^1(\mathbb{A})$
- $\sum_{\xi \in k} f(a(x+\xi))$ and $\sum_{\xi \in k} f(a(x+\xi))$ converge absolutely for each idele a and adele x, and the convergence is uniform for ordered pairs $(a, x) \in K \times D$, where K is a compact subset of $\mathbb{I}$ and D is the additive fundamental domain of k in $\mathbb{A}$.
- For $\sigma > 1$, $f(a)|a|^\sigma, \hat{f}(a)|a|^\sigma \in L^1(\mathbb{I})$

The first two conditions ensure that we may apply the Riemann-Roch theorem. The latter allows us to define zeta functions that are holomorphic on a subset of each equivalence class.

Definition 5.17. For a function f with the above properties, we define $\zeta(f, c)$, a function of quasi-characters c, as

$$\zeta(f, c) = \int f(a)c(a)\, d^*a$$

Again, because the third condition assures that this integral will be absolutely convergent for c with exponent > 0, we immediately see that $\zeta(f, c)$ is holomorphic in the domain of quasi-characters of positive exponent.

Finally, we state and prove the *Main Theorem* of the global theory, giving an analytic continuation for the zeta functions to the entirety of the domain of quasi-characters.

Theorem 5.18. *As in the local theory, let $\hat{c}(a) = |a|c^{-1}(a)$. Then our zeta functions have an analytic continuation from the domain of characters of exponent > 1 to the domain of all characters. The continuation is entire on every equivalence class except for the class of unramified characters. On this equivalence class, the function has simple poles at $s = 0$ and $s = 1$ of residues $\kappa f(0)$ and $-\kappa \hat{f}(0)$. The continuation satisfies the functional equation*

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c})$$

Proof. Recall our decomposition $\mathbb{I} = J \times T$. We set

$$\zeta_t(f, c) = \int_J f(tb)c(tb)\, d^*b$$

in order to write (using Fubini)

$$\zeta(f, c) = \int f(a)c(a)\, d^*a = \int_0^\infty \left(\int_J f(tb)c(tb) d^*b \right) \frac{dt}{t} = \int_0^\infty \zeta_t(f, c) \frac{dt}{t}$$

The key step of the proof involves using the Riemann-Roch theorem to find a functional equation for ζ_t

Lemma 5.19. *For all quasi-characters c, we have the functional equation:*

$$\zeta_t(f,c) + f(0)\int_E c(tb)\,d^*b = \zeta_{1/t}(\hat{f},\hat{c}) + \hat{f}(0)\int_E \hat{c}\left(\frac{1}{t}b\right)\,d^*b$$

Proof. Because E is a fundamental domain for k^* in J, we have

$$\zeta_t(f,c) + f(0)\int_E c(tb)\,d^*b = \sum_{\alpha\in k^*}\int_{\alpha E} f(tb)c(tb)\,d^*b + f(0)\int_E c(tb)\,d^*b$$

Using the substitution $b \mapsto \alpha b$ in each integral (under which the measure d^*b is invariant), and using the fact that $c(\alpha) = 1$, we have

$$= \sum_{\alpha\in k^*}\int_E f(\alpha tb)c(\alpha tb) + f(0)\int_E c(tb)\,d^*b$$

$$= \sum_{\alpha\in k^*}\int_E f(\alpha tb)c(tb) + f(0)\int_E c(tb)\,d^*b$$

$$= \int_E \left(\sum_{\alpha\in k^*} f(\alpha tb)\right) c(tb)\,d^*b + f(0)\int_E c(tb)\,d^*b$$

Where the interchange is justified because the sum is uniformly convergent by hypothesis. Since $k = k^*\cup\{0\}$, we have

$$= \int_E \left(\sum_{\xi\in k} f(\xi tb)\right) c(tb)\,d^*b$$

We apply the Riemann-Roch theorem to find:

$$= \int_E \left(\sum_{\xi\in k} \hat{f}\left(\frac{\xi}{tb}\right)\right) \frac{1}{|tb|}c(tb)\,d^*b$$

Finally, applying the substitution $b \mapsto 1/b$, under which d^*b is invariant, leaves

$$= \int_E \left(\sum_{\xi\in k} \hat{f}\left(\frac{\xi b}{t}\right)\right) \hat{c}\left(\frac{1}{t}b\right)\,d^*b$$

