Universal Coefficient Theorem for Homology

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Contents

Abstract

This paper will give a brief introduction to homological algebra. Starting with various exact sequences, we will define tensor product and projective modules, which will lead to the object of interest: homology groups, a more computable alternative to homotopy groups in higher dimensions. Given a chain complex of free abelian groups C_n, is it possible to compute the homology groups $H_n(C; G)$ of the associated chain complex of tensor product with G just in terms of G and $H_n(C)$? The Universal Coefficient Theorem for Homology provides an algebraic formula that answers this question.

1 Introduction

In algebraic topology, we can distinguish various topological spaces using singular homology. Nonetheless we may want to calculate homology of arbitrary coefficients, so we need a theorem which will establish the relationship between homology of arbitrary coefficients and homology with $\mathbb{Z}$ coefficients.

In Section 2, we will give the necessary algebra background. In Section 3 we will define Tor and prove the Universal Coefficient Theorem for Homology. In the last section, we will compute two examples.

2 Background in Algebra

2.1 Exact Sequences

Definition 2.1. A pair of homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact* at B if im $(f) = \ker(g)$. A sequence $\cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots$ is exact if it is exact at every A_i that is between two homomorphisms.

Proposition 2.2. *A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective. On the other hand, a sequence $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective.*

Proof. Exactness at A implies that ker f is equal to the image of the homomorphism $0 \to A$, which is zero. This is equivalent to the injectivity of homomorphism f. Similarly, the kernel of zero homomorphism $C \to 0$ is C, and $g(B) = C$ if and only if g is surjective. $\square$

Corollary 2.3. *A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if f is injective, g is surjective, and im f = ker g. We say B is an* extension *of C by A. This exact sequence is called a* short exact sequence.

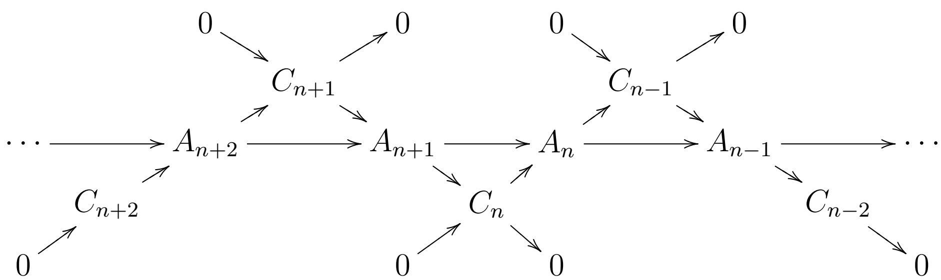
Example 2.4. Given two $\mathbb{Z}$-modules $A = \mathbb{Z}$ and $C = \mathbb{Z}/n\mathbb{Z}$, we can construct two different short exact sequences. First, $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \xrightarrow{g} \mathbb{Z}/n\mathbb{Z} \to 0$ where $f(a) = (a, 0)$ and $g(a, c) = c$.

The $\mathbb{Z}$-module $\mathbb{Z}$ is also an extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$. Consider $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/n\mathbb{Z} \to 0$, where n maps z to nz, while p is the projection map. Note that even though A and C are the same modules in the example,
$Z \oplus Z/n\mathbb{Z}$ is not isomorphic to $\mathbb{Z}$, making the two exact sequences not equivalent.

The significance of short exact sequence shows up when we try to break down a long exact sequence into short exact sequences. Consider the exact sequence of R-modules $\cdots \to A_{n+2} \to A_{n+1} \to A_n \to A_{n-1} \to A_{n-2} \to \cdots$. Let

$$C_n \cong \ker\,(A_n \to A_{n-1}) \cong \mathrm{im}\,(A_{n+1} \to A_n).$$

As the algebraic structure underlying R-module is abelian group, the cokernel of each homomorphism exists such that $C_n \cong \operatorname{coker}(A_{n+2} \to A_{n+1})$. Then we obtain the following commutative diagram, in which all the diagonal sequences are short exact:



(2.5)

Conversely, given any short exact sequences overlapped in this way, their middle terms form an exact sequence.

