FUNDAMENTAL GROUPS AND THE VAN KAMPEN'S THEOREM

SAMUEL BLOOM

Abstract. In this paper, we define the fundamental group of a topological space and explore its structure, and we proceed to prove Van-Kampen's Theorem, a powerful result useful for calculating the fundamental groups of spaces which decompose into spaces whose fundamental groups are already known. With these tools, we show that the circle, 2-sphere, torus, and figure-8 space are topologically distinct. We also use a result from covering space theory to prove two important results outside of Topology, including the Fundamental Theorem of Algebra.

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1. Basic Definitions: Homotopies, Loops, and the Fundamental Group

We begin with a few definitions:

Definition 1.1. Let X and Y be sets and $f, g : X \to Y$ be continuous. Then, f and g are **homotopic** if there exists a continuous function $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ $\forall x \in X$. F is called a **homotopy** between f and g, and we write $f \simeq g$. If f is homotopic to a constant map, then f is **nulhomotopic**.

Definition 1.2. Suppose $f, g : [0,1] \to X$ are continuous. Then, we call f, g **paths** in X, with initial point x_0 and ending point x_1. Suppose also that $f(0) = g(0) = x_0$, $f(1) = g(1) = x_1$, and $f \simeq g$, where the homotopy F between f and g satisfies

$$F(0,t) = x_0 \text{ and } F(1,t) = x_1 \ \forall t \in [0,1] \tag{1.3}$$

Then, f and g are **path homotopic**, and F is a **path homotopy** (or **basepoint-preserving homotopy**) between f and g. We write $f \simeq_P g$.

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The relations $\simeq$ and $\simeq_P$ are equivalence relations, with the latter stronger than the former. We write the path-homotopy equivalence class of a path f as $[\,f\,]$. Note also that, for compactness of notation, we may write a homotopy as $f_t(s) = F(s,t)$.

Example 1.4. Let X be a convex set and f, g be paths in X such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then, $f \simeq_P g$ by the "straight-line" path homotopy

$$F(s,t) = (1-t) \cdot f(s) + t \cdot g(s), \tag{1.5}$$

and $g \in [\,f\,]$.

We now define an operation on paths and their equivalence classes.

Definition 1.6. Let f, g be paths in X such that $f(1) = g(0)$. We define the product $f * g$ by:

$$f * g(s) = \begin{cases} f(2s) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{1.7}$$

Definition 1.8. Let f, g be paths in X such that $f(1) = g(0)$. Then, we define the product $[\,f\,] * [\,g\,] = [\,f * g\,]$. This product is associative. We define the **constant path** at x_0 as $e_{x_0}(s) = x_0 \ \forall s \in [0,1]$. Then, $[\,f\,] * \left[\,e_{f(1)}\,\right] = \left[\,e_{f(0)}\,\right] * [\,f\,] = [\,f\,]$. We also define the **reverse** of a path f from x_0 to x_1 as $\overline{f}(s) = f(1-s)$. Then, $[\,f\,] * \left[\,\overline{f}\,\right] = [\,e_{x_1}\,]$ and $\left[\,\overline{f}\,\right] * [\,f\,] = [\,e_{x_0}\,]$.

A basic theorem that will be used later in the paper is that $[\,f\,]$ can be decomposed into the equivalence classes of the segments composing f. That is,

Theorem 1.9. *Let f be a path in X, and let $0 = a_0 < a_1 < \ldots < a_n = 1$. Let $f_i : I \to X$ be paths such that $f_i(s) = f(a_{i-1} + s(a_i - a_{i-1}))$. Then,*

$$[\,f\,] = [\,f_1\,] * [\,f_2\,] * \cdots * [\,f_n\,] \tag{1.10}$$

We now define a fundamental notion of algebraic topology: the *fundamental group.*

Definition 1.11. Let X be a space and $x_0 \in X$. A **loop** in X is a path from x_0 to x_0, and we say that x_0 is the **basepoint** of the loop. Then, the **fundamental group** of X relative to x_0 is $\pi_1(X, x_0) = \{[\,f\,] \,|\, f \text{ is a loop based at } x_0\}$. The fundamental group is also called the **first homotopy group** of X.

Theorem 1.12. *$\pi_1(X, x_0)$ is a group.*

Proof. See (1.8). □

The fundamental group of a space is nearly independent of the choice of basepoint x_0:

Theorem 1.13. *If X is path-connected, then $\forall x_0, x_1 \in X$, $\pi_1(X, x_0) \approx \pi_1(X, x_1)$.*

Proof. Let α be a path in X from x_0 to x_1. α exists by hypothesis. Define $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha} = [\,\overline{\alpha}\,] * [\,f\,] * [\,\alpha\,]\,. \tag{1.14}$$

It is clear that $\hat{\alpha}$ is an isomorphism. □

Corollary 1.15. *$\pi_1(X, x_0)$ depends only on the path-connected component of X that contains x_0.*

From now on, we will work only with path-connected spaces, so that each space has a unique fundamental group. An especially nice category of spaces is the *simply-connected* spaces:

Definition 1.16. A path-connected space X is **simply-connected** if $\pi_1(X, x_0)$ is trivial, i.e. $\pi_1(X, x_0) = \{e_{x_0}\}$.

Examples 1.17. Any convex set is simply-connected. $\mathbb{R}^n$ is simply-connected.

