A TREATMENT OF THE DIRICHLET INTEGRAL VIA THE METHODS OF REAL ANALYSIS

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Abstract. Herein I present multiple solutions to an improper integral using elementary calculus and real analysis. The integral, sometimes known as the Dirichlet integral, is often evaluated using complex-analytic methods, e.g. via contour integration. While the proofs presented here may not be as direct as certain complex-analytic approaches, they do illustrate the unique real variable techniques for dealing with this type of problem.

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1. Elementary Approach to the Dirichlet Integral

The integral we work with here is

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

Our claim is that $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$. We first show that the integral converges. Noting that the integrand is even, a simple parity check implies that we may work with $\int_0^{\infty} \frac{\sin x}{x}\, dx$. If we let

$$I = \int_0^{\infty} \frac{\sin x}{x}\, dx,$$

then equivalently,

$$I = \lim_{\substack{a \searrow 0 \\ b \to \infty}} I_{ab}$$

where

$$I_{ab} = \int_a^b \frac{\sin x}{x}\, dx.$$

We demonstrate convergence through a simple integration by parts argument. First, note that if the upper limit of our integral I is finite, then the integral is convergent since $\frac{\sin x}{x}$ is continuous for all finite x (for $x = 0$, we have $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$). In

Date: June 26, 2009.

other words, for the purpose of showing convergence, we may consider $\frac{\sin x}{x}$ to be a continuous function at zero whose value at zero is 1. Note that

$$I_{ab} = \int_a^b \frac{\sin x}{x}\, dx = \int_a^b \frac{1}{x} \frac{d(1-\cos x)}{dx}\, dx.$$

Integration by parts gives

$$\begin{aligned} I_{ab} &= \frac{1-\cos b}{b} - \frac{1-\cos a}{a} + \int_a^b \frac{1-\cos x}{x^2}\, dx \\ &= \frac{1-\cos b}{b} - \sin a \cdot \frac{\sin a}{a} \cdot \frac{1}{1+\cos a} + \int_a^b \frac{1-\cos x}{x^2}\, dx. \end{aligned}$$

Now

$$I = \int_0^\infty \frac{\sin x}{x}\, dx = \lim_{\substack{a \searrow 0 \\ b \to \infty}} I_{ab} = \int_0^\infty \frac{1-\cos x}{x^2}\, dx,$$

where the last integral on the right is convergent (one way to see this is to apply the integral test to $\int_1^\infty \frac{1-\cos x}{x^2}\, dx$ and consider $\int_0^1 \frac{1-\cos x}{x^2}\, dx$ as the integral of a continuous function). Having established convergence, we now calculate the value. First, we state and prove a special case of the Riemann-Lebesgue lemma.

Lemma 1.1. *If f is continuous on $[a,b]$, then*

$$\lim_{\lambda \to \infty} \int_a^b f(t) \sin \lambda t\, dt = 0.$$

Proof. Let n be a natural number and $P = \{t_0, t_1, \ldots, t_n\}$ be the partition that divides $[a,b]$ into n equal subintervals, where

$$\begin{aligned} a = t_0 < t_1 < ... < t_n = b, \\ t_i - t_{i-1} = \frac{b-a}{n}. \end{aligned}$$

Then

$$\begin{aligned} \int_a^b f(t) \sin \lambda t\, dt &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f(t) \sin \lambda t\, dt \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} [f(t) - f(t_i)] \sin \lambda t\, dt + \sum_{i=1}^n f(t_i) \int_{t_{i-1}}^{t_i} \sin \lambda t\, dt. \end{aligned}$$

Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$. Given $\varepsilon > 0$, there is some $\delta > 0$ such that, for all x and y in $[a,b]$,

