CLASSIFICATION OF ROOT SYSTEMS

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ABSTRACT. In this paper I address and explore the basic concept of a root system. First, its origins in the theory of Lie algebras are introduced, and then an axiomatic definition is provided. Bases, Weyl groups, and the transitive action of the latter on the former are explained and proven, respectively. Finally, the Cartan matrix and Dynkin diagram are introduced to suggest the multiple applications of root systems to other fields of study.

CONTENTS

1. INTRODUCTION

To anchor our discussion of root systems, let us begin with a general overview of their occurrence in the theory of Lie algebras. A complete understanding of this section will not be necessary for the purposes of this paper. A **Lie algebra** may be understood as a vector space with an additional bilinear operation known as the **commutator** $[,]$ defined for all elements and satisfying certain properties. A Lie algebra is called **simple** if its only ideals are itself and 0, and specifically the derived algebra $\{[xy]|x, y \in L\} = [LL] \neq 0$. (This is analogous to the commutator subgroup of a group being nontrivial.) Let the Lie algebra L be semisimple, i.e. decomposible as the direct product of simple Lie algebras. Then we define a **toral subalgebra** as the span of some semisimple elements of L. (Note that the existence of such elements is guaranteed by the abstract Jordan decomposition; see e.g. [4].) It is natural to consider a **maximal toral subalgebra** H which is not properly contained in any other. It turns out that L may then be written as the direct sum of H (which is its own centralizer) and the subspaces $L_\alpha = \{x \in L|[hx] = \alpha(h)x \forall h \in H\}$, where α ranges over all elements of H^*. The nonzero α for which $L_\alpha \neq 0$ are called the **roots** of L relative to H. [4]

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Root systems thus provide a relatively uncomplicated way of completely characterizing simple and semisimple Lie algebras [4]. Via correspondence, they may also be used to inform other areas of study, such as quiver representations [1, 2, 3], invariant theory [5], and Coxeter groups. It is the goal of this paper to show that root systems may be themselves completely characterized (up to isomorphism) by their Cartan matrices.

2. Root systems

Humphreys defines a root system as a subset Φ of the Euclidean space E which satisfies the following properties:

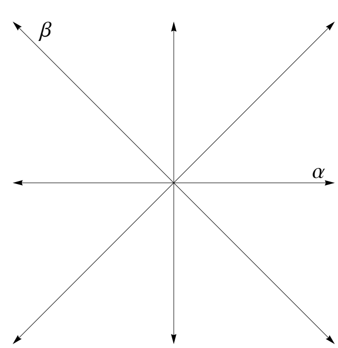
(1) Φ is finite and spans E, with $0 \notin E$
(2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
(3) For $\alpha \in \Phi, \sigma_\alpha$ leaves Φ invariant
(4) For $\alpha, \beta \in \Phi, \langle \alpha, \beta \rangle \in \mathbb{Z}$

Since the number occurs in several contexts, we adopt the notation of [4] and denote by $\langle \alpha, \beta \rangle$ the fraction $2(\beta, \alpha)/(\alpha, \alpha)$ where (,) denotes the usual inner product. It will be useful later to note that item (4) in the definition above restricts the possible values of $\langle \alpha, \beta \rangle$ rather strictly. Recall that $(\alpha, \beta) = ||\alpha|| * ||\beta|| cos\theta$. Then $\langle \alpha, \beta \rangle = 2\frac{||\beta||}{||\alpha||} cos\theta$, but we also have $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4cos^2\theta \in \mathbb{Z}$. Then the only possible values for $\langle \alpha, \beta \rangle$ are $0, \pm1, \pm2, \pm3$.

3. Bases

From property (1) above, it is clear that any root system also contains a basis for E. There arises thus a natural notion of a **base** for the root system, i.e. a set of roots which form a basis for the vector space and from which all other roots may be formed via linear combination. Moreover, the scalar coefficients used in obtaining these linear combinations are either all nonpositive or all nonnegative. We may thus consider each element of Φ to be of the form $\beta = \sum_{\alpha \in \Delta} k_\alpha$ with either $k_\alpha \geq 0$ or $k_\alpha \leq 0$. In order to easily distinguish between the two cases, we will denote the former by $\beta \succ 0$ and the latter by $\beta \prec 0$.

Example 3.1. For convenience, we introduce the root system B2 by way of providing an uncomplicated example for future reference. Note that α and β as labeled form a base for B2.



The roots which are part of a given basis are called **simple**. It follows from the simple roots' status as a basis that the **rank** of the base, i.e. the number of simple roots, is equal to the dimension of the Euclidean space E. The existence of such a base for any given root system may be proven in such a way as to determine an algorithm for finding a base given a root system. Let a root in Φ be called **indecomposable** if it may not be written as a linear combination of any other roots. By selecting all the indecomposable roots whose inner product with a predetermined vector γ in E is positive, one obtains a set of linearly independent roots α which lie entirely on the same side of the hyperplane normal to γ. Then $-\alpha$ is not contained in the set for all α, and in fact these roots both span E and give rise to all other roots, i.e. define a base for Φ. For a more rigorous treatment, see [4].