Lemma 1.18. *If X is simply-conneted, then any two paths in X are path-homotopic.*

Proof. $[\, f * \overline{g}\,] = [\, e_{x_0}\,] \implies [\, f\,] = [\, g\,]$. □

Now, we define a way for a continuous map $h : X \to Y$ to act on elements of $\pi_1(X, x_0)$.

Definition 1.19. Suppose $h : X \to Y$ is continuous and $h(x_0) = y_0$. (We write this as $h : (X, x_0) \to (Y, y_0)$.) Then, define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by

$$h_*([\, f\,]) = [\, h \circ f\,] \tag{1.20}$$

This is the **homomorphism induced by h**, relative to x_0.

Lemma 1.21. *$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism.*

Proof. Let $f, g \in \pi_1(X, x_0)$. Then,

$$\begin{aligned} h_*([\, f\,]) * h_*([\, g\,]) &= [\, h \circ f\,] * [\, h \circ g\,] = [\, (h \circ f) * (h \circ g)\,] \\ &= [\, h \circ (f * g)\,] = h_*([\, f * g\,]) \end{aligned}$$

□

Notation 1.22. Suupose $x_0, x_1 \in X$ and $h(x_1) = y_1$. To avoid confusion, we write $(h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $(h_{x_1})_* : \pi_1(X, x_1) \to \pi_1(Y, y_1)$.

Theorem 1.23. *Suppose $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous. Then, $(k \circ h)_* = k_* \circ h_*$. Also, suppose $i : (X, x_0) \to (X, x_0)$ is the identity map. Then, i_* is the identity map in $\pi_1(X, x_0)$.*

Theorem 1.24. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism, and $\pi_1(X, x_0) \approx \pi_1(Y, y_0)$.*

Proof. By hypothesis, $h : (X, x_0) \to (Y, y_0)$ and $h^{-1} : (Y, y_0) \to (X, x_0)$ are continuous; thus, h_* and $(h^{-1})_*$ are homomorphisms. But $h_* \circ (h^{-1})_* = (h \circ h^{-1})_* = i_* \implies (h^{-1})_* = h_*^{-1}$ is a homomorphism, hence h_* is an isomorphism. □

2. Results from $\pi_1(S^1) \approx \mathbb{Z}$: Fundamental Theorem of Algebra, Brouwer's Fixed Point Theorem

It is a well-known theorem that the fundamental group of the circle is isomorphic to the additive group of the integers, i.e.

$$\pi_1(S^1) \approx \mathbb{Z} \tag{2.1}$$

This is a result proved by covering space theory, the details of which this paper will not cover for sake of brevity. We will, however, prove a few important results of this theorem. We begin with the Fundamental Theorem of Algebra.

Theorem 2.2 (Fundamental Theorem of Algebra)**.** *Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \in \mathbb{C}\,[x]$. Then, $\exists z \in \mathbb{C}$ such that $p(z) = 0$.*

Proof. Without loss of generality, we may assume that $a_n = 1$. We proceed by contradiction: if p has no roots in $\mathbb{C}$, then $\forall r \geq 0$, the function

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

defines a loop in $S^1 \subset \mathbb{C}$ since $|f_r(s)| = 1\ \forall r \geq 0, \forall s \in [0,1]$, and this loop begins and ends at $1 \in \mathbb{C}$. As r varies, f_r is a path-homotopy of loops since p is continuous. But f_0 is the trivial loop, so for all $r \geq 0$, the class $[\, f_r\,] \in \pi_1(S^1, 0)$ is equal to $[\, f_0\,] \approx 0 \in \mathbb{Z}$.

Fix a large value of r such that $r > |a_{n-1}| + |a_{n-2}| + \ldots + |a_0|$. Then, for $|z| = r$, we have

$$|z^n| = r^n = r \cdot r^{n-1} > (|a_{n-1}| + |a_{n-2}| + \ldots + |a_0|) \cdot \left|z^{n-1}\right| \geq \left|a_{n-1}z^{n-1} + \ldots + a_0\right|.$$

From this inequality, we have that the polynomials

$$p_t(z) = z^n + t \cdot (a_{n-1}z^{n-1} + \ldots a_0)$$

have no roots in $\mathbb{C}$. We replace p in the formulation of f_r by the polynomials p_t and let t go from 1 to 0, and we obtain a homotopy from f_r to the loop

$$f_{r,0}(z) = \frac{z^n}{|z^n|} \implies f_{r,0}(e^{2\pi i\theta}) = e^{2n\pi i\theta},$$

which, when θ runs from 0 to 2π, is the loop in S^1 which is equivalent to $n \in \mathbb{Z}$, i.e. it runs n times counter-clockwise around the origin.

But $f_{r,0}$ is homotopic to the trivial loop f_0, so $n = 0$; thus, $p(z) = a_0$, the constant polynomial. $\square$

We continue to another important result.

Theorem 2.3 (Brouwer's Fixed Point Theorem)**.** *Every continuous map $h : D^2 \to D^2$ has a fixed point.*

Proof. Suppose for sake of contradiction that $h(x) \neq x\, \forall x \in D^2$. Then, define a map $f : D^2 \to S^1$ as follows: draw a ray beginning at $h(x)$ and passing through x, and let $f(x)$ be the point of S^1 which intersects the ray. Continuity of f is clear; small movements of x produce small movements of $h(x)$ and thus small movements of the ray passing between them. Note also that $\forall x \in S^1$, $f(x) = x$.