Definition 2.6. Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be two short exact sequences of modules. A *homomorphism of short exact sequences* is a triple f, g, h of module homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0. \end{array} \tag{2.7}$$

If f, g, h are all isomorphisms, then this is an *isomorphism of short exact sequences*, where B and B' are isomorphic extensions. The two exact sequences are *equivalent* if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \approx}\downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0. \end{array} \tag{2.8}$$

Definition 2.9. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules. The sequence is *split* if $B = A \oplus C$ up to isomorphism. A map $s : C \to B$ is called a *section* of g if $g \circ s =$id. If s is also a homomorphism, then it is a *splitting homomorphism*.

Splitting is equivalent to either of the following statements:
(a) There is a homomorphism $p : B \to A$ such that $p \circ f = 1 : A \to A$.
(b) There is a homomorphism $s : C \to B$ such that $g \circ s = 1 : C \to C$.

Example 2.10. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is split, by definition. In contrast, the sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is not split because there is not a nontrivial homomorphism from $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$.

2.2 Tensor Product of Modules

Definition 2.11. For a ring R, let M be a right module, and N be a left module. The *tensor product* $M \otimes N$ over R is the abelian group $M \times N$ quotiented by

$$(m_1 + m_2, n) \sim (m_1, n) + (m_2, n)$$

$$(m, n_1 + n_2) \sim (m, n_1) + (m, n_2)$$

$$(mr, n) \sim (m, rn)$$

for $m, m_1, m_2 \in M, n, n_1, n_2 \in N$ and $r \in R$.

Theorem 2.12. *Let L, M, N be right modules, and D be a left module. If*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

is exact, then the associated sequence of abelian groups

$$L \otimes_R D \xrightarrow{\psi \otimes 1} M \otimes_R D \xrightarrow{\varphi \otimes 1} N \otimes_R D \to 0$$

is exact.

Proof. To show the surjectivity of $\varphi \otimes 1$, we know φ is surjective. Then for some $m \in M, n = \varphi(m)$. But $n \otimes d = \varphi(m) \otimes d = \varphi(m \otimes d) \otimes 1$. This implies $(\varphi \otimes 1)$ is a surjective homomorphism from $M \otimes D$ to $N \otimes D$, which are abelian groups. For exactness at $M \otimes_R D$, it is sufficient to show that $\pi : M \otimes D/\text{im}(\psi \otimes 1) \to N \otimes D$ is an isomorphism. To construct the inverse of π, define a map

$$p : N \times D \to M \otimes D/\text{im}(\psi \otimes 1)$$

by $p(n, d) = m \otimes d$ where $\varphi(m) = n$. If $\varphi(m) = \varphi(m') = n$, then $m - m' = \psi(l)$ for some $l \in L$ by the exactness at M. This implies $m \otimes d - m' \otimes d = (m - m') \otimes d = \psi(l) \otimes d \in \text{im}(\psi \otimes 1)$, so p is well-defined. Since p is constant on each equivalence class, p induces $\tilde{p} : N \otimes D \to M \otimes D/\text{im}(\psi \otimes 1)$, which is a homomorphism and an inverse to π. □

Definition 2.13. A left R-module D is called *flat* if it satisfies one of the two following equivalent conditions:
(1) For any right modules L, M, N, if $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is exact, then $0 \to L \otimes D \xrightarrow{\psi \otimes 1} M \otimes D \xrightarrow{\varphi \otimes 1} N \otimes D \to 0$ is also exact.
(2) For any right module L, M, if ψ is injective, then $\psi \otimes 1$ is injective.

Corollary 2.14. *Free modules are flat; projective modules are flat.*

In conclusion, for a left R-module D, the functor $- \otimes D$ from the category of right R-module to the category of abelian group is right exact; it is exact if and only if D is a flat module. Here are some convenient facts to know:

Corollary 2.15. *(1) For any left R-module D,*

$$\mathbb{Z} \otimes_{\mathbb{Z}} D = D. \tag{2.16}$$

(2) For $m, n \in \mathbb{Z}$,

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}, \tag{2.17}$$

where d is the g.c.d. of m and n.
(3) Let M, M' be right R-modules and let N, N' be left R-modules. Then there are unique group isomorphisms

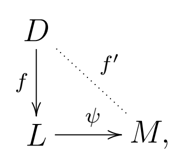
$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N) \tag{2.18}$$

such that $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$. The isomorphism is defined similarly for $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$.

2.3 Projective Modules

Let R be a ring with unity, and let $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ be a short exact sequence of R-modules. We would like to find out whether the properties of L and N imply the related properties of M. First we will consider whether a homomorphism from an R-module D to L or N implies the existence of a homomorphism from D to M.