4. Weyl groups

The **Weyl group** $\mathcal{W}$ of a root system consists of all the reflections σ_α generated by elements α of the root system. For a given root α, the reflection σ_α fixes the hyperplane normal to α and maps $\alpha \to -\alpha$. We may write $\sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha$.

The hyperplanes fixed by the elements of $\mathcal{W}$ partition E into **Weyl chambers**. For a given base Δ of E, the unique Weyl chamber containing all vectors γ such that $(\gamma, \alpha) \geq 0 \forall \alpha \in \Delta$ is called the **fundamental Weyl chamber**.

We first prove the statement for $\mathcal{W}'$, the subgroup of $\mathcal{W}$ generated by only those rotations arising from the simple roots of a given base.

Theorem 4.1. *Given Δ and Δ' bases of a root system Φ, $\Delta' = \sigma(\Delta)$ for some $\sigma \in \mathcal{W}'$.*

Proof. First, note that a base for a given root system may be uniquely determined by its fundamental Weyl chamber. (This is clear from the algorithm for finding and determining all bases of a root system which is given in [4].) Then we may represent the fundamental Weyl chamber by selecting a vector γ which does not lie inside any of the fixed hyperplanes. Furthermore, it is thus sufficient to prove that $\mathcal{W}'$ acts transitively on Weyl chambers, i.e. that for any basis Δ, $\exists \sigma \in \mathcal{W}$ with $(\sigma(\gamma), \alpha) > 0 \forall \alpha \in \Delta$.

Define $\delta = \frac{1}{2} \sum_{\alpha \succ 0} \alpha$, intuitively the "average" location of all the positive roots. Then, choosing $\sigma \in \mathcal{W}'$ so that $(\sigma(\gamma), \delta)$ is as large as possible, we have $(\sigma(\gamma), \delta) \geq (\sigma_\alpha \sigma(\gamma), \delta)$ for all simple roots α.

From [4], we know that the reflections generated by simple roots merely permute the other positive simple roots, and thus $\sigma_\alpha(\delta) = \delta - \alpha \forall \alpha \in \Delta$. Then the linearity of $(,)$ yields:

$(\sigma(\gamma), \delta) \geq (\sigma_\alpha \sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha)$

Then we must have $(\sigma(\gamma), \alpha) \geq 0$; however, if $(\sigma(\gamma), \alpha) = 0$ then $(\gamma, \sigma^{-1}\alpha) = 0$, i.e. γ must be orthogonal to a simple root, which is absurd. Then $(\sigma(\gamma), \alpha) > 0 \forall \alpha \in \Delta$, and an arbitrary γ is sent to the fundamental Weyl chamber of our chosen base, just as desired. □

To see that the Weyl group itself acts transitively on bases, it remains only to show that $\mathcal{W}$ is indeed generated by a set of simple rotations. We undertake exactly this in the following two lemmas.

Lemma 4.2. *For all $\alpha \in \Phi, \exists \sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.*

Proof. Since $\mathcal{W}'$ acts transitively on bases, one need only show that every α appears in some base. Since only $\pm\alpha$ appear in Φ as multiples of α, the hyperplanes P_β fixed by σ_β for $\beta \neq \pm\alpha$ are distinct from the P_α, i.e. the hyperplane fixed by $\sigma_{\pm\alpha}$. Then we may take $\gamma \in P_\alpha, \notin P_\beta \forall \beta \neq \pm\alpha$ (which must exist lest a contradiction arise). Taking γ' very near γ s.t. $(\gamma', \alpha) = \epsilon > 0$ while $(|\gamma', \beta)| > \epsilon \forall \beta \neq \pm\alpha$. α cannot be decomposable, otherwise we would have β_1, β_2 s.t. $(\gamma', \alpha) = (\gamma', \beta_1 + \beta_2)$, leading to a contradiction. Then α must belong to the base with fundamental Weyl chamber containing γ'. □

Lemma 4.3. *$\mathcal{W}' = \{\sigma_\alpha$ arising from $\alpha \in \Delta\}$ generates $\mathcal{W}$.*

Proof. From the previous proof, it is straightforward to show that $\mathcal{W} \subset \mathcal{W}'$ and thus infer the equivalence of the two groups. Given $\alpha \in \Phi$, the preceding theorem provides us with $\sigma \in \mathcal{W}'$ such that $\beta = \sigma(\alpha) \in \Delta$. Then $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$, so $\sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in \mathcal{W}'$. □

This completes our proof that the Weyl group acts transitively on all possible bases Δ of Φ. We will now utilize this result by way of introducing one form in which root systems attain more general usefulness.