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}.$$

If $n > \frac{b-a}{\delta}$, then for t such that $t_{i-1} \le t \le t_i$, $|t - t_i| \le |t_i - t_{i-1}| = \frac{b-a}{n} < \delta$ $(i = 1, ..., n)$ implies

$$|f(t) - f(t_i)| < \frac{\varepsilon}{2(b-a)}.$$

Consequently,

$$\begin{aligned}\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}[f(t)-f(t_i)]\sin\lambda t\,dt\right| &\le \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|f(t)-f(t_i)|\cdot|\sin\lambda t|\;dt\\ &\le \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|f(t_i)-f(t_{i-1})|\;dt\\ &< \frac{\varepsilon}{2(b-a)}\cdot(b-a)=\frac{\varepsilon}{2}.\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}\left|\int_{t_{i-1}}^{t_i}\sin\lambda t\,dt\right| &= \left|-\frac{1}{\lambda}\left[\cos\lambda t_i-\cos\lambda t_{i-1}\right]\right|\\ &\le \frac{|\cos\lambda t_i|+|\cos\lambda t_{i-1}|}{\lambda}\le\frac{2}{\lambda}.\end{aligned}$$

Since f is continuous on $[a,b]$, f is bounded above, i.e., there is some M such that $|f(x)|\le M$ for all x in $[a,b]$. Hence,

$$\begin{aligned}\left|\sum_{i=1}^{n}f(t_i)\int_{t_{i-1}}^{t_i}\sin\lambda t\,dt\right| &\le M\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}\sin\lambda t\,dt\right|\\ &\le M\sum_{i=1}^{n}\frac{2}{\lambda}=\frac{2Mn}{\lambda}.\end{aligned}$$

Choose a real number R so that $R>\frac{4Mn}{\varepsilon}$. If $\lambda>R$,

$$\begin{aligned}\left|\int_a^b f(t)\sin\lambda t\,dt\;-\;0\right| &\le \left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}[f(t)-f(t_i)]\sin\lambda t\,dt\right|\\ &\quad+\left|\sum_{i=1}^{n}f(t_i)\int_{t_{i-1}}^{t_i}\sin\lambda t\,dt\right|\\ &< \frac{\varepsilon}{2}+\frac{2Mn}{\lambda}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Since ε was arbitrary, it follows that $\lim_{\lambda\to\infty}\int_a^b f(t)\sin\lambda t\,dt=0$, as desired. □

We want to write our original integral in a form that allows us to apply the lemma. Let c be an arbitrary positive number. The substitution $x=\lambda t$, $dx=\lambda\,dt$ gives

$$\lim_{\lambda\to\infty}\int_0^c\frac{\sin\lambda t}{t}\,dt=\lim_{\lambda c\to\infty}\int_0^{\lambda c}\frac{\sin x}{x}\,dx=I. \qquad (1.2)$$

Hence, the integral we want to evaluate can be rewritten as a limit of integrals over a fixed interval. However, we cannot simply apply our lemma to $f(t)=\frac{1}{t}$. Instead, we evaluate our integral in terms of an expression we can easily determine. Keeping the details of the lemma in mind, we try to determine a function g such that the function f, whose values are defined by

$$f(x)=\frac{1}{x}-\frac{1}{g(x)}=\frac{g(x)-x}{xg(x)}\quad x\ne 0,$$

is continuous on a closed interval $[0, c]$, $c > 0$. In particular, we require $\lim_{x\to 0} f(x)$ to exist (and equal $f(0)$). Let us assume for now that $\lim_{x\to 0} g(x) = 0$ and g' exists. Then we may apply l'Hopital's rule to get

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{g(x) - x}{xg(x)} = \lim_{x\to 0} \frac{g'(x) - 1}{g(x) + g'(x)x}.$$

Now if we further assume that $\lim_{x\to 0} g'(x) = 1$ and g'' exists, then applying l'Hopital's rule again gives

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{g'(x) - 1}{g(x) + g'(x)x} = \lim_{x\to 0} \frac{g''(x)}{g'(x) + g''(x)x + g'(x)}.$$

Finally, if we assume that $\lim_{x\to 0} g''(x) = 0$, and recall that we want f continuous, then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{g''(x)}{2} = 0 = f(0).$$

Moreover, our set of assumptions for g implies that $\lim_{x\to 0} g(x) + g''(x) = 0$. These conditions suggest that $g = \sin$ is a candidate, and it is easy to check that sine does satisfy our assumptions for g. If we define

$$f(t) = \begin{cases} \frac{1}{t} - \frac{1}{\sin t} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0 \end{cases},$$

then it follows from the above discussion that f is continuous on the interval $[0, c]$, provided that we restrict c so that $0 \le c < \pi$. We may apply Lemma 1.1 to f to get

$$\lim_{\lambda\to\infty} \int_0^c \frac{\sin \lambda t}{t}\, dt = \lim_{\lambda\to\infty} \int_0^c \frac{\sin \lambda t}{\sin t}\, dt. \tag{1.3}$$