Let $f : D \to L$ and $\psi : L \to M$. Then the composition of f and ψ defines a homomorphism $f' : D \to M$ such that $f' = \psi \circ f$. It is equivalent to the following diagram commutes:



or ψ induces a homomorphism between abelian groups:

$$\begin{aligned} \psi' : \text{Hom}_R(D, L) &\longrightarrow \text{Hom}_R(D, M) \\ f &\longmapsto f' = \psi \circ f. \end{aligned} \tag{2.19}$$

Proposition 2.20. *Let D, L, M be R-modules. Let $\psi : L \to M$ induces $\psi' : Hom_R(D, L) \to Hom_R(D, M)$. If $\psi : L \to M$ is injective, then ψ' is also injective, i.e. if $0 \to L \xrightarrow{\psi} M$ is exact, then $0 \to Hom_R(D, L) \xrightarrow{\psi'} Hom_R(D, M)$ is also exact.*

Proof. Let f, g be two distinct homomorphisms in $\text{Hom}_R(D, L)$. Consider the composites $\psi \circ f, \psi \circ g : D \to M$. Since ψ is injective, $\psi \circ f$ is distinct from $\psi \circ g$ for any distinct $f, g \in \text{Hom}_R(D, L)$. Thus the induced homomorphism ψ' is injective. □

It should be noted that exactness at N does *not* guarantee that $\text{Hom}_R(D, M) \xrightarrow{\phi} \text{Hom}_R(D, N) \to 0$ is exact. An obvious example is the exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/n\mathbb{Z} \to 0$. Let $D = \mathbb{Z}/n\mathbb{Z}$, and let $f \in \text{Hom}_R(D, N)$ be the identity map. Since $\mathbb{Z}$ contains no element of finite order except zero, there is only the zero homomorphism $F : D \to M$. Thus $p \circ F = 0 \neq f$. However, we have the following theorem.

Theorem 2.21. *Let D, L, M, N be R-modules. If a sequence*

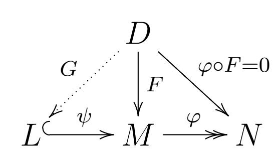
$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

is exact, then its associated sequence

$$0 \to Hom_R(D, L) \xrightarrow{\psi'} Hom_R(D, M) \xrightarrow{\varphi'} Hom_R(D, N)$$

is also exact.

Proof. It remains to prove that $\ker(\psi') = \text{im}(\varphi')$. To show $\ker(\varphi') \subset \text{im}(\psi')$, choose $F \in \text{Hom}_R(D, M)$ such that $\varphi \circ F = 0$.

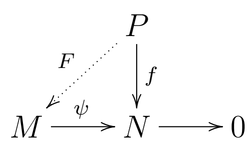


For any element $d \in D, \varphi(F(d)) = 0$, which means $F(d)$ is in the kernel of φ. Since $\ker(\varphi) = \text{im}(\psi)$, $\psi(l) = F(d)$ for some $l \in L$. Furthermore the injectivity of ψ guarantees the uniqueness of l, which gives a well-defined map $G : D \to L$ where

$G(d) = l$. Check that G respects the R-module structures of D and L. Since the left triangle commutes, $F = \psi'(G)$ for some $G \in \mathrm{Hom}_R(D, L)$, making $F \in \mathrm{im}\ (\psi')$. Hence $\ker\ (\varphi') \subset \mathrm{im}\ (\psi')$.

Conversely, if F is in the image of ψ', then $F = \psi'(G)$ for some $G \in \mathrm{Hom}_R(D, L)$, so $\varphi(F(d)) = \varphi(\psi(G(d)))$ for all $d \in D$. By exactness however, $\varphi \circ \psi = 0$, which implies $\varphi(F(d)) = 0\ \forall\ d \in D$, i.e. F is in the kernel of φ', proving $\mathrm{im}\ \psi' \subset \ker \varphi'$. $\square$

Definition 2.22. An R-module P is *projective* if it satisfies any of the following equivalent conditions:
(1) If $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is exact, then $0 \to \mathrm{Hom}_R(P, L) \xrightarrow{\psi'} \mathrm{Hom}_R(P, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(P, N) \to 0$ is exact.
(2) Let $M \xrightarrow{\varphi} N \to 0$ be an exact sequence of modules. Every $f : P \to N$ lifts to $F \in \mathrm{Hom}_R(P, M)$ such that the following diagram commutes:



(3) If P is a quotient of the R-module M, then any short exact sequence $0 \to L \to M \to P \to 0$ splits.
(4) P is a direct summand of a free R-module.

