ISOMETRIES OF THE PLANE

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Abstract. First we define different kinds of isometries of the plane. Then we prove that isometries of the plane are determined by one of two equations. Using these two equations, we can determine the only four possible types of isometries of the plane: translations, rotations, reflections, and glide-reflections.

Contents

1. What is an isometry?

An isometry is a distance-preserving transformation. In this paper, we consider isometries of the plane $\mathbb{C}$.

Definition 1.1. A transformation $\alpha : \mathbb{C} \to \mathbb{C}$ is an *isometry* if for any two points $p, q \in \mathbb{C}$, the Euclidean distance $|\alpha(p) - \alpha(q)| = |p - q|$.

Surprisingly, there are only four types of isometries of the Euclidean plane, which together form a group under composition known as the *Euclidean group of the plane*. The next sections will formally define these four isometries and prove the classification theorem, which states that every isometry of the plane must be of one of these four types.

2. Translations and Rotations

Definition 2.1. A *translation* τ is a mapping from $\mathbb{C}$ to $\mathbb{C}$ defined by equations of the form

$$\begin{cases} x' = x + a \\ y' = y + b. \end{cases}$$

For any point $x + iy$ in the plane, let $x' + iy' = \tau(x + iy) = x + a + i(y + b)$. We can rewrite this as $\tau : z \mapsto z + c$, where $z = x + iy$ and $c = a + ib$. The transformation τ moves each point the same distance in the same direction.

Proposition 2.2. *A translation τ is an isometry of the plane.*

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Proof. Let $p, q \in \mathbb{C}$. Suppose $\tau : z \mapsto z + c$. Then

$$|\tau(p) - \tau(q)| = |(p+c) - (q+c)| = |p - q|\,.$$

$\square$

Definition 2.3. A *rotation* $\rho_{0,\theta}$ is a transformation that rotates about the origin through an angle of rotation θ.

Such a rotation is given by the equation $\rho_{0,\theta} : z \mapsto e^{i\theta} z$. Let $z = x + iy$.

$$\begin{aligned}\rho_{0,\theta}(z) = e^{i\theta} z &= (\cos\theta + i\sin\theta)(x + iy)\\ &= (x\cos\theta + iy\cos\theta + ix\sin\theta - y\sin\theta)\\ &= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta).\end{aligned}$$

Rotations around points other than the origin will be discussed later.

Proposition 2.4. *Rotations about the origin are isometries of the plane.*

Proof. Let $p, q \in \mathbb{C}$. Suppose $\rho_{0,\theta} : z \mapsto e^{i\theta} z$. Then

$$\begin{aligned}|\rho_{0,\theta}(p) - \rho_{0,\theta}(q)| &= \left|(e^{i\theta}p) - (e^{i\theta}q)\right|\\ &= \left|e^{i\theta}(p-q)\right|\\ &= \left|e^{i\theta}\right||p-q|\\ &= |p-q|\,.\end{aligned}$$

$\square$

3. Reflections and Glide-Reflections

Consider the map $\sigma : z \mapsto \overline{z}$. This map σ fixes points on the real line because $z = x + iy = x - iy = \overline{z}$ if and only if $y = 0$. Now consider point $z = x + iy$ where $y \neq 0$. Then $\sigma(z) = \overline{z} \neq z$. The segment connecting z and $\overline{z}$ consists points of the form $x + it$, for $t \in [-y, y]$. The point x on the real line is the midpoint of this segment, and since this segment is vertical, we can conclude that the real line is the perpendicular bisector of this segment between z and $\overline{z}$.

We can generalize this kind of transformation to define a reflection in a line m.

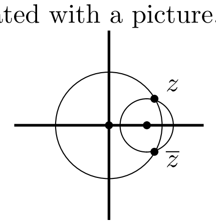
Definition 3.1. A *reflection* σ_m *in a line* m is the mapping defined by

$$\sigma_m(z) = \begin{cases} z, \text{ if point } z \text{ is on } m\\ z', \text{ if } z \text{ is not on } m \text{ and } m \text{ is the perpendicular bisector}\\ \qquad \text{of the segment between } z \text{ and } z'.\end{cases}$$

We will not prove it here, but a reflection is an isometry of $\mathbb{C}$.