We know from (1.2) that the expression on the left-hand side of this last equality is equivalent to the integral we want to evaluate. If we can evaluate the expression on the right-hand side of (1.3), then we are done. Note that we may replace λ with an expression that tends to infinity. To complete the problem, we prove the following trigonometric identity:

$$1 + 2\cos 2t + 2\cos 4t + ... + 2\cos 2nt = \frac{\sin(2n+1)t}{\sin t}. \tag{1.4}$$

From the identity $\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$, we have

$$\sin(2k+1)t - \sin(2k-1)t = 2\sin(t)\cos(2kt).$$

Hence,

$$\begin{aligned} 1 + 2\cos 2t + ... + 2\cos 2nt &= 1 + \frac{1}{\sin t}\left[\sum_{k=1}^{n} \sin(2k+1)t - \sin(2k-1)t\right] \\ &= 1 + \frac{1}{\sin t}\left[\sin(2n+1)t - \sin t\right] \\ &= \frac{\sin(2n+1)t}{\sin t}, \end{aligned}$$

as desired.

Now notice the similarity between the integrand of the expression on the right-hand side of (1.3) and the right-hand side of (1.4). In fact, we can safely replace λ with $2n+1$ in (1.3) since $2n+1 \to \infty$ as $n \to \infty$. Since (1.3) holds for $0 \le c < \pi$, we may integrate both sides of (1.4) over the interval $\left[0, \frac{\pi}{2}\right]$ to get

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t}\,dt &= \int_0^{\frac{\pi}{2}} (1 + 2\cos 2t + 2\cos 4t + \ldots + 2\cos 2nt)\,dt \\ &= \frac{\pi}{2} + \left[\sin 2t + \frac{\sin 4t}{2} + \ldots + \frac{\sin 2nt}{n}\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2}. \end{aligned}$$

Hence (1.3) in the case $c = \frac{\pi}{2}$ implies

$$\lim_{\lambda\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin \lambda t}{t}\,dt = \lim_{\lambda\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin \lambda t}{\sin t}\,dt = \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t}\,dt = \frac{\pi}{2}.$$

We recall from (1.2) that

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = I = \lim_{\lambda\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin \lambda t}{t}\,dt = \frac{\pi}{2}.$$

Another parity check shows that $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$, as desired.

2. The Dirichlet Integral and Fourier Series

Many of the ideas used in the previous section arise naturally in the basic analysis of Fourier series. We introduce the basic definitions and then prove a theorem that implicitly contains the solution to the Dirichlet integral.

If f is any function that is integrable on $[-\pi, \pi]$, the numbers

$$a_v = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos vt\,dt, \quad b_v = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin vt\,dt$$

are called the Fourier coefficients of f. We can write down the series

$$\frac{a_0}{2} + \sum_{v=1}^{\infty} (a_v \cos vx + b_v \sin vx),$$

which is called the Fourier series corresponding to f. We want to know under what conditions does the Fourier series corresponding to f actually converge to f. We assume that f is also 2π periodic. Let us consider the partial sums

$$S_n(x) = \frac{a_0}{2} + \sum_{v=1}^{n} (a_v \cos vx + b_v \sin vx).$$

By substituting the expressions for the coefficients a_v and b_v found above and interchanging the order of integration and summation we get

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{v=1}^{n} (\cos vt \cos vx + \sin vt \sin vx)\right] dt.$$

or,

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{v=1}^{n} \cos v(t-x)\right] dt,$$

Since $\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$,

$$\cos v(t-x) = \frac{\sin{(v+\frac{1}{2})(t-x)} - \sin{(v-\frac{1}{2})(t-x)}}{2\sin{(\frac{t-x}{2})}}.$$

If we sum this last expression from 1 to n, we would get a telescoping sum due to the right-hand side. Consequently,

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin{(n+\frac{1}{2})(t-x)}}{\sin{(\frac{t-x}{2})}}\,dt.$$