Reversing the steps completes the proof. □

With this in hand, it seems natural to calculate $\int_E c(tb)$. We note that $\int_E c(tb) = c(t)\int_E c(b)$. This is the integral over J/k^* of a character on this group, so the integral is 0 if the character is nontrivial on J (c is ramified) and the volume of E ($=\kappa$) if c is unramified. In the former case, we have $c(t) = |t|^s = t^s$, so that we have:

$$\int_E c(tb)\,d^*b = \begin{cases}\kappa t^s & \text{if } c \text{ unramified,}\\ 0 & \text{if } c \text{ ramified}\end{cases} \tag{5.20}$$

Finally, to attack the main proof, we write

$$\zeta(f,c) = \int_0^1 \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(f,c)\frac{dt}{t}$$

The second integral comes to $\int_{|a|>1} f(a)c(a)\, d^*a$. The first integral clearly converges where the exponent of c is > 1 because $\zeta(f, c)$ converges there. Meanwhile, it will converge even more easily if the exponent is lower since we integrate where $|a| > 1$. Therefore, it is convergent for all c and defines a holomorphic function in this domain.

To handle the first integral, we use lemma 5.19 and equation 5.20 to write

$$\int_0^1 \zeta_t(f, c)\frac{dt}{t} = \int_0^1 \zeta_{1/t}(\hat{f}, \hat{c})\frac{dt}{t} + \left[\left[\kappa\hat{f}(0)\int_0^1 \left(\frac{1}{t}\right)^{1-s}\frac{dt}{t} - \kappa f(0)\int_0^1 t^s \frac{dt}{t}\right]\right]$$

where the statement in brackets is to be included only on the unramified equivalence class. In particular, for characters of exponent > 1, we have $\mathrm{Re}(s) > 1$, so that the two integrals inside the brackets converge. Calculating these integrals, and applying the substitution $t \mapsto 1/t$ in the integral outside the brackets (under which the measure dt/t is invariant) yields

$$\int_0^1 \zeta_t(f, c)\frac{dt}{t} = \int_1^\infty \zeta_t(\hat{f}, \hat{c})\frac{dt}{t} + \left[\left[\frac{\kappa\hat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right]\right]$$

The same logic that showed the first integral to be holomorphic also shows the second integral to be holomorphic. Therefore, this expression gives us an analytic continuation from the domain of quasi-characters of exponent > 1 to the entire domain. Since the bracketed expression comes into play only on the unramified equivalence class, then ζ is entire on all ramified equivalence classes, and has simple poles at 0 and 1 of residue $-\kappa f(0)$ and $\kappa\hat{f}(0)$ respectively.

Moreover, noting that the expression is completely unchanged by the transformation $(f, c) \mapsto (\hat{f}, \hat{c})$ (since this also takes s to $1 - s$), the zeta functions satisfy the functional equation:

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c})$$

To quote Tate: 'The Main Theorem is proved!' □

6. Connections to Classical Theory, and Why We Care

We will not fully expound upon the connection between Tate's thesis and the 'classical' theory of zeta functions on number fields. The reader interested in pursuing the full version should see the final chapter of Tate's thesis itself, where he provides an explanation.

As usual, we begin with a definition.

Definition 6.1. Let k be a number field. Then we define

$$\zeta_k(s) = \sum_a \frac{1}{\mathcal{N}a^s}$$

where the sum is taken over all ideals a in the ring of integers $\mathcal{O}$.

The absolute convergence of this sum in the domain $\mathrm{Re}(s) > 1$ is a keystone of the classical theory. Tate does not prove this fact himself. Rather, he cites Landau's 'Algebraische Zahlen', second edition, pages 55 and 56. Even so, if we accept the absolute convergence of the sum, we may express the zeta function as an Euler product:

$$\zeta_k(s) = \prod_{\wp} \frac{1}{1 - \mathcal{N}\wp^{-s}}$$

where the product is taken over all prime ideals $\wp \subset \mathcal{O}$.

The 'big idea' is that we can pick a special function f such that $\zeta(f, c)$ looks like $\zeta_k(s)$. In this case, let c be in the unramified equivalence class of quasi-characters on the idele group $\mathbb{I}$, so that $c(a) = |a|^s$. We may also write c as a product of local quasi-characters $c_\wp(a_\wp) = |a_\wp|^s$.