Note that free modules are projective. A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module. Every module is a quotient of a projective module.

We have defined projective modules for the purpose of defining the homology groups $\mathrm{Tor}_n^R(D, B)$ using *projective resolution*.

Definition 2.23. Let B be any R-module. A *projective resolution* of B is an exact sequence

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \to 0$$

such that each P_i is a projective R-module.

Every R-module has a projective resolution. Let P_0 be a free R-module on a set of generators of B. Then B is a direct summand of P_0. Let $\epsilon : P_0 \to B$ be the projection map. Since ϵ is surjective, $\epsilon(P_0) = B = \ker(0)$, making the sequence $P_0 \xrightarrow{\epsilon} B \to 0$ exact. For the following homomorphism d_1, define $d_1(P_1) = \ker(\epsilon)$, and let P_1 be a free R-module mapping onto $\ker(\epsilon) \subset P_0$. This gives the exactness at P_0. By repeating these steps, we get a resolution of B that is free (thus projective) at every P_i, which we call a *free resolution.*

3 Universal Coefficient Theorem

3.1 $\mathbf{Tor}_n^R(B, D)$

Let B be a right R-module. Take a projective resolution of B

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \to 0. \tag{3.1}$$

Then tensor with D to obtain

$$\cdots \to P_n \otimes D \xrightarrow{d_n \otimes 1} P_{n-1} \otimes D \to \cdots \xrightarrow{d_1 \otimes 1} P_0 \otimes D \xrightarrow{\epsilon \otimes 1} B \otimes D \to 0. \tag{3.2}$$

Since im $(d_{n+1} \otimes 1) < \ker(d_n \otimes 1), (d_n \otimes 1) \circ (d_{n+1} \otimes 1) = (d_n \otimes 1)(\mathrm{im}(d_{n+1} \otimes 1)) = 0$ for all n, making (3.2) a chain complex, so we may construct its homology groups.

Definition 3.3. Let D be a left R-module, and let B be a right R-module. For any projective resolution of B by right R-module as above, take tensor product with D, and define $d_n \otimes 1 : P_n \otimes D \to P_{n-1} \otimes D$ for all $n \geq 1$. Then

$$\mathrm{Tor}_n^R(B, D) = \ker(d_n \otimes 1)/\mathrm{im}(d_{n+1} \otimes 1), \tag{3.4}$$

which we call the *nth homology group derived from the functor* $- \otimes D$. When $R = \mathbb{Z}$ the group $\mathrm{Tor}_n^{\mathbb{Z}}(B, D)$ is also denoted by simply $\mathrm{Tor}_n(B, D)$. $\mathrm{Tor}_0^R(B, D)$ is the 0^{th} homology of $P_1 \otimes D \xrightarrow{d_1 \otimes 1} P_0 \otimes D \xrightarrow{x} 0$.

Thus $\mathrm{Tor}_n^R(B, D)$ is the n^{th} homology group of the chain complex obtained from (3.2) by removing the term $B \otimes D$. The next proposition is special for 0^{th} homology group.

Proposition 3.5. *For any right R-module B, $Tor_0^R(B, D) \cong B \otimes D$.*

Proof. Let $\cdots \to P_0 \xrightarrow{\epsilon} B \to 0$ be any projective resolution of B.

$$\begin{aligned}
\mathrm{Tor}_0^R(B, D) &= \ker(x)/\mathrm{im}(d_1 \otimes 1) && \text{by Equation (3.4)} \\
&= P_0 \otimes D/\mathrm{im}(d_1 \otimes 1) && P_0 \otimes D \text{ is annihilated by } x \\
&= P_0 \otimes D/\ker(\epsilon \otimes 1) && \text{exactness at} P_0 \otimes D \text{ in(3.2)}
\end{aligned}$$

By right exactness of tensor product, the sequence $P_1 \otimes D \xrightarrow{d_1 \otimes 1} P_0 \otimes D \xrightarrow{\epsilon \otimes 1} B \otimes D \to 0$ is exact, so $B \otimes D = \mathrm{im}(\epsilon \otimes 1)$. However, $\ker(\epsilon \otimes 1) = \mathrm{im}\ (d_1 \otimes 1) = (P_0 \otimes D)/(B \otimes D)$. Thus $P_0 \otimes D/\ \ker\ (\epsilon \otimes 1) \cong B \otimes D$. □

The following two propositions are important, as they together guarantee the homology group $\mathrm{Tor}_n^R(B, D)$ of any R-module B is well-defined. For proof, see Hatcher Section 3.A.