Substituting $u = (t-x)$ and recalling the periodicity of the integrand, we have

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin{(n+\frac{1}{2})u}}{\sin\frac{u}{2}}\,dt. \tag{2.1}$$

Note that this form of the partial sum $S_n(x)$ was largely obtained by applying the same argument used to derive the equation in (1.4). In fact, if we replace t with $\frac{t}{2}$ in (1.4) and divide both sides by 2, we can define the expression

$$D_n(x) = \frac{1}{2} + \sum_{v=1}^{n}\cos vx = \frac{\sin{(n+\frac{1}{2})}}{2\sin\frac{x}{2}},$$

as the Dirichlet kernel. Clearly, the Dirichlet kernel will play an important role in showing that $S_n(x)$ tends to $f(x)$. Before we can impose conditions on f so that $S_n(x)$ will indeed converge and represent $f(x)$, we introduce the following notions.

A function f is sectionally smooth on an interval if it is itself sectionally continuous (i.e., continuous on the interval except at a finite number of jump discontinuities; also referred to as piecewise continuous) and if its first derivative f' is sectionally continuous.

Now recall that we defined f to be periodically extended beyond the interval $[-\pi,\pi]$. But for each point at which $f(x)$ has a jump discontinuity, we will, if needed, write

$$f(x) = \frac{1}{2}\left[f(x^-) + f(x^+)\right],$$

where $f(x^-)$ and $f(x^+)$ are the limits of $f(y)$ as y approaches x from the left and from the right, respectively. This last equation clearly holds for each point x at which f is continuous.

We want to prove that if f is sectionally smooth and satisfies the last equation, then the partial sum $S_n(x)$ in the form shown in (2.1), tends to $f(x)$. The special case of the Riemann-Lebesgue lemma (Lemma 1.1) seems applicable here, but we asserted that f was to be continuous in the statement of the lemma. But in fact, the exact same proof would demonstrate the conclusion of the Riemann-Lebesgue lemma for a sectionally continuous function f. This follows from the fact that we could always express the integral of a piecewise continuous function as a finite number of integrals that appear in the limit of Lemma 1.1. However, we can easily generalize the Riemann-Lebesgue lemma to integrable functions.

Lemma 2.2. *If f is integrable on $[a,b]$, then*

$$\lim_{\lambda\to\infty}\int_a^b f(t)\sin\lambda t\,dt = 0 \text{ and } \lim_{\lambda\to\infty}\int_a^b f(t)\cos\lambda t\,dt = 0.$$

Proof. We prove the first limit. The second limit follows from the same method. A function s is called a step function if there is a partition $P = \{t_0, \ldots, t_n\}$ of $[a, b]$ such that s is a constant on each (t_{i-1}, t_i). We will use the fact that if f is integrable on $[a, b]$, then for any $\varepsilon > 0$ there is a step function $s \le f$ with $\int_a^b f - \int_a^b s < \varepsilon$. This means that

$$\begin{aligned} \left| \int_a^b f(x) \sin \lambda x \, dx - \int_a^b s(x) \sin \lambda x \, dx \right| &= \left| \int_a^b [f(x) - s(x)] \sin \lambda x \, dx \right| \\ &\le \int_a^b [f(x) - s(x)] \cdot |\sin \lambda x| \, dx \\ &\le \int_a^b [f(x) - s(x)] \, dx < \varepsilon. \end{aligned}$$

If s has the values s_i on (t_{i-1}, t_i), then

$$\lim_{\lambda \to \infty} \int_a^b s(x) \sin \lambda x \, dx = \lim_{\lambda \to \infty} \sum_{i=1}^n s_i \int_{t_{i-1}}^{t_i} \sin \lambda x \, dx = 0.$$

Hence,

$$\lim_{\lambda \to \infty} \left| \int_a^b f(x) \sin \lambda x \, dx \right| < \varepsilon$$

Since ε was arbitrary, we conclude that $\lim\limits_{\lambda \to \infty} \int_a^b f(t) \sin \lambda t \, dt = 0$, as desired. □

Now we state and prove the main theorem of this section.