For all archimedian places, pick $f_\wp$ so that the local zeta function $\zeta_\wp(f_\wp, c_\wp)$ is nontrivial on the unramified equivalence class , and on the discrete places, let $f_\wp$ be the characteristic function of $\mathcal{O}_\wp$. Let $f(x) = \prod_\wp f_\wp(x_\wp)$. Then proposition 4.7 gives us that

$$\zeta(f, c) = \prod_\wp \zeta_\wp(f_\wp, c_\wp)$$

For all discrete places, we have that

$$\zeta_\wp(f_\wp, |\cdot|^s) = \sum_{v=0}^{\infty} \mu^*(A_v)(\mathcal{N}\wp)^{-vs} = \frac{\mathcal{N}\mathcal{D}^{-1/2}}{1 - (\mathcal{N}\wp)^{-s}}$$

Therefore, we have

$$\begin{aligned} \zeta(f, |\cdot|^s) &= \prod_{\wp \text{ archimedian}} \zeta_\wp(f_\wp, |\cdot|^s) \prod_{\wp \text{ discrete}} \frac{\mathcal{N}\mathcal{D}^{-1/2}}{1 - (\mathcal{N}\wp)^{-s}} \\ &= \zeta_k(s) \prod_{\wp \text{ archimedian}} \zeta_\wp(f_\wp, |\cdot|^s) \prod_{\wp \text{ discrete}} \mathcal{N}\mathcal{D}^{-1/2} \end{aligned}$$

Since all measures were chosen to be self dual, we have $\hat{f} = \prod_\wp \hat{f}_\wp$, and then applying proposition 4.7 again gives us

$$\zeta(\hat{f}, \widehat{|\cdot|^s}) = \prod_\wp \zeta_\wp(\hat{f}_\wp, \widehat{|\cdot|^s})$$

Since, at the discreet places, $\hat{f}_\wp$ is $\mathcal{N}\mathcal{D}^{1/2}$ times the charactersitic function of $\mathcal{D}^{-1}$, then calculating $\zeta(\hat{f}, \hat{c})$ by summing a geometric series as above yields

$$\zeta(\hat{f}_\wp, \widehat{|\cdot|^s}) = \zeta(\hat{f}_\wp, |\cdot|^{1-s}) = \mathcal{N}\mathcal{D}^{-1/2}\mathcal{N}\mathcal{D}^{1-s}\frac{\mathcal{N}\mathcal{D}^{-1/2}}{1 - (\mathcal{N}\wp)^{1-s}} = \frac{\mathcal{N}\mathcal{D}^{-s}}{1 - (\mathcal{N}\wp)^{1-s}}$$

So that

$$\begin{aligned} \zeta(\hat{f}, \widehat{|\cdot|^s}) &= \prod_{\wp \text{ archimedian}} \zeta(\hat{f}, \widehat{|\cdot|^s}) \prod_{\wp \text{ discrete}} \frac{\mathcal{N}\mathcal{D}^{-s}}{1 - (\mathcal{N}\wp)^{1-s}} \\ &= \zeta_k(1-s) \prod_{\wp \text{ archimedian}} \zeta(\hat{f}, \widehat{|\cdot|^s}) \prod_{\wp \text{ discrete}} \mathcal{N}\mathcal{D}^{-s} \end{aligned}$$

Using the functional equations $\zeta(f, c) = \zeta(\hat{f}, \hat{c})$ and $\zeta_\wp(f_\wp, c_\wp) = \rho_\wp(c_\wp)\zeta(\hat{f}_\wp, \hat{c}_\wp)$, and the values for $\rho(|\cdot|^s)$ expressed in section 3.7, we have

$$\zeta_k(1-s) = \zeta_k(s) \prod_{\wp \text{ discrete}} (\mathcal{N}\mathcal{D})_\wp^{s-1/2} \prod_{\wp \text{ archimedean}} \rho_\wp(|\cdot|^s)$$

$$= \zeta_k(s) \left(2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\right)^{r_1} \left(-i\frac{(2\pi)^{1-s}\Gamma(s+1/2)}{(2\pi)^s\Gamma((1-s)+1/2)}\right)^{r_2} \prod_{\wp \text{ discrete}} (\mathcal{N}\mathcal{D})_\wp^{s-1/2}$$

These functional equations, as well as the functional equations for more general 'classical' zeta functions Tate describes in the final chapter of his thesis, were already known. Why, then, should we care about Tate's thesis?

Even though Tate's thesis dealt with an 'old' subject, the means by which Tate explored the material was new. Tate was not the first to work with adele groups or idele groups on fields. However, his thesis was among the first to examine Fourier analysis on these groups and to use Fourier analysis as a means to examining zeta functions.

Harmonic analysis over number fields is now an important area of number theory, and is studied by our own Paul Sally.

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