Proposition 3.6. *The homology groups $Tor_n^R(B, D)$ are independent of the choice of projective resolution of B.*

Proposition 3.7. *Let $f : B \to B'$ be an R-module homomorphism, and take projective resolution of B and B', respectively. Then for each $n \geq 0$ there is an induced map $\eta_n : Tor_n^R(B, D) \to Tor_n^R(B', D)$ on the homology groups obtained from these resolutions, depending only on f.*

Let $0 \to L. \xrightarrow{\psi} M. \xrightarrow{\varphi} N. \to 0$ be a short exact sequence of chain complexes, i.e. a sequence of homomorphism of complexes such that $0 \to L_i \xrightarrow{\psi} M_i \xrightarrow{\varphi} N_i \to 0$ is short exact for every i, or equivalently, the following diagram commutes:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_{i+1} & \xrightarrow{d_{i+1}} & L_i & \xrightarrow{d_i} & L_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \psi_{i+1}} & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_{i-1}} & & \\
\cdots & \longrightarrow & M_{i+1} & \xrightarrow{d_{i+1}} & M_i & \xrightarrow{d_i} & M_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i-1}} & & \\
\cdots & \longrightarrow & N_{i+1} & \xrightarrow{d_{i+1}} & N_i & \xrightarrow{d_i} & N_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.8}$$

It can be stretched out into a long exact sequence of homology groups:

$$\cdots \to H_{i+1}(N) \to H_i(L) \to H_i(M) \to H_i(N) \to H_{i-1}(L) \to \cdots .$$

Now for simplicity, we will vary the original setup a little. For a short exact sequence of right R-modules $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$, find a free resolution for each of the modules L, M, and N:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P_1'' & \longrightarrow & P_0'' & \xrightarrow{\epsilon''} & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \psi & & \\ \cdots & \longrightarrow & P_1' & \longrightarrow & P_0' & \xrightarrow{\epsilon'} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi & & \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\epsilon} & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.9}$$

such that $0 \to P_i'' \to P_i' \to P_i \to 0$ is split short exact for every i. (Their existences are guaranteed by Horseshoe Lemma. See Weibel Section 2.2.) When we tensor with a left R-module D and remove the end terms, we get

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P_2'' \otimes D & \longrightarrow & P_1'' \otimes D & \longrightarrow & P_0'' \otimes D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \psi & & \\ \cdots & \longrightarrow & P_2' \otimes D & \longrightarrow & P_1' \otimes D & \longrightarrow & P_0' \otimes D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi & & \\ \cdots & \longrightarrow & P_2 \otimes D & \longrightarrow & P_1 \otimes D & \longrightarrow & P_0 \otimes D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.10}$$

where each short exact sequence is split because the original ones were. Then the long exact sequence of homology of this short exact sequence of chain complexes is as we desired:

$$\begin{aligned} \cdots \to \operatorname{Tor}_2^R(N, D) \xrightarrow{\delta_1} \operatorname{Tor}_1^R(L, D) \xrightarrow{\psi_*} \operatorname{Tor}_1^R(M, D) \xrightarrow{\varphi_*} \\ \operatorname{Tor}_1^R(N, D) \xrightarrow{\delta_0} L \otimes D \xrightarrow{\psi_*} M \otimes D \xrightarrow{\varphi_*} N \otimes D \to 0. \end{aligned} \tag{3.11}$$

The maps δ_i are called *connecting homomorphisms.*

To define the connecting homomorphism $\delta_i : \mathrm{Tor}_i(N, D) \to \mathrm{Tor}_{i-1}(L, D)$, suppose $n \in \ker\, d_i$. Since φ_i is surjective, there exists some $m \in M_i$ such that $\varphi_i(m) = n$. Based on the following calculation

$$\begin{aligned} \varphi_{i-1}(d_i(m)) &= d_i(\varphi_i(m)) && \text{diagram(3.8)commutes} \\ &= d_i(n) = 0 && n \text{ is in the kernel of } d_i \end{aligned}$$