Theorem 2.3. *If the function f is sectionally smooth and at each point of discontinuity of x satisfies the equation $f(x) = \frac{1}{2}[f(x^-) + f(x^+)]$, then the Fourier series corresponding to f(x) converges at every point and represents the function and we have*

$$\lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin (n + \frac{1}{2})t}{\sin \frac{t}{2}} \, dt = f(x).$$

Proof. We first subdivide the interval of integration at the origin. If we fix the values of x, the function

$$s(t) = \frac{f(x+t) - f(x^+)}{2sin\frac{t}{2}}$$

is sectionally continuous on $[0, \pi]$. This is clear when $t \in (0, \pi]$, but the continuity of s at $t = 0$ follows from the existence of the right-hand derivative

$$\begin{aligned} \lim_{t \to 0,\ t > 0} \frac{f(x+t) - f(x^+)}{t} &= \lim_{t \to 0} \frac{f(x+t) - f(x^+)}{2 \sin \frac{t}{2}} \cdot \frac{2 \sin \frac{t}{2}}{t} \\ &= \lim_{t \to 0} \frac{f(x+t) - f(x^+)}{2 \sin \frac{t}{2}}. \end{aligned}$$

By Lemma 2.2 (Riemann-Lebesgue),

$$\frac{1}{\pi} \int_0^{\pi} s(t) \sin \lambda t \, dt = \frac{1}{2\pi} \int_0^{\pi} f(x+t) \frac{\sin \lambda t}{\sin \frac{t}{2}} - \frac{1}{2\pi} \int_0^{\pi} f(x^+) \frac{\sin \lambda t}{\sin \frac{t}{2}}$$

tends to zero as $\lambda = n + \frac{1}{2}$ increases without bound. We can take the factor $f(x^+)$ out of the second integral on the right. Moreover, a familiar calculation using the

Dirichlet kernel shows that for $\lambda = n + \frac{1}{2}$, the integral $\int_0^\pi \frac{\sin \lambda t}{2 \sin \frac{t}{2}}$ is equal to $\frac{\pi}{2}$. It follows immediately that

$$\lim_{\lambda \to \infty} \frac{1}{2\pi} \int_0^\pi f(x+t) \frac{\sin \lambda t}{\sin \frac{t}{2}} = \frac{1}{2} f(x^+).$$

Similarly, we obtain

$$\lim_{\lambda \to \infty} \frac{1}{2\pi} \int_{-\pi}^0 f(x+t) \frac{\sin \lambda t}{\sin \frac{t}{2}} = \frac{1}{2} f(x^-)$$

for the interval $[-\pi, 0]$. Adding these last two equations gives

$$\lim_{\lambda \to \infty} \frac{1}{2\pi} \int_{-\pi}^\pi f(x+t) \frac{\sin \lambda t}{\sin \frac{t}{2}} = f(x),$$

as desired. □

Corollary 2.4. $\int_{-\infty}^\infty \frac{\sin x}{x} \, dx = \pi$.

Proof. In the above theorem, we obtained the equation

$$\frac{1}{2\pi} \int_0^\pi f(x+t) \frac{\sin \lambda t}{\sin \frac{t}{2}} = \frac{1}{2} f(x^+).$$

If we let $x = 0$, $f(t) = \frac{\sin \frac{t}{2}}{t}$ in this equation, and substitute $u = \lambda t$, then

$$\lim_{\lambda \to \infty} \int_0^{\lambda \pi} \frac{\sin u}{u} \, du = \pi \cdot \lim_{y \searrow 0} \frac{\sin \frac{y}{2}}{y} = \frac{\pi}{2}.$$

By parity symmetry, we find that $\int_{-\infty}^\infty \frac{\sin x}{x} \, dx = \pi$, as desired. □

This approach to the Dirichlet integral is perhaps more structured than the previous approach. By attempting to prove a theorem in the setting of Fourier series, the Dirichlet kernel arises more naturally. Yet both approaches are inevitably very similar due to the usage of the same theorems and equations.

3. The Dirichlet Integral as a Generalized Function

We may also think of the Dirichlet integral in terms of a generalized function. The notion of a generalized function, also called a distribution, was originally introduced to solve problems in which the classical idea of a function was inadequate. A generalized function is a continuous linear functional on a vector space of so called "test functions." We define a generalized function by its action on all of the test functions in the vector space. We can choose the space K of test functions φ in many ways, but we often require these test functions to satisfy certain smoothness conditions.