Proposition 3.2. *Consider an isometry α which fixes the points* 0 *and* 1*. Then either $\alpha : z \mapsto z$ for all z, or $\alpha : z \mapsto \overline{z}$ for all z.*

Proof. This proof can be illustrated with a picture.



Let z be any other point of $\mathbb{C}$. Draw a circle around 0 that passes through z and a second circle around 1 that also passes through z. These two circles intersect at z and at $\overline{z}$, as seen in the picture above. The map α must preserve the distance between z and the two fixed points, 0 and 1. To preserve these distances, $\alpha(z)$ must lie on both of these circles. Hence, α must either fix z or send z to its complex conjugate $\overline{z}$ in order to preserve the distance between z and the two fixed points, 0 and 1. Both maps $\alpha : z \mapsto z$ and $\alpha : z \mapsto \overline{z}$ preserve these distances, and from the picture it is clear that these are the only two maps that preserve these distances.

We need to makes sure that either $\alpha(z) = z$ for all z, or $\alpha(z) = \overline{z}$ for all z. Suppose there are points z and w such that $\alpha(z) = \overline{z} \neq z$ and $\alpha(w) = w$. Because α is an isometry,

$$|\alpha(z) - \alpha(w)| = |z - w|\,.$$

From our assumptions about α we also know

$$|\alpha(z) - \alpha(w)| = |\overline{z} - w|\,.$$

From these two equalities we can conclude that w lies on the perpendicular bisector of the segment connecting z and $\overline{z}$. We know from before that the real line is the perpendicular bisector of such a segment, so w is a real number and is fixed by the map $\alpha : z \mapsto \overline{z}$. From this we can conclude that if there exists any point z such that $\alpha(z) \neq z$, then $\alpha : z \mapsto \overline{z}$ for all z. If α fixes every point of $\mathbb{C}$, then $\alpha : z \mapsto z$ for all z. □

Now consider an isometry α that fixes the point 0. Since α is an isometry, we require $|\alpha(1) - \alpha(0)| = |1 - 0| = 1$. Since 0 is fixed, $|\alpha(1)| = 1$. Note that all unit vectors in $\mathbb{C}$ are of the form $\cos\theta + i\sin\theta = e^{i\theta}$. Thus, for some θ, $\alpha(1) = e^{i\theta}$.

Proposition 3.3. *Let α be an isometry fixing 0 and let $\alpha(1) = e^{i\theta}$. If $\beta : z \mapsto e^{i\theta} z$, then $\beta^{-1}\alpha$ fixes both 0 and 1. Then either $\alpha : z \mapsto e^{i\theta} z$ or $\alpha : z \mapsto e^{i\theta}\overline{z}$.*

Proof. Since β fixes 0, β^{-1} fixes 0. We assume that α fixes 0, so $\beta^{-1}\alpha$ must also fix 0. We now need to show that $\beta^{-1}\alpha$ fixes 1. It follows from our assumptions on α and β that

$$\begin{aligned} \beta^{-1}\alpha(1) &= \beta^{-1}(e^{i\theta}) \\ &= 1. \end{aligned}$$

The transformation $\beta^{-1}\alpha$ fixes both 0 and 1, so by Proposition 3.2, either $\beta^{-1}\alpha : z \mapsto z$ for all z, in which case

$$\begin{aligned} \beta^{-1}\alpha(z) &= z \\ \alpha(z) &= \beta(z) \\ &= e^{i\theta} z, \end{aligned}$$

or $\beta^{-1}\alpha : z \mapsto \overline{z}$, in which case

$$\begin{aligned} \beta^{-1}\alpha(z) &= \overline{z} \\ \alpha(z) &= \beta(\overline{z}) \\ &= e^{i\theta}\overline{z}. \end{aligned}$$

