HILBERT'S NULLSTELLENSATZ AND ITS APPLICATION IN GRAPH THEORY

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Abstract. *Hilbert's Nullstellensatz*, one of the fundamental theorems of Algebraic Geometry, is a powerful algebraic technique that has extensive applications in Graph Theory. In this paper, we prove *Combinatorial Nullstellensatz*, a localization of *Hilbert's Nullstellensatz*, which asserts a stronger conclusion. We then present its applications in demonstrating the results on the existence of regular subgraphs, the choosability of directed graphs, and the cube covering by hyperplanes.

Contents

1. Hilbert's Nullstellensatz

Theorem 1.1. *(Hilbert's Nullstellensatz) For F an algebraically closed field, let f, g_1,...g_m be polynomials in $F[x_1, ..., x_n]$. If $g_i(x) = 0$ for all $1 \leq i \leq m$ implies $f(x) = 0$, then there exists $k \in \mathbb{N}$ and $h_1, ..., h_m \in F[x_1, ...x_n]$ such that $f^k = \sum_{i=1}^{m} h_i g_i$.*

For its combinatorial application, one considers the special case with the following conditions:

(1) $m = n$
(2) Each g_i is univariate in the form of $\prod_{s \in S_i}(x_i - s)$
(3) The condition on algebraic closure is loosened

Theorem 1.2. *(Combinatorial Nullstellensatz) For F a field and $S_1, .., S_n$ nonempty subsets of F, let $g_i(x_i) = \prod_{s \in S_i}(x_i - s)$. If, for $f \in F[x_1, ..., x_n]$, $f(s_1, ..., s_n) = 0$ $\forall s_i \in S_i$, then there exists $h_1, ..., h_n \in F[x_1, ...x_n]$ such that $deg(h_i) \leq deg(f) - deg(g_i)$ and $f = \sum_{i=1}^{n} h_i g_i$. Furthermore, if $f, g_1, ..., g_n \in R[x_1, ..., x_n]$ for R a subring of F, then $h_1, ..., h_n \in R[x_1, ...x_n]$*

In order to prove *Combinatorial Nullstellensatz*, we need the following lemma.

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Lemma 1.3. *For F a field and $p \in F[x_1, ..., x_n]$, suppose that the degree of p as a polynomial in x_i is at most t_i and $S_i \subset F$ such that $|S_i| \geq t_i + 1$ $\forall i$. If $p(x_1, ..., x_n) = 0$ $\forall (x_1, ...x_n) \in S_1 \times ... \times S_n$, then $p \equiv 0$*

Proof. (Induction on n)

(1) Base case: For $n = 1$, a non-zero single-variable polynomial of degree t_1 can have at most t_1 distinct zeros by the *Fundamental Theorem of Algebra.* Hence, $p \equiv 0$.

(2) Inductive case: For $n \geq 2$, assume true for $n - 1$.
Let $p = p(x_1, ..., x_n)$ be a polynomial satisfying the hypotheses. p can be written as a polynomial in x_n:

$$p = \sum_{i=0}^{t_n} p_i(x_1, ..., x_{n-1}) x_n^i$$

Note that the degree of each p_i as a polynomial in x_j is at most t_j. For each fixed $(n-1)$-tuple $(c_1, ..., c_{n-1}) \in S_1 \times ... \times S_{n-1}$,

$$p(x_n) = \sum_{i=0}^{t_n} p_i(c_1, ..., c_{n-1}) x_n^i = 0 \text{ for all } x_n \in S_n$$

Thus, $p_i(x_1, ...x_n) = 0$ for all $(x_1, ..., x_{n-1}) \in S_1 \times ... \times S_{n-1}$. Then, by the induction hypothesis, $p_i \equiv 0$ for all i, and, consequently, $p \equiv 0$.

□

Proof of Combinatorial Nullstellensatz. . Let $t_i = |S_i| - 1$ for all i.

$$f(x_1, ..., x_n) = 0 \text{ for all n-tuple } (x_1, ..., x_n) \in S_1 \times ...S_n$$

For each i, $1 \leq i \leq n$, g_i can be written as follows:

$$g_i(x_i) = \prod_{s \in S_i} (x_i - s) = x_i^{t_i+1} - \sum_{j=0}^{t_i} g_{ij} x_i^j$$

Hence, for $s \in S_i$,

$$x_i^{t_i+1} = \sum_{j=0}^{t_i} g_{ij} x_i^j$$

Using the above relation, one can repeatedly replace x_i^j in f where $1 \leq i \leq n$ and $j > t_i$ by a linear combination of lower degree terms. Let $\bar{f}$ be the resulting polynomial whose degree in x_i is at most t_i. Then, $\bar{f}$ is f subtracted by the linear combination $\sum_{i=1}^{n} h_i g_i$ where $h_i \in F[x_1, ..., x_n]$ and $deg(h_i) \leq deg(f) - deg(g_i)$ for $1 \leq i \leq n$. Moreover, the coefficients of each h_i are in the smallest ring containing the coefficients of f and g_1,...,g_n. Since

$$\bar{f}(x_1, ..., x_n) = f(x_1, ..., x_n) = 0 \text{ for all n-tuple } (x_1, ..., x_n) \in S_1 \times ...S_n,$$

$\bar{f} \equiv 0$, by Lemma 1.3. Thus, $f = \sum_{i=1}^{n} h_i g_i$. □

Corollary 1.4. *For F an arbitrary field, let f be a polynomial in $F[x_1, ..., x_n]$. Suppose $deg(f) = \sum_{i=1}^{n} t_i$ where $t_i \geq 0$ for all i and the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. If S_1,...,S_n are subsets of F such that $|S_i| > t_i$, then there exists an n-tuple $(s_1, ..., s_n) \in S_1 \times ... \times S_n$ such that $f(s_1, ..., s_n) \neq 0$.*