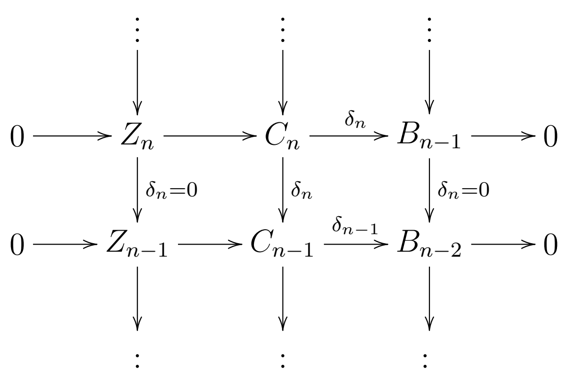
we know that $d_i(m) \in \ker\, \varphi_{i-1} = \mathrm{im}\, \psi_{i-1}$. Thus there exists a unique $l \in L_{i-1}$ such that $\psi_{i-1}(l) = d_i(m)$. Note that

$$\begin{aligned} \psi_{i-1} \circ d_i(l) &= d_i \circ \psi_{i-1}(l) && \text{diagram(3.8)commutes} \\ &= d_i \circ d_i(m) && \psi_{i-1} \text{ is injective} \\ &= 0 && d_i(m) = 0 \end{aligned}$$

Now define $\delta_i : Tor_i(N, D) \to Tor_{i-1}(L, D)$ by $\delta_i[n] = [l]$. We will leave to the readers to prove the map is well-defined and a homomorphism and that the sequence is exact.

3.2 Proof of Universal Coefficient Theorem

Next we will give a variation of the construction above, which will be used in the Universal Coefficient Theorem for Homology. Let $\cdots \to C_n \xrightarrow{\delta_n} C_{n-1} \to \cdots$ be a chain complex of free abelian groups. Let $B_n = \mathrm{im}(\delta_n) \subset Z_n = \ker(\delta_n) \subset C_n$. Since $\delta_n|_{Z_n} = 0$ and $\delta_n|_{B_n} = 0$, we can regard B and C as subcomplexes of Z with trivial boundary maps. This gives rise to a short exact sequence of chain complexes



(3.12)

The subgroups of a free abelian group are free abelian, so $C_n \cong Z_n \oplus B_{n-1}$. Be careful that the chain complex C is not direct sum of chain complexes Z and B, for the boundary maps in C are not necessarily trivial, unlike the trivial ones in B and N. Next take the tensor product with G to get

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n \otimes G & \longrightarrow & C_n \otimes G & \xrightarrow{\delta_n \otimes 1} & B_{n-1} \otimes G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_n \otimes 1} & & \downarrow{\scriptstyle \delta_n \otimes 1} & & \downarrow{\scriptstyle \delta_n \otimes 1} & & \\
0 & \longrightarrow & Z_{n-1} \otimes G & \longrightarrow & C_{n-1} \otimes G & \xrightarrow{\delta_{n-1} \otimes 1} & B_{n-2} \otimes G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{3.13}$$

Tensor product commutes with direct sums by 2.18, so the rows in the second diagram are split exact. By the construction in 3.11, we can pass to a long exact sequence of homology groups:

$$\cdots \to H_n(Z;G) \to H_n(C;G) \to H_n(B;G) \to H_{n-1}(Z;G) \to \cdots . \tag{3.14}$$

In the original sequence, the chain complex of Z_n has only zero homomorphisms, so $H_n(Z;G) = Z_n \otimes G/0 = Z_n \otimes G$ for all n. Similarly $H_n(B;G) = B_n \otimes G$. The long exact sequence 3.14 is isomorphic to

$$\cdots \to B_n \otimes G \to Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \to Z_{n-1} \otimes G \to \cdots . \tag{3.15}$$

Take $b \otimes g \in B_{n-1} \otimes G$. Since $\delta_n \otimes 1$ is surjective, we can find $c \otimes g \in C_n \otimes G$ such that $\delta_n \otimes 1(c \otimes g) = b \otimes g$. Apply $\delta_n \otimes 1$ to $c \otimes g$, we get back $b \otimes g$, now in $C_{n-1} \otimes G$. Since $B_{n-1} \otimes G \subset Z_{n-1} \otimes G$, $b \otimes g \in B_{n-1} \otimes G \implies b \otimes g \in Z_{n-1} \otimes G$. Thus we may define the boundary map as $i_n \otimes 1 : B_n \otimes G \to Z_n \otimes G$ where i_n is the inclusion map from $B_n \to Z_n$.