Now, let g be a loop in S^1 with basepoint x_0. Because D^2 is convex and thus simply-connected, there is a homotopy in D^2 between g and the constant loop at x_0; call the homotopy g_t $(0 \leq t \leq 1)$. Then, $f \circ g$ is a homotopy from f to the constant loop at $x_0 \in S^1$; but $\pi_1(S^1, x_0)$ is non-trivial, contradiction. $\square$

We end this section with one means to calculate the fundamental group of a space that is the product of spaces whose fundamental groups we already know:

Proposition 2.4. $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$ *if X and Y are path-connected spaces.*

Proof. A map $f : Z \to X \times Y$ is continuous iff the associated maps $g : Z \to X$ and $h : Z \to Y$ defined by $f(z) = (g(z), h(z))$ are continuous. So, a loop f in $X \times Y$, based at (x_0, y_0), is equivalent to a pair of loops g in X and h in Y, based at x_0 and y_0 respecitvely. Similarly, a homotopy f_t of a loop in $X \times Y$ is equivalent to a pair of homotopies g_t and h_t of the corresponding loops in X and Y, respectively.

Thus, we have a bijection $[\,f\,] \overset{\phi}{\longmapsto} ([\,g\,], [\,h\,])$. ϕ is clearly a group homomorphism, hence it is an isomorphism, so that $\pi_1(X \times Y, (x_0, y_0)) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$. □

Corollary 2.5. *The fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

Proof. The torus is defined as $S^1 \times S^1$. Apply the above lemma. □

3. Deformation Rectractions and Homotopy Equivalence

We consider a type of related spaces and the relation between their fundamental groups. We begin with a lemma:

Lemma 3.1. *Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic, and if the basepoint x_0 remains fixed during the homotopy, then $h_* = k_*$.*

Proof. Let f be a loop in X based at x_0. By assumption, $\exists$ a homotopy $H : I \times I \to Y$ between h and k such that $H(x_0, t) = y_0$ $\forall t \in [0, 1]$. Then, the composite

$$I \times I \xrightarrow{f \times \mathrm{id}} X \times I \xrightarrow{H} Y$$

is a homotopy between $h \circ f = h_*$ and $k \circ f = k_*$, and it is a path-homotopy because H is basepoint-preserving by assumption, so that $H(\{x_0\} \times I) = y_0$. □

As a result, we have the following theorem:

Theorem 3.2. *The inclusion map $j : S^n \hookrightarrow \mathbb{R}^{n+1} \backslash \boldsymbol{0}$ induces an isomorphism of fundamental groups.*

Proof. Let $X = \mathbb{R}^{n+1} \backslash \mathbf{0}$ and $b_0 = (1, 0, \ldots, 0) \in X$. Let $r : X \to S^n$ be the map

$$r(x) = \frac{x}{\|x\|}$$

Then, $r \circ j = \mathbf{1}_{S^n}$, so that $r_* \circ j_* = \mathbf{1}_{\pi_1(S^n, b_0)}$.

Now, consider $j \circ r$:

$$X \xrightarrow{r} S^n \overset{j}{\hookrightarrow} X$$

This map is homotopic to the identity map of X by the straight-line homotopy $H(x, t) = (1 - t)x + t \cdot \frac{x}{\|x\|}$. Because the coefficient on x in H is between 1 and $\frac{1}{\|x\|}$, $H(x, t) \neq 0$ for all x, t. Note also that $H(b_0, t) = b_0$ since $\|b_0\| = 1$.

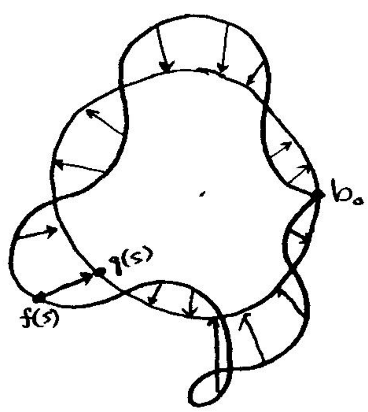

Figure 1. Retraction of a loop in $\mathbb{R}^2 \backslash \mathbf{0}$ into $S^1 \subset \mathbb{R}^2 \backslash \mathbf{0}$.

Thus, by (3.1), $j_* \circ r_* = \mathbf{1}_{\pi_1(X,b_0)}$, so that j_* and its inverse are homomorphisms, i.e $j_* : \pi_1(S^n, b_0) \to \pi_1(X, b_0)$ is an isomorphism. Though we do not know the fundamental group of S^n yet, we will soon. □

This relation of the fundamental groups of one space (S^n) embedded in another space ($\mathbb{R}^{n+1} \backslash \mathbf{0}$) can be generalized to a type of subsets of a space, called deformation retracts:

Definition 3.3. Let x be a space and $A \subset X$. A is a **deformation retract** of X if $\mathbf{1}_X$ is homotopic to a continuous map ($X \xrightarrow{r} A \overset{j}{\hookrightarrow} X$) such that every point of A is fixed during the homotopy. That is, $\exists$ continuous $H : X \times I \to X$ such that $H(x, 0) = x \ \forall x \in X$, $H(x, 1) \in A \ \forall x \in X$, and $H(a, t) = a \ \forall a \in A$, $\forall t \in [0, 1]$.