Let K be the set of all finite functions φ on $(-\infty, \infty)$ which are also infinitely differentiable, where each function $\varphi \in K$, being finite, vanishes outside some finite interval depending on the choice of φ. K is then a linear space, and we introduce the notion of convergence on K as follows:

Definition 3.1. A sequence (φ_n) of functions in K is said to converge to a function $\varphi \in K$ provided that there exists an interval outside which all the functions φ_n vanish and that the sequence $(\varphi_n^{(k)})$ of derivatives of order k converges uniformly on this interval to $\varphi^{(k)}$ for each $k = 0, 1, 2, \ldots$.

The linear space K just introduced is called the test space, and the functions in K are called test functions.

Definition 3.2. Every continuous linear functional $T(\varphi)$ on the test space K is called a generalized function (or distribution) on $(-\infty, \infty)$, where continuity of $T(\varphi)$ means that $\varphi_n \to \varphi$ in K implies $T(\varphi_n) \to T(\varphi)$

If $f(x)$ is a locally integrable function (a function integrable on every compact set of its domain), then $f(x)$ generates a generalized function via the expression

$$T_f(\varphi) = (f, \varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx,$$

which is a continuous linear functional on K. Many of the operations defined on smooth functions with compact support can also be defined for distributions. However, the notion we need most to compute the Dirichlet integral is presented in the following theorem

Theorem 3.3. *Let X be an open set in $\mathbb{R}^m$, and Ω be a measure space. Given $f(x,\omega)$, for each $\omega \in \Omega$, a generalized function of $x \in X$, define:*

$$\left(\int_\Omega f(\cdot,\omega)\,d\omega, \varphi\right) := \int_\Omega (f(\cdot,\omega)\varphi)\,d\omega, \quad \varphi \in D(X).$$

Assuming that the integral above is well-defined and generates a distribution, we have

$$\frac{\partial}{\partial x_i}\int_\Omega f(x,\omega)\,d\omega = \int_\Omega \frac{\partial}{\partial x_i} f(x,\omega)\,d\omega.$$

This operation is known as differentiation under the integral sign. To justify the usage of this operation to compute the Dirichlet integral, we first note that the expression $\int_0^\infty \frac{\sin\alpha\cdot x}{x}\,dx$ is locally integrable and hence a parity argument shows that $\int_{-\infty}^\infty \frac{\sin\alpha\cdot x}{x}\,dx$ generates a distribution. Moreover, we have already demonstrated the convergence of $\int_{-\infty}^\infty \frac{\sin x}{x}\,dx$ in the ordinary sense. Hence, we should be able to apply Theorem 3.3 to $\int_{-\infty}^\infty \frac{\sin\alpha\cdot x}{x}\,dx$.

Furthermore, it is certainly permissible to differentiate $\int_{-\infty}^\infty \frac{\sin\alpha\cdot x}{x}\,dx$ under the integral sign to determine if the resulting expression is a continuous linear functional. If this is true (which we soon find out is the case), we may conclude that the antiderivative was indeed a distribution and differentiating under the integral sign was allowable. Hence we work with $\frac{\partial}{\partial\alpha}\int_{-\infty}^\infty \frac{\sin\alpha\cdot x}{x}\,dx$ in a distributional sense. We rewrite the integrand to introduce Fourier phases as follows

$$\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}\frac{\sin\alpha\cdot x}{x}\,dx = \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}-e^{-i\alpha x}}{2ix}\,dx.$$

Differentiating under the integral sign gives

$$\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}\frac{\sin\alpha\cdot x}{x}\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial\alpha}\frac{e^{i\alpha x}-e^{-i\alpha x}}{2ix}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{i\alpha x}+e^{-i\alpha x}\right)dx.$$

Noting that the Fourier transform (non-unitary angular frequency convention) of the constant function $f(x) = 1$, is the Dirac delta function multiplied by 2π, we have

$$\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}\frac{\sin\alpha\cdot x}{x}\,dx = 2\pi\delta(\alpha).$$

The Dirac delta "function" is a continuous linear functional, and it is the distributional derivative of the Heaviside step function. Hence, integrating both sides of the last equation with respect to α from $-c$ to c and noting the parity symmetry shows that in the sense of generalized functions, we have

$$\int_{-\infty}^{\infty} \frac{\sin c \cdot x}{x} \, dx = \pi \operatorname{sgn}(c).$$

Once again, we have already demonstrated convergence in the ordinary sense of the left hand side for $c = 1$. The same argument shows convergence in this sense for c in a neighborhood of 1. By integrating against test functions supported about $c = 1$, we conclude that the Dirichlet integral evaluates to π, as before.