Referring to Diagram 2.5, we can view $H_n(C;G)$ as A_n. It follows that $B_n \otimes G = A_{n+2}, Z_n \otimes G = A_{n+1}$, et cetera. Then we can construct an extension of C_{n+1} by C_n. We will define C_n and C_{n+1} in terms of maps we know, i.e. $i_n \otimes 1$ and $i_{n-1} \otimes 1$, respectively.

$$C_n \cong \operatorname{coker}(A_{n+2} \to A_{n+1}) = \operatorname{coker}(B_n \otimes G \to Z_n \otimes G) = \operatorname{coker}(i_n \otimes 1), \tag{3.16}$$

while $C_{n-1} \cong \ker(A_{n-1} \to A_{n-2}) = \ker(B_{n-1} \otimes G \to Z_{n-1} \otimes G) = \ker(i_{n-1} \otimes 1)$. Together they make a short exact sequence:

$$0 \to \operatorname{coker}(i_n \otimes 1) \to H_n(C;G) \to \ker(i_{n-1} \otimes 1) \to 0, \tag{3.17}$$

and $\text{coker}(i_n\otimes 1) = Z_n\otimes G/\text{im }(i_n\otimes 1)$. Now our task is to find out what $\text{coker}(i_n\otimes 1)$ and $\ker(i_{n-1}\otimes 1)$ are.

In general, $A \xrightarrow{f} B \to \text{coker} f \to 0$ is exact by the definition of coker $f = B/f(A)$. In the case of $B_n \xrightarrow{i_n} Z_n \to \text{coker}(i_n) \to 0$, we actually know that $\text{coker}(i_n) = H_n(C)$. Right exactness of tensor product says that

$$\begin{array}{ccccccc} B_n\otimes G & \xrightarrow{i_n\otimes 1} & Z_n\otimes G & \longrightarrow & \text{coker}(i_n\otimes 1) & \longrightarrow & 0 \\ \| & & \| & & \mid\approx & & \\ B_n\otimes G & \xrightarrow{i_n\otimes 1} & Z_n\otimes G & \longrightarrow & H_n(C)\otimes G & \longrightarrow & 0. \end{array} \tag{3.18}$$

That is, $\text{coker}(i_n\otimes 1) \cong H_n(C)\otimes G$, and it does not depend on choice of B_n or Z_n.

Now to find $\ker(i_{n-1}\otimes 1)$, or equivalently $\ker(i_n\otimes 1)$, consider the free resolution of $H_n(C)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \xrightarrow{i_n} & Z_n & \longrightarrow & H_n(C) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & P_1 & & P_0 & & H & & \end{array} \tag{3.19}$$

Tensoring with G, we get

$$0 \to P_1\otimes G \xrightarrow{i_n\otimes 1} P_0\otimes G \to H\otimes G \to 0. \tag{3.20}$$

Then calculate $\text{Tor}_1^{\mathbb{Z}}$ of $H\otimes G$, which equals $H_1(P_1\otimes G) = \ker\,(i_n\otimes 1)$. Combining the two results, we proved the first half of the Universal Coefficient Theorem:

Theorem 3.21. *If C is a chain complex of free abelian groups, then there are natural short exact sequences*

$$0 \to H_n(C)\otimes G \to H_n(C;G) \to \ Tor_1(H_{n-1}(C), G) \to 0 \tag{3.22}$$

for all n and all G, and these sequences split, though not naturally.

To prove the splitting, we go back to the split short exact sequence $0 \to Z_n \xrightarrow{f} C_n \xrightarrow{g} B_{n-1} \to 0$. Splitting implies that there is $p : C_n \to Z_n$ such that $p\circ f = 1_{Z_n}$. Further p can be extended to p', making the following diagram commutes:

$$\begin{array}{ccc} C_n & & \\ {\scriptstyle p}\downarrow & \searrow^{p'} & \\ Z_n & \xrightarrow{q} & H_n(C). \end{array} \tag{3.23}$$

To get a chain map $F : C. \to H.(C)$, we make $H.$ a chain complex by adding trivial boundary maps between them. Tensor with G, which yields $F \otimes 1 : C. \otimes G \to H.C \otimes G$. When we take the homology of $C. \otimes G$, we get the usual $H_n(C; G)$. When we take the homology of $H.C \otimes G$, however, it gives us $H_n(C) \otimes G$, due to the zero homomorphisms. Thus we have the induced homomorphism on homology $F_* : H_n(C; G) \to H_n(C) \otimes G$, which proves the desired splitting.