The homotopy H is called a **deformation retraction** of X onto A. The homotopy is between the identity map of X and the map $j \circ r$, where $r : X \to A$ is the retraction $r(x) = H(x, 1)$, and $j : A \hookrightarrow X$ is inclusion.

We then have the following relationship between the fundamental groups of a deformation retract of a space and that space:

Theorem 3.4. *Let A be a deformation retract of X, and let $x_0 \in A$. Then, the inclusion map $j : (A, x_0) \hookrightarrow (X, x_0)$ induces an isomorphism of fundamental groups.*

Proof. This is a direct generalization of the above proof. □

Example 3.5. Let B denote the z-axis in $\mathbb{R}^3$. Consider the space $\mathbb{R}^3 \backslash B$; it has the punctured plane $\mathbb{R}^2 \backslash \{0\}$ as a deformation retract via the deformation retraction

$$H\left((x, y, z), t\right) = \left(x, y, (1 - t)z\right),$$

which is a "flattening" of $\mathbb{R}^3 \backslash B$ onto $\mathbb{R}^2 \backslash \{0\}$. Thus, we have $\pi_1(\mathbb{R}^3 \backslash B) \approx \pi_1(\mathbb{R}^2 \backslash \{0\})$.

But we also have that S^1 is a deformation retract of $\mathbb{R}^2 \backslash \{0\}$ by the deformation retraction

$$G((x, y, 0), t) = (1 - t) \cdot (x, y, 0) + t \cdot \frac{(x, y, 0)}{\|(x, y, 0)\|}.$$

Thus, we conclude that $\pi_1(\mathbb{R}^3 \backslash B) \approx \pi_1(S^1) \approx \mathbb{Z}$.

There is an even more general way to show that two spaces have isomorphic fundamental groups, however. Consider, for instance, a space X with deformation retracts Y and Z such that neither is a deformation retract of the either. Yet Y and Z still have isomorphic fundamental groups! We introduce the notion of homotopy equivalence:

Definition 3.6. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps such that $f \circ g$ is homotopic to $\mathbf{1}_X$ and $g \circ f$ is homotopic to $\mathbf{1}_Y$. Then, the maps f and g are called **homotopy equivalences**, and each is said to be a **homotopy inverse** of the other.

If $f : X \to Y$ and $h : Y \to Z$ are homotopy equivalences (of X with Y, and of Y with Z, respectively), then it is clear that $h \circ f : X \to Z$ is a homotopy equivalence of X with Z. Thus, homotopy equivalence is, in fact, an equivalence relation. We say that two spaces that homotopy equivalent have the same **homotopy type**.

Example 3.7. A deformation retract of a space has the same homotopy type as that space. Two spaces which are deformation retracts of the space space have the same homotopy type.

We would like to show that spaces of the same homotopy type do, indeed, have isomorphic fundamental groups. We begin with a lemma.

Lemma 3.8. *Let $h, k : X \to Y$ be continuous maps with $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, then $\exists$ a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$, i.e. $k_*([\,f\,]) = [\,\overline{\alpha}\,] * h_*([\,f\,]) * [\,\alpha\,]$. More specifically, if $H : X \times I \to Y$ is the homotopy between h and k, then $\alpha(t) = H(x_0, t)$.*

Proof. Let $f : I \to X$ be a loop in X based at x_0. We must show that $k_*[\,f\,] = [\,\overline{\alpha}\,] * h_*([\,f\,]) * [\,\alpha\,]$, or $[\,\alpha\,] * [\,k \circ f\,] = [\,h \circ f\,] * [\,\alpha\,]$. This is the equation we shall verify.

Consider the loops in the space $x \times I$ given by $f_0(s) = (f(s), 0)$ and $f_1(s) = (f(s), 1)$. Also, consider the path $c(t) = (x_0, t)$. Then,

$$H \circ f_0 = h \circ f \text{ and } H \circ f_1 = k \circ f,$$

and we choose $\alpha = H \circ c$. Also, let $F : I \times I \to X \times I$ be the map $F(s,t) = (f(s), t)$.

Define the following paths in $I \times I$, running along the edges:

$$\beta_0(s) = (s, 0) \text{ and } \beta_1(s) = (s, 1),$$
$$\gamma_0(t) = (0, t) \text{ and } \gamma_1(t) = (1, t).$$

Then, $F \circ \beta_0 = f_0$ and $F \circ \beta_1 = f_1$, while $F \circ \gamma_0 = F \circ \gamma_1 = (x_0, t) = c$. Then, the concatenations $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ from $(0,0)$ to $(1,1)$; since $I \times I$ is convex, there exists a path homotopy $G : I^2 \times I \to I^2$ between them.

Then, $F \circ G$ is a path homotopy in $X \times I$ between $f_0 * c$ and $c * f_1$, and $H \circ (F \circ G)$ is a path homotopy in Y between $H(f_0 * c) = (h \circ f) * \alpha$ and $H(c * f_1) = \alpha * (k \circ f)$, as desired. $\square$

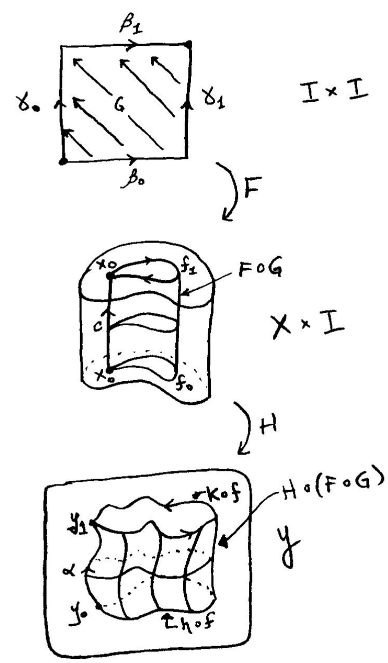

FIGURE 2. The defined paths and homotopies of the above proof.