This approach to the Dirichlet integral demonstrates the usefulness of differentiation under the integral sign. It is actually possible to calculate the value of the Dirichlet integral by applying Fourier inversion techniques to the sinc function, but this approach is more difficult to justify and perhaps less elegant. By working with generalized functions, we were lead to a quick solution to a concrete problem.

4. Concluding Remarks

Although we have refrained from using complex analysis to evaluate the Dirichlet integral, the solutions presented above are not necessarily less elegant. Perhaps one difference is that contour integration via the Cauchy integral formula and the method of residues is a direct technique for evaluating definite integrals, whereas the solutions above may seem relatively indirect. For example, the first approach, when divorced from the setting of Fourier series, may seem like a series of tricks. Considering the first two approaches together, both are perhaps rather indirect methods to obtaining the answer to our problem. Although the generalized functions approach may seem more direct than the previous approaches, the value of the Dirichlet integral is often used to establish theorems in distribution theory and Fourier transforms. For instance, the value of the Dirichlet integral can be used to actually prove that the Fourier transform of $f(x) = 1$ is the Dirac delta. Hence, there is perhaps an element of circularity in our approaches.

While the complex-analytic route may be seem more straightforward for integrals such as the Dirichlet integral, there are more direct approaches to our problem that use real analysis. For the sake of having an elementary approach that leads to a direct evaluation of the Dirichlet integral, we conclude with the following solution.

We continue with the line of thought introduced in our initial demonstration of the convergence of the Dirichlet integral. Recall that we showed that

$$I = \int_0^{\infty} \frac{\sin x}{x} \, dx = \int_0^{\infty} \frac{1 - \cos x}{x^2} \, dx.$$

Now the trick is to rewrite $\frac{1}{x^2}$ as $\int_0^{\infty} t e^{-tx} \, dt$ so that

$$I = \int_0^{\infty} \frac{1 - \cos x}{x^2} \, dx = \int_0^{\infty} \int_0^{\infty} (1 - \cos x) \cdot t e^{-tx} \, dt \, dx.$$

Since everything is positive, Tonelli's theorem says that we may reverse the order of integration so that

$$I = \int_0^\infty \int_0^\infty (1-\cos x)\cdot te^{-tx}\,dt\,dx = \int_0^\infty \int_0^\infty \left(te^{-tx} - t\cos xe^{-tx}\right)dx\,dt.$$

A simple computation shows that

$$\int t\cos xe^{-tx}\,dx = \frac{1}{t^2+1}\left(t\sin xe^{-tx} - t^2\cos xe^{-tx}\right)$$

so we have

$$\begin{aligned} I &= \int_0^\infty \left[-e^{-tx} - \left(\frac{t}{1+t^2}\sin xe^{-tx} - \frac{t^2}{1+t^2}\cos xe^{-tx}\right)\right]_{x=0}^{x=\infty} dt \\ &= \int_0^\infty \frac{1}{1+t^2}\,dt = [\arctan t]_0^\infty = \frac{\pi}{2}. \end{aligned}$$

It follows that $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, as desired.

Acknowledgments. It is a pleasure to thank my mentors, David Chudzucki and Zachary Madden for their suggestions and their careful reading of this paper.

References

[1] R. Churchill and J. Brown. Fourier Series and Boundary Value Problems. McGraw-Hill, Inc., Fifth Edition, 1993.
[2] R. Courant. Differential and Integral Calculus, Vol. I. Transl. by E. J. McShane. Interscience, New York, 2d ed., 1937.
[3] A.N. Kolmogorov and S.V. Fomin. Introductory Real Analysis. Dover Publications, 1975.
[4] M. Spivak. Calculus. Publish or Perish, Inc., Third Edition, 1994.