□

We already showed that the map $\alpha : z \mapsto e^{i\theta}z$ is a rotation about the origin. Now we want to understand the map $\alpha : z \mapsto e^{i\theta}\overline{z}$. Let r be a real number. The image of the complex number $re^{i\theta/2}$ under this map α is

$$\alpha\left(re^{i\theta/2}\right) = \left(e^{i\theta}\right)\left(re^{-i\theta/2}\right) = re^{i\theta/2}.$$

Hence, α fixes points $re^{i\theta/2}$ where r is a real number. These fixed points lie on the line that goes through the origin and the point $e^{i\theta/2}$.

The image of the complex number $re^{i\phi}$ under α is $\alpha(re^{i\phi}) = (e^{i\theta})(re^{-i\phi}) = re^{i(\theta-\phi)}$. This map fixes points such that

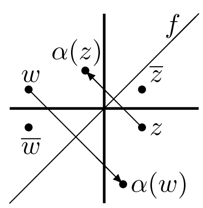
$$\phi = \theta - \phi.$$

In other words, α fixes a point $re^{i\phi}$ if

$$\phi = \frac{\theta}{2}.$$

This implies that α only fixes the line through the origin and $e^{i\theta/2}$.

An explicit example will illustrate exactly what this map, $\alpha : z \mapsto e^{i\theta}\overline{z}$, looks like on the plane. Let $\theta = \frac{\pi}{2}$. First find the line of fixed points. Because $\theta = \frac{\pi}{2}$, α will fix points of the form $re^{i\pi/4}$. The line of fixed points includes 0 and $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$, so we can express this line as $y = \left(\tan\frac{\pi}{4}\right)x = x$. Call this line f, as in the picture below.



For a point z not on f, z is reflected across the line f. First, α takes z to $\overline{z}$ and then rotates $\overline{z}$ by $\theta = \frac{\pi}{2}$. A second example is shown with point w.

Now we show that a rotation can be described as a product of two reflections.

Proposition 3.4. *The rotation $z \mapsto e^{i\theta}z$ is the product of two reflections with axes through the origin.*

Proof. This rotation is equivalent to the reflection $z \mapsto \overline{z}$ followed by a reflection over the line that makes an angle $\frac{1}{2}\theta$ with the real line. □

We will conclude this section with a definition of the fourth type of plane isometry, the glide-reflection. This isometry will be discussed in more detail when it appears in the proof of the classification of plane isometries.

Definition 3.5. A *glide-reflection* is an isometry that is the product of a reflection and a translation in the direction of the axis of the reflection.

4. Classification of Plane Isometries

Theorem 4.1. *Every isometry of the plane, other than the identity, is either a translation, a rotation, a reflection, or a glide-reflection.*

Proof. Let α be any isometry, let $\alpha(0) = c$, and let τ be the translation $\tau : z \mapsto z+c$. By definition,

$$\tau^{-1}\alpha(0) = \tau^{-1}(c) = 0.$$

Thus, the map $\tau^{-1}\alpha$ fixes 0.

Because $\tau^{-1}\alpha$ is an isometry,

$$\left|\tau^{-1}\alpha(1) - \tau^{-1}\alpha(0)\right| = |1-0| = 1.$$

Since $\tau^{-1}\alpha$ fixes 0, it follows that

$$\left|\tau^{-1}\alpha(1)\right| = 1.$$

Thus, there exists some θ such that $\tau^{-1}\alpha(1) = e^{i\theta}$. From Proposition 3.3, it follows that either $\tau^{-1}\alpha : z \mapsto e^{i\theta}z$ or $\tau^{-1}\alpha : z \mapsto e^{i\theta}\overline{z}$. In the first case, $\alpha : z \mapsto e^{i\theta}z + c$ and in the second case, $\alpha : z \mapsto e^{i\theta}\overline{z} + c$. Since α is any isometry, these two equations must define all isometries of the plane. Now we will analyze these two equations and show that they define four types of possible isometries, which were introduced in the previous sections.