Corollary 3.9. *Let $h, k : X \to Y$ be homotopic continuous maps. If h_* is injective, surjective, or trivial, then so is k_*. Specifically, if $h : X \to X$ is nulhomotopic, then h_* is the identity map in $\pi_1(X, x_0)$.*

We now prove our desired relationship:

Theorem 3.10. *Let $f : X \to Y$ be a homotopy equivalence, with $f(x_0) = y_0$. Then*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Proof. Let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1),$$

where $x_1 = g(y_0)$ and $y_1 = f(y_1)$. We have the corresponding homomorphisms:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(f_{x_0})_*"] \arrow[d, "\hat{\alpha}"'] & \pi_1(Y, y_0) \arrow[dl, "g_*"'] \\ \pi_1(X, x_1) \arrow[r, "(f_{x_1})_*"] & \pi_1(Y, y_1) \end{tikzcd}$$

Now, by hypothesis, the map $g \circ f : (X, x_0) \to (X, x_1)$ is homotopic to the identity map in X, so there exists a path α in X such that $(g \circ f)_* = \hat{\alpha} \circ (i_X)_* = \hat{\alpha}$, which is an isomorphism; thus, g_* must be surjective.

Similarly, because $f \circ g$ is homotopic to the identity map in Y, $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism, so that g_* must also be injective. Thus, g_* is an isomorphism.

Applying the first equation again,

$$(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha},$$

so that $(f_{x_0})_* = f_*$ is also an isomorphism, as desired. □

Examples 3.11. Any convex subset of $\mathbb{R}^n$ is homotopy equivalent to a single point. The cylinder ($S^1 \times I$) and solid torus ($S^1 \times D^2$) are homotopy equivalent to S^1. If C is a contractable space and X is any space, then $X \times C$ is homotopy equivalent to X.

4. Preliminaries for Van Kampen's Theorem: Free Products, First Isomorphism Theorem

We provide the definitions and lemmas necessary to prove the main result in this paper, the Van Kampen's Theorem. We begin with the definintion of a *free product*. This paper assumes knowledge of basic group theory, e.g. what a group is.

Definition 4.1. Let $\{G_\alpha\}$ be a family of groups. Then, the **free product** on $\{G_\alpha\}$ is the set of all "words" $g_1 g_2 \dots g_m$ of arbitrary finite length $m \geq 0$, where each g_i and g_{i+1} belong to different groups G_{α_i} and $G_{\alpha_{i+1}}$, and each g_i is not the identity in the group G_{α_i}. The set includes the "empty word" $\emptyset$, which will be the identity. The group operation is **juxtaposition**, i.e. $(g_1 g_2 \dots g_m) * (h_1 h_2 \dots h_n) = g_1 \dots g_m h_1 \dots h_n$, though this word might not be **reduced**, i.e. satisfy the conditions above, namely that g_m and h_1 might be in the same group. In this case, words can be reduced by writing the word $g_m h_1$ as the element $k = g_m \cdot h_1 \in G_\alpha$. If the words juxtaposed are reduced words, then the word $g_1 \dots g_{m-1} k h_1 \dots h_n$ will be reduced. If not, then we can continue canelling letters like this until a reduced word is reached.

The free product on $\{G_\alpha\}$ is written as $*_\alpha \{G_\alpha\}$. Verification that juxtaposition defines a group operation on $*_\alpha \{G_\alpha\}$ is messy, especially with respect to associativity; see [Hatcher 41-42].

Example 4.2. An example of a free product is $\mathbb{Z} * \mathbb{Z}$, i.e. the free product on two infinite cyclic groups. These take the form $a^{n_1} b^{n_2} a^{n_3} \dots a^{n_m}$ for some $m \geq 0$ and $n_k \in \mathbb{Z}$. One important aspect of the free product to note is that it is *not* an Abelian group, e.g. $a^2 b^3 a^4 \neq b^3 a^6$.

Lemma 4.3. *Suppose $\{G_\alpha\}$ is a set of groups and H is a group, and suppose $h_\alpha : G_\alpha \to H$ are homomorphisms. Then, these homomorphisms extend uniquely to a homomorphism $h : *_\alpha \{G_\alpha\} \to H$.*

Proof. By assumption, h must map one-letter words whose letter comes from G_α to that element's image in H under h_α. Then, we must uniquely have

$$h(g_1 g_2 \dots g_m) = h_{\alpha_1}(g_1) h_{\alpha_2}(g_2) \dots h_{\alpha_m}(g_m), \tag{4.4}$$

which is well-defined, since the different groups G_α are considered disjoint. $\square$

We also require the following group theoretic result:

Lemma 4.5 (First Isomorphism Theorem)**.** *If $f : G \to H$ is a homomorphism, then it induces an isomorphism $G/ker(f) \approx im(f)$. Specifically, if f is surjective, then $G/ker(f) \approx H$.*

Proof. Let K denote the kernel of f.