5. The Cartan matrix

For a root system $\Phi = \alpha_i, \alpha_j, ...$ one may define a matrix C by $C_{ij} = \langle \alpha_i, \alpha_j \rangle$. This is the **Cartan matrix** of Φ. Clearly, the Cartan matrix is not symmetric; however, Cartan matrices do possess several immediately observable and distinctive features. For example the main diagonal always consists of 2's, and off-diagonal entries are restricted to integers of absolute value ≤ 3. (To understand why, recall the definition of $\langle , \rangle$.)

Example 5.1. The Cartan matrix for the root system B2, introduced previously, has the following form:

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$

6. Characterization of Φ

As a consequence of the transitive action of the Weyl group on bases, it may be shown that the Cartan matrix of a root system Φ is independent of the base chosen.

Theorem 6.1. *Given two root systems $\Phi \subset E$ and $\Phi' \subset E'$ with bases $\Delta = \{\alpha_i, \alpha_j, ..., \alpha_l\}$ and $\Delta' = \{\alpha'_i, \alpha'_j, ..., \alpha'_l\}$, with identical Cartan matrices, i.e. $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for $1 \leq i, j \leq l$. Then this bijection extends to an isomorphism $f : E \to E'$ which maps $\Phi \to \Phi'$ and satisfies $\langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle \forall \alpha, \beta \in \Phi$.*

Proof. Again, we follow [4] in noting that each base is also a basis for its Euclidean space, and so we have a unique isomorphism from E to E' (as vector spaces) which sends α_i to α'_i. Then, for $\alpha, \beta \in \Delta$:

$\sigma_{f(\alpha)}(f(\beta)) = \sigma_{\alpha'}(\beta') = \beta' - \langle \beta', \alpha' \rangle \alpha' = f(\beta) - \langle \beta, \alpha \rangle f(\alpha) = f(\beta - \langle \beta, \alpha \rangle \alpha) = f(\sigma_\alpha(\beta))$

Then the following diagram commutes $\forall \alpha \in \Delta$:

$$\begin{CD} E @>f>> E' \\ @V{\sigma_\alpha}VV @VV{\sigma_{f(\alpha)}}V \\ E @>>f> E' \end{CD}$$

Since the Weyl groups $\mathcal{W}, \mathcal{W}'$ are generated by simple reflections (as we proved above), this commutativity gives rise to an isomorphism between the two groups, sending $\sigma \to f \circ \sigma \circ f^{-1}$. Then, as $\forall \beta \in \Phi \exists \sigma \in \mathcal{W}$ such that $\sigma(\beta) \in \Delta$, we have $f(\beta) = (f \circ \sigma \circ f^{-1})(f(\alpha)) \in \Phi'$. Then f sends all $\beta \in \Phi$ to Φ'. Finally, since $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ and $\langle \beta, \alpha \rangle$ is preserved by f, we see that the C_{ij} are fixed under f. □

7. Dynkin Diagrams

As mentioned previously, irreducible root systems provide a simple means of classifying Lie algebras. However, the root systems may themselves be classified according to their **Dynkin diagrams**. Each such diagram belongs to one of finitely many families of graphs with a variety of connections to e.g. quiver representations. This correspondence between Cartan matrices and Dynkin diagrams may be explicitly understood as follows.

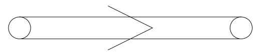
Each vertex of the Dynkin diagram corresponds to a root α_i. Clearly, if $C_{ij} = C_{ji} = 0$, no edge exists between the vertices for α_i and α_j. If the C_{ij}th and C_{ji}th entries in the Cartan matrix are both ± 1, a single edge connects the vertices corresponding to α_i and α_j. If the C_{ij}th or C_{ji}th entry is ± 2 or ± 3, two or three edges, respectively, connect the two vertices in question. In order to distinguish the relative lengths of the roots, an arrow pointing towards the shorter of the two is drawn over the vertex in question.

The properties of the Cartan matrices place a number of restrictions on possible Dynkin diagrams, which we enumerate below. In fact, these properties, enumerated below, lead to a complete description of all possible Dynkin diagrams, which may be found in [4].

(1) If some of the vertices of the Dynkin diagram are omitted along with all their attached edges, the remaining graph is also possible as a Dynkin diagram.
(2) The number of vertex pairs connected by at least one edge is strictly less than the order of the root system. It follows that no Dynkin diagram may contain a cycle.
(3) No more than three edges can connect to a single vertex. Thus, the only Dynkin diagrams containing a triple edge contain exactly those two vertices it connects.
(4) If a Dynkin diagram contains as a subgraph a simple chain, the graph obtained by reducing that chain to a point also forms a Dynkin diagram. This prohibits several possible arrangements of terminal vertices from cooccuring within a diagram, lest the preceding restriction be violated.

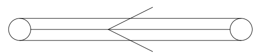
Example 7.1. The Dynkin diagram for our familiar root system, B2, is as follows. Recall that e.g. β is longer than α.

B_2



Example 7.2. The only Dynkin diagram with a triple edge, G2, has the following form:

G_2



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