Case 1: $\alpha : z \mapsto e^{i\theta}z + c$

If $e^{i\theta} = 1$, then $\alpha : z \mapsto 1 \cdot z + c$, so α a translation. Now suppose $e^{i\theta} \neq 1$. Then $\alpha(p) = p$ if and only if $p = \frac{c}{1-e^{i\theta}}$. For this p,

$$\begin{aligned} \alpha(z) - p &= \alpha(z) - \alpha(p) \\ &= \left(e^{i\theta}z + c\right) - \left(e^{i\theta}p + c\right) \\ &= e^{i\theta}\left(z - p\right). \end{aligned}$$

This map α has the form of a rotation, but around point p rather than at the origin. We have shown that an isometry defined by $\alpha : z \mapsto e^{i\theta}z + c$ is either a translation or a rotation.

Case 2: $\alpha : z \mapsto e^{i\theta}\overline{z} + c$

First note that $\alpha^2 : z \mapsto z$, in other words, α^2 is the identity map, if and only if

$$e^{i\theta}\left(\overline{e^{i\theta}\overline{z} + c}\right) + c = z.$$

Simplifying, we see that this equation is equivalent to

$$\begin{aligned} e^{i\theta}\left(\overline{e^{i\theta}\overline{z} + c}\right) + c &= e^{i\theta}\left(e^{-i\theta}z + \overline{c}\right) + c \\ &= z + e^{i\theta}\overline{c} + c. \end{aligned}$$

It follows that α^2 is the identity map if and only if

$$e^{i\theta}\overline{c} + c = 0. \tag{4.2}$$

Proposition 4.3. *Let $\alpha : z \mapsto e^{i\theta}\overline{z} + c$. If equation (4.2) holds, then α fixes the midpoint of the line segment between 0 and $\alpha(0)$.*

Proof. Since $\alpha(0) = c$, the midpoint of the line segment between 0 and $\alpha(0)$ is $\frac{c}{2}$. By definition of α,

$$\alpha\left(\frac{c}{2}\right) = e^{i\theta}\left(\frac{\overline{c}}{2}\right) + c.$$

From equation (4.2), we find that

$$\left(e^{i\theta}\left(\frac{\overline{c}}{2}\right) + \frac{c}{2}\right) + \frac{c}{2} = 0 + \frac{c}{2},$$

and thus

$$\alpha\left(\frac{c}{2}\right) = \frac{c}{2}.$$

∎

Using this fact, we can show that for any real r,

$$\begin{aligned}\alpha\left(\frac{c}{2}+re^{i\theta/2}\right) &= e^{i\theta}\left(\overline{\frac{c}{2}+re^{i\theta/2}}\right)+c\\ &= \left(e^{i\theta}\frac{\overline{c}}{2}+c\right)+\left(re^{i\theta/2}\right)\\ &= \frac{c}{2}+re^{i\theta/2}.\end{aligned}$$

Hence, if equation (4.2) holds, α fixes the line $\frac{c}{2}+re^{i\theta/2}$.

Proposition 4.4. *If an isometry α has a line of fixed points, then α is either the identity map or a reflection with a line of fixed points as its axis of reflection.*

Proof. This proof is similar to the proof of Proposition 3.2 that involves circle intersections. □

This shows that the map $\alpha : z \mapsto e^{i\theta}\overline{z}+c$ defines a reflection or the identity when α^2 is the identity.

Now suppose $\alpha : z \mapsto e^{i\theta}\overline{z}+c$ is not a reflection. Consider the isometry α^2.

$$\begin{aligned}\alpha^2(z) &= e^{i\theta}\left(\overline{e^{i\theta}\overline{z}+c}\right)+c\\ &= z+\left(e^{i\theta}\overline{c}+c\right)\end{aligned}$$

This is a translation. In fact, $\alpha^2=\tau^2$ where τ is the translation

$$\tau : z \mapsto z+\frac{1}{2}\left(e^{i\theta}\overline{c}+c\right).$$

It follows that

$$\alpha\tau(z) = e^{i\theta}\left(\overline{z+\frac{1}{2}\left(e^{i\theta}\overline{c}+c\right)}\right)+c.$$