(1): *K is a normal subgroup of G.*

Let $g \in G$ and $k \in K$. Then,

$$\begin{aligned} f(g^{-1} k g) &= f(g)^{-1} f(k) f(g) \\ &= f(g)^{-1} f(g) \\ &= 1 \implies g^{-1} k g \in K \implies K \triangleleft G, \end{aligned}$$

so that the group G/K is well-defined.

Let $\beta : G/K \to \operatorname{im}(f)$ be the function $gK \overset{\beta}{\longmapsto} f(g)$. We wish to show that β is an isomorphism.

(2): *β is well-defined.*

Let $g_1, g_2 \in G$ such that $g_1 K = g_2 K$. Then,

$$\begin{aligned} K = (g_1^{-1} g_2) K &\implies g_1^{-1} g_2 \in K \implies f(g_1^{-1} g_2) = 1 \\ &\implies f(g_1)^{-1} f(g_2) = 1 \implies f(g_1) = f(g_2), \end{aligned}$$

so β is well-defined.

(3): *β is a homomorphism.*

Let $g_1 K, g_2 K \in G/K$. Then,

$$\begin{aligned} \beta(g_1 K \cdot g_2 K) &= f(g_1 \cdot g_2) \\ &= f(g_1) f(g_2) \\ &= \beta(g_1 K) \beta(g_2 K). \end{aligned}$$

(4): *β is an isomorphism.*

First, the kernel of β in G/K is exactly the one coset K because β is well-defined and K consists exactly of all elements of G that map to the identity in $\mathrm{im}(f)$. Then, let $g_1K, g_2K \in G/K$ such that $\beta(g_1K) = \beta(g_2K)$. Then,

$$\begin{aligned} f(g_1) = f(g_2) \quad &\Longrightarrow \quad f(g_2^{-1})f(g_1) = f(g_2)^{-1}f(g_1) = 1 \\ &\Longrightarrow \quad g_2^{-1}g_1 \in K \implies g_1K = g_2K, \end{aligned}$$

so β is injective.

Let $h \in \mathrm{im}(f)$ and g its pre-image under f. Then, $\beta(gK) = f(g) = h$, so that β is surjective, and thus β is an isomorphism, as desired. $\square$

5. Van Kampen's Theorem

We now prove a theorem powerful for finding the fundamental group of a space composed of simpler spaces whose fundamental groups we know. For instance, what is the fundamental group of the "figure-8" space, composed of two copies of S^1 with one point in common? Or, what is the fundamental group of S^n, the n-sphere, composed of two copies of n-discs, D^n, whose boundaries are glued together? We now prove that, given broad hypothesis, this fundamental group is nearly isomorphic to the free product on the fundamental groups of the spaces that compose the larger space.

Theorem 5.1. *Let X be a space that is the union of path-connected open sets A_α, each containing the same basepoint x_0. If each intersection $A_\alpha \cap A_\beta$ is path-conected, then the homomorphism $\Phi : *_\alpha\pi_1(A_\alpha) \to \pi_1(X)$ that is the extension of the homomorphisms $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \to X$, is surjective.*

If, in addition, each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all of the elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$, where $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is induced by $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$. Thus, Φ induces an isomorphism $\pi_1(X) \approx *_\alpha\pi_1(A_\alpha)/N$.

$$\begin{tikzcd}
& \pi_1(A_\alpha) \arrow[d, "j_\alpha"] \arrow[rrd, bend left] & & \\
\pi_1(A_\alpha \cap A_\beta \cap A_\gamma) \arrow[ru, "i_{\alpha\beta}"] \arrow[r] \arrow[rd, "i_{\beta\alpha}"'] \arrow[rdd, "i_\gamma"'] & \pi_1(X) & *_\alpha\pi_1(A_\alpha)/N \arrow[l, Rightarrow, two heads, "\Phi'"'] & *_\alpha\pi_1(A_\alpha) \arrow[l, "(f\mapsto fN)"'] \arrow[ll, dashed, bend left, two heads, "\Phi"] \\
& \pi_1(A_\beta) \arrow[u, "j_\beta"'] \arrow[rru, bend right] & & \\
& \pi_1(A_\gamma) \arrow[uu, bend right, "j_\gamma"'] \arrow[rruu, bend right] & &
\end{tikzcd}$$

Proof. (1): **Φ is surjective.**

Given a loop $f : I \to X$ based at x_0, there is a partition $0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$ of $[0,1]$ such that each interval $[s_i, s_{i+1}]$ is mapped to a single A_α. This is because each $s \in [0,1]$ has a neighborhood that is mapped completely to one A_{α_s} in X; we may even take these neighborhoods to be closed. Compactness of I implies that there exists a finite covering by these neighborhoods, and their boundaries determine our partition.

Denote the A_α containing $f([s_{i-1,s_i}])$ by A_i, and let $f_i = f|_{[s_{i-1},s]}$. Then, $f = f_1 * f_2 * \cdots * f_m$, with each path being considered a path in the corresponding A_i.