4 Applications

We will demonstrate two calculations of homology with arbitrary coefficients. Recall

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < i < n \text{ and } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Example 4.2. Let $C = \mathbb{R}P^n$ and $G = \mathbb{Z}/2\mathbb{Z}$. Calculate the homology $H_i(C; G)$.

By the Universal Coefficient Theorem, $H_i(C; G) \cong H_i(C) \otimes G \oplus \mathrm{Tor}_1(H_{i-1}(C), G)$. We will consider each case separately. For $i = 0$,

$$\begin{aligned} H_0(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) &\cong H_0(\mathbb{R}P^n) \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_{-1}(\mathbb{R}P^n), \mathbb{Z}/2\mathbb{Z}) && \text{by 3.22} \\ &= \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) && \text{by 4.1} \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) && \text{by 2.16} \\ &= \mathbb{Z}/2\mathbb{Z} && \text{trivial Tor} \end{aligned}$$

For $i = 1$,

$$\begin{aligned} H_1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) &\cong H_1(\mathbb{R}P^n) \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_0(\mathbb{R}P^n), \mathbb{Z}/2\mathbb{Z}) && \text{by 3.22} \\ &= \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) && \text{by 4.1} \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) && \text{by 2.16} \\ &= \mathbb{Z}/2\mathbb{Z} && \text{see below} \end{aligned}$$

A free short exact sequence of $\mathbb{Z}/2\mathbb{Z}$ is $0 \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/2\mathbb{Z} \to 0$. After we take the tensor product, we get $0 \xrightarrow{d_2 \otimes 1} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{d_1 \otimes 1} (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z} \xrightarrow{\epsilon \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z} \to 0$, which is equal to the original sequence. Thus $\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$.

For $1 < i < n$ such that i is an odd integer,

$$\begin{aligned}
H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) &\cong H_i(\mathbb{R}P^n) \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_{i-1}(\mathbb{R}P^n), \mathbb{Z}/2\mathbb{Z}) && \text{by 3.22} \\
&= \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) && \text{by 4.1} \\
&= \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) && \text{by 2.17} \\
&= \mathbb{Z}/2\mathbb{Z} && \text{as in } i = 0
\end{aligned}$$

For $0 < i < n$ such that i is even,

$$\begin{aligned}
H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) &\cong H_i(\mathbb{R}P^n) \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_{i-1}(\mathbb{R}P^n), \mathbb{Z}/2\mathbb{Z}) && \text{by 3.22} \\
&= 0 \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) && \text{by 4.1} \\
&= 0 \oplus \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

For $i = n$ even, we have $H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = 0 \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. If i is odd, it is $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}_1(0, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ instead. In summary, we have the answer $H_n(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, which is even simpler than the homology with $\mathbb{Z}$ coefficients. The next example is also about $\mathbb{R}P^n$, but we will change G to $\mathbb{Q}$.

Example 4.3. Let $C = \mathbb{R}P^n$ and $G = \mathbb{Q}$. Calculate the homology $H_i(C; G)$

For $i = 0$, $H_i(C; G) = \mathbb{Z} \otimes \mathbb{Q} \oplus 0 = \mathbb{Q}$.
For odd i between 0 and n,

$$\underbrace{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Q}}_{=0} \oplus \mathrm{Tor}(0, \mathbb{Q}) = 0.$$

$\mathbb{Z}/2\mathbb{Z}$ is a torsion abelian group, while $\mathbb{Q}$ under addition is a divisible group. Thus their tensor product is zero. For $a \in \mathbb{Z}/2\mathbb{Z}$ and $b \in \mathbb{Q}$, $0 = 0 \otimes b = 2 \otimes b = (1 \cdot 2) \otimes b = 1 \otimes 2b$. By similar argument, $\mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) = 0$, and $\mathrm{Tor}(0, \mathbb{Q}) = 0$.
For even i between 0 and $n, 0 \otimes \mathbb{Q} \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) = 0$.
For $i = n$ even,

$$0 \otimes \mathbb{Q} \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) = 0.$$

If i is odd instead, then

$$\mathbb{Z} \otimes \mathbb{Q} \ \oplus \mathrm{Tor}(0, \mathbb{Q}) = \mathbb{Q}.$$

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