Simplifying this expression, we find that

$$\begin{aligned}e^{i\theta}\left(\overline{z+\frac{1}{2}\left(e^{i\theta}\overline{c}+c\right)}\right)+c &= e^{i\theta}\overline{z}+\frac{1}{2}c+\frac{1}{2}e^{i\theta}\overline{c}+c\\ &= e^{i\theta}\overline{z}+c+\frac{1}{2}\left(e^{i\theta}\overline{c}+c\right)\\ &= \tau\alpha(z).\end{aligned}$$

Hence $\alpha\tau=\tau\alpha$, and we can multiply on the left and the right with τ^{-1} to find $\tau^{-1}\alpha=\alpha\tau^{-1}$. It follows that $\left(\tau^{-1}\alpha\right)^2=\left(\alpha\tau^{-1}\right)^2$ and thus

$$\begin{aligned}\left(\tau^{-1}\alpha\right)^2 &= \alpha\tau^{-1}\alpha\tau^{-1}\\ &= \alpha^2\tau^{-2}\\ &= \mathrm{id}.\end{aligned}$$

Since $\left(\tau^{-1}\alpha\right)^2$ is the identity, where $\tau^{-1}\alpha : z \mapsto e^{i\theta}\overline{z}+\frac{1}{2}\left(c-e^{i\theta}\overline{c}\right)=e^{i\theta}\overline{z}+k$, we know from equation (4.2) that $e^{i\theta}\overline{k}+k=0$.

From Proposition 4.3, it follows that $\tau^{-1}\alpha$ fixes the midpoint of 0 and $\tau^{-1}\alpha(0)$, and furthermore, $\tau^{-1}\alpha$ fixes all points of the form $\frac{1}{2}k+re^{i\theta/2}$ for real r. Since $\tau^{-1}\alpha$ has a line of fixed points, then by Proposition 4.4, $\tau^{-1}\alpha$ is either the identity or a reflection with the line of fixed points as an axis.

Now we consider these two possibilities. If $\tau^{-1}\alpha$ is the identity map then $\alpha = \tau$ which means α is a translation. However, we showed in Case 1 that a translation must be of the form $\alpha : z \mapsto e^{i\theta}z + c$. Hence, $\tau^{-1}\alpha \neq 1$, and so $\tau^{-1}\alpha = \eta$ for η a reflection. Moreover, starting with the fact $\tau\alpha = \alpha\tau$, we can substitute in $\alpha = \tau\eta$ and multiply on the left by τ^{-1} to find $\tau\eta = \eta\tau$.

Proposition 4.5. *Let α, τ, and η be defined as above. If A is a point such that $\eta(A) = A$, then $\eta(\tau(A)) = \tau(A)$. Thus α is a glide-reflection.*

Proof. Since η and τ commute by the above calculation,

$$\begin{aligned} \eta(\tau(A)) &= \tau\eta(A) \\ &= \tau(A). \end{aligned}$$

A is fixed by η, so A is on the axis of reflection, and since $\tau(A)$ is fixed, $\tau(A)$ is also on the axis of reflection. This shows that α is the composition of a reflection and a translation in the direction of the axis of reflection, so α is a glide-reflection. □

To show that τ is uniquely defined, note that

$$\begin{aligned} \alpha^2 &= (\eta\tau)^2 \\ &= \eta^2\tau^2 \\ &= \tau^2, \end{aligned}$$

since η is a reflection and thus $\eta^2 = 1$. This equation shows that τ is uniquely defined, and since $\eta = \tau^{-1}\alpha$, η is also uniquely defined by α.

We have shown that the equation $\alpha : z \mapsto e^{i\theta}\overline{z} + c$ defines either a reflection or a glide-reflection.

Combining our two results, we find that an isometry α of the plane is either the identity map or a translation, a rotation, a reflection, or a glide-reflection. □

References

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