Since $A_i \cap A_{i+1}$ is path-connected, $\exists$ a path g_i from x_0 to $f(s_i) \in A_i \cap A_{i+1}$. Consider the loop

$$(f_1 * \overline{g_1}) * (g_1 * f_2 * \overline{g_2}) * \cdots * (g_{m-2} * f_m * \overline{g_{m-1}}) * (g_{m-1} * f_m) \simeq_P f.$$

This loop is a composition of loops, each lying entirely in a single A_i. Thus, $[\,f\,]$ is in the image of Φ, as it is the product of equivalence classes of loops, which is the image under Φ of a word in $*_\alpha \pi_1(A_\alpha)$. Thus, Φ is surjective.

(2): **The kernel of Φ is N.**

Define a *factorization* of $[\,f\,] \in \pi_1(X)$ as a formal product $[\,f_1\,] \cdot [\,f_k\,]$, where

(1) Each f_i is a loop in some A_i with basepoint x_0, and
(2) f is homotopic to $f_1 * f_2 * \cdots f_k$ in X.

Thus, a factorization is a word in $*_\alpha \pi_1(A_\alpha)$, possibly unreudced, that is mapped to $[\,f\,]$ by Φ. Because Φ is surjective, every $[\,f\,]$ has a factorization.

We consider two factorizations equivalent if they are related via two operations or their inverses:

(1) Combine adjacent terms $[\,f_i\,][\,f_{i+1}\,]$ into a single term $[\,f_i * f_{i+1}\,]$ if $[\,f_i\,]$ and $[\,f_{i+1}\,]$ are elements of $\pi_1(A_\alpha)$, or
(2) Regard the term $[\,f_i\,] \in \pi_1(A_\alpha)$ as lying in $\pi_1(A_\beta)$ instead, if f_i is a loop in $A_\alpha \cap A_\beta$.

The first "move" does not change the element in $*_\alpha \pi_1(A_\alpha)$, and hte second does not change the image of this element in the quotient group $Q = *_\alpha \pi_1(A_\alpha)/N$ by the definition of N. So, equivalent factorizations give the same element in Q.

If we can show that any two factorizations are equivalent, then this will mean that the map $Q \to \pi_1(X)$ induced by Φ, i.e.

$$([\,f_1\,][\,f_2\,] \ldots [\,f_k\,])\,N \mapsto \Phi([\,f_1\,][\,f_2\,] \ldots [\,f_k\,])\,,$$

is injective, so that $\ker(\Phi) = N$, as we desire. So:

Let $[\,f_1\,] \ldots [\,f_k\,]$ and $[\,f_1'\,] \ldots [\,f_l'\,]$ be two factorizations of $[\,f\,]$. Let $F : I \times I \to X$ be the homotopy between $f_1 * \cdots * f_k$ and $f_1' * \cdots * f_l'$. Let $0 = s_0 < s_1 < \ldots < s_m = 1$ and $0 = t_0 < \ldots < t_n = 1$ be the partitions of $[0,1]$ such that each rectangle $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped onto a single A_α, which we will label $A_{i,j}$. These partitions may be obtained by covering $I \times I$ with finitely many rectangles $[a,b] \times [c,d]$, each mapping to a single A_α (which exist since $I \times I$ is compact), then partitioning $I \times I$ by the union of all vertical and horizontal lines containing edges of these rectangles.

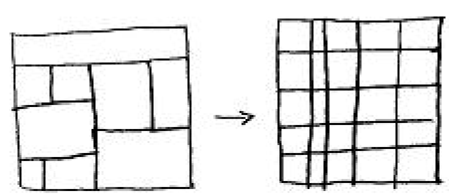

FIGURE 3. The desired partition of $I \times I$.

We may assume the s-partition is a refinement of the partitions giving the products $f_1 * \cdots * f_k$ and $f_1' * \cdots * f_l'$. (If it is not, take the common refinement of the three partitions.)

Since F maps a neighborhood of $R_{i,j}$ to $A_{i,j}$, we may perturb the vertical sides of the rectnagles so that each point of $I \times I$ lies in at most three distinct rectangles. We may also assume that there are at least three rows of rectangles, so that we need not perturb the top and bottom rows, as in Figure 4. Let us relable the rectangles $R_1, R_2, \ldots, R_{mn}$, as in Figure 5.

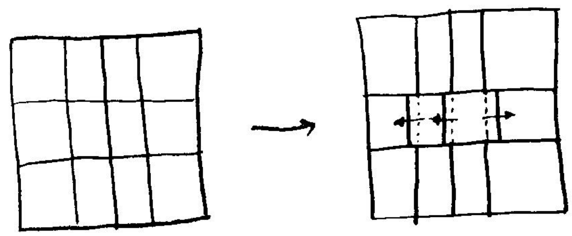

FIGURE 4. Perturbation of the partitions of $I \times I$.

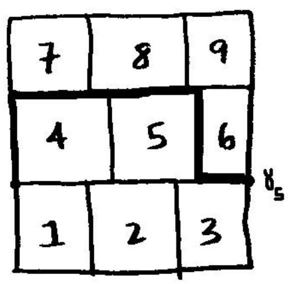

FIGURE 5. Relabelling of the partitions, and an example path.

If γ is a path in $I \times I$ from the left edge to the right edge, then the restriction $F|_\gamma$ is a loop at the basepoint x_0, since $F(\{0\} \times I) = F(\{1\} \times I) = x_0$. Let γ_r be the path separating the first r rectangles from the rest. Thus, γ_0 is the bottom edge of $I \times I$, and γ_{mn} is the top edge. We pass from γ_r to γ_{r+1} by "pushing" across R_{r+1}, i.e. applying a homotopy in $I \times I$ from γ_r to γ_{r+1}, which exists because $I \times I$ is simply-connected.

Let us call the corners of the rectangles *vertices*. For each vertex v such that $F(v) \neq x_0$, let g_v be a path from x_0 to $F(v)$. We can choose g_v to lie in the intersection of the two or three $A_{i,j}$'s corresponding to the R_r's containing v because, by asumption, $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected and contains x_0.

If we insert into $F|_{\gamma_r}$ the appropriate paths $\overline{g_v} \cdot g_v$ at successive vertices, as we did for the proof that Φ is surjective, we obtain a factorization of $[\, F|_{\gamma_r}]$ by regarding the loop corresponding to a horizontal or vertical segment between adjacent vertices as lying in the $A_{i,j}$ corresponding to either of the R_r's containing the segment. Different choices of R_r's here change the factorization of $[\, F|_{\gamma_r}]$ to an equivalent factorization via the second "move". Also, the factorizations associated with paths γ_r and γ_{r+1} are equivalent because pushing γ_r across R_{r+1} to γ_{r+1} homotopes $F|_{\gamma_r}$

to $F|_{\gamma_{r+1}}$ by a homotopy within the $A_{i,j}$ corresponding to R_{r+1}, and we can choose this $A_{i,j}$ for all segments of γ_r and γ_{r+1} in R_{r+1}.

We can arrange that the factorization associated to γ_0 is equivalent to our first factorization $[\,f_1\,] * \cdots * [\,f_k\,]$ by choosing the path g_v (from x_0 to $F(v)$) for each vertex v along the lower edge of $I \times I$ to lie not only in the two $A_{i,j}$'s corresponding to the R_r and R_{r+1} containing v, but also to lie in the A_α that is the image space of the f_i whose domain includes v. In case v is the common endpoint of two domains of consecutive f_i's, we have $F(v) = x_0$ (since $x_0 \in A_\alpha\ \forall\alpha$), so no path g_v needs to be chosen. In a similar manner, we can assume that the factorization associated with γ_{mn} is equivalent to $[\,f'_1\,] * \cdots [\,*f'_l\,]$.

Since all of the factorizations associated with the γ_r's are equivalent, the factorizations $[\,f_1\,] * \cdots * [\,f_k\,]$ and $[\,f'_1\,] * \cdots [\,*f'_l\,]$ are equivalent. Thus, $\ker\Phi = N$, as desired. It follows from (4.5) that since Φ is surjective, Φ induces an isomorphism $*_\alpha \pi_1(A_\alpha)/N \to \pi_1(X)$, and thus, $*_\alpha\pi_1(A_\alpha) \approx \pi_1(X)$. $\square$

Corollary 5.2. *Assume the hypotheses of the Van-Kampen's Theorem. If $\cap_\alpha A_\alpha$ is simply-connected, then*

$$*_\alpha\pi_1(A_\alpha) \approx \pi_1(X).$$

Corollary 5.3. *Assume the hypotheses of the Van-Kampen's Theorem for $X = A_1 \cup A_2$. If A_2 is simply-connected, then*

$$\pi_1(A_1)/N' \approx \pi_1(X), \tag{5.4}$$

where N' is the least normal subgroup of $\pi_1(A_1)$ containing the image of the homomorphism $i_{1,2} : \pi_1(A_1 \cap A_2) \to \pi_1(A_1)$.

We now have the power to calculate the fundamental groups of a number of spaces and show that they are topologically distinct.

Corollary 5.5. *$\pi_1(S^n)$ is trivial for $n \geq 2$.*

Proof. $S^n = D^n \cup D^n$, with the intersection of the D^n's being their boundary S^{n-1}, which is path-connected. The D^n's are convex and thus simply-connected, i.e. have trivial fundamental groups.

Thus, N is trivial, since its generators are trivial. By the Van Kampen's Theorem, $\pi_1(S^n)$ is trivial. $\square$

Corollary 5.6. *The figure-8 space, which is $S^1 \vee S^1$, has fundamental group $\pi_1(S^1 \vee S^1) \approx \mathbb{Z} * \mathbb{Z}$.*

Proof. $S^1 \vee S^1$ is decomposable as the union of $\big(S^1 \vee \big(S^1\backslash p\big)\big) \cup \big(\big(S^1\backslash q\big) \vee S^1\big)$, where p and q are in different copies of S^1. The intersection is contractable, so it is simply-connected. Thus, by (5.2),

$$\pi_1(S^1 \vee S^1) \approx \pi_1(S^1) * \pi_1(S^1) \approx \mathbb{Z} * \mathbb{Z}.$$

More generally, the n-fold wedge sum of S^1 has fundamental group isomorphic to $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}$. $\square$

Corollary 5.7. *The circle, 2-sphere, torus, and n-fold wedge sums of circles are topologically distinct.*

The Van-Kampen's Theorem may be applied to numerous other spaces which are decomposable into the spaces A_α required by the hypotheses of the theorem. There are more ways to show that two spaces are topologically distinct, however; for instance, $\mathbb{R}^n$ and $\mathbb{R}^m$ are both simply-connected, but for $m \neq n$, they are topologically distinct. For this, more homotopy theory is required and is beyond the scope of this paper. Yet, the fundamental group, as we have seen, is a very powerful tool in algebraic topology.

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