Proof. Without the loss of generality, assume that $|S_i| = t_i + 1$. Suppose that there exists no such n-tuple, and let $g_i(x_i) = \prod_{s \in S_i}(x_i - s)$. By Theorem 1.2, there exists polynomials $h_1,...h_n$ in $F[x_i, ..., x_n]$ such that $f = \sum_{i=1}^n h_i g_i$ and $deg(h_i) \leq deg(f) - deg(g_i)$. For each $1 \leq i \leq n$, the degree of $h_i g_i = h_i \prod_{s \in S_i}(x_i - s)$ is at most $deg(f)$, and if there are any monomials of degree $deg(f)$, they are products of $x_i^{t_i+1}$. Hence, the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in the right side is zero, which leads to a contradiction. □

2. Regular Subgraphs

Definition 2.1. A **graph** G is an ordered pair (V, E) consisting of V, the set of **vertices**, and E, the set of **edges** which are unordered pairs of vertices in V.

Definition 2.2. A graph $G = (V, E)$ is called **regular** if every vertex has the same degree. G is d**-regular** if every vertex is of degree d.

Definition 2.3. A graph $G = (V, E)$ is called **simple** if G contains no multiple edges, i. e. each pair of vertices can have at most one edge that is incident to them.

Theorem 2.4. *Every simple* 4*-regular contains a* 3*-regular subgraph.*

The preceding theorem was conjectured by Berge and Sauer and, later, proved by Taśkinov. However, if the assumption on simplicity is relaxed, the result is false, as demonstrated by a 3-vertex graph with two edges between each pair of vertices. In this case, one extra edge is sufficient to ensure the existence of a 3-regular subgraph. *Hilbert's Nullstellensatz* shows not only the existence of a 3-regular subgraph within an "almost" 4-regular graph but also more generalized result stated as follows:

Theorem 2.5. *For p a prime, any loopless graph $G = (V, E)$ with average degree greater than $2p-2$ and maximum degree at most $2p-1$ contains a p-regular subgraph.*

Proof. For $v \in V$ and $e \in E$, let $a_{v,e} = 1$, if $v \in e$, and $a_{v,e} = 0$, otherwise. Consider the following polynomial in $(x_e)_{e \in E}$ over $GF(p)$:

$$F((x_e)_{e \in E}) = \prod_{v \in V}[1 - (\sum_{e \in E} a_{v,e}x_e)^{p-1}] - \prod_{e \in E}(1 - x_e)$$

Due to the assumption on average degree,

$$\frac{2|E|}{|V|} > 2p - 2$$
$$|E| > (p-1)|V|$$

Hence, the degree of F is $|E| = \sum_{e \in E} t_e$ where $t_e = 1$ for all $e \in E$. The coefficient of $\prod_{e \in E} x_e^{t_e} = \prod_{e \in E} x_e = (-1)^{|E|+1} \neq 0$. Let $S_e = \{0, 1\}$ for all $e \in E$. Since $|S_e| > t_e$ for all $e \in E$, by Corollary 1.4, there exists $(s_e)_{e \in E}$ such that $s_e \in S_e$ for all $e \in E$ and $F((s_e)_{e \in E}) \neq 0$. Such vector is not the zero vector, since $F(\mathbf{0}) = \prod_{v \in V} 1 - \prod_{e \in E} 1 = 1 - 1 = 0$. Thus, $\prod_{e \in E}(1 - s_e) = 0$. Then, for each $v \in V$, $\sum_{e \in E} a_{v,e}s_e \equiv 0$ (mod p). Otherwise, by *Fermat's Little Theorem*, $\sum_{e \in E} a_{v,e}s_e \equiv 1$ (mod p), and $F((s_e)_{e \in E}) = 0$. Thus, in the subgraph consisting of all edges $e \in E$ such that $s_e = 1$, all the endpoints of the edges are of degree p, since the maximum degree is at most $2p - 1$. □

One can easily see that the special case $p = 3$ of the above result demonstrates the existence of a 3-regular subgraph within a 4-regular graph with an extra edge.

3. Colorings of Directed Graphs

Definition 3.1. An **orientation** of an undirected graph $G = (V, E)$ is a map $D : E \mapsto E'$ which maps an unordered pair of vertices into an ordered pair of the same vertices, thus, assigning a direction to each edge of G.

Hence, an orientation transforms its underlying undirected graph into a directed graph, or a digraph, whose definition is formally stated as follows:

Definition 3.2. A **directed graph**, or **digraph**, D is an ordered pair (V, E) consisting of V, the set of **vertices**, and E, the set of **directed edges** which are ordered pairs of vertices in V.

Definition 3.3. A **vertex coloring** of a directed or an undirected graph $G = (V, E)$ is a map $c : V \mapsto \mathbb{Z}$ that assigns a color to each vertex of G.

Definition 3.4. A vertex coloring $c : V \mapsto Z$ of a directed or an undirected graph $G = (V, E)$ is **proper** if $c(v_i) \neq c(v_j)$ for all $(v_i, v_j) \in E$.

Definition 3.5. A directed or an undirected graph $G = (V, E)$ is d**-colorable** if there exists a proper vertex coloring $c : V \mapsto \{1, ..., d\}$.

Definition 3.6. For a directed or an undirected graph $G = (V, E)$ and a function $f : V \mapsto \mathbb{N}$, G is f**-choosable** if for every assignment of a set of integers $S(v) \subset \mathbb{Z}$ to each vertex $v \in V$ where $|S(v)| = f(v)$ for each $v \in V$, there exists a proper vertex coloring $c : V \mapsto Z$ such that $c(v) \in S(v)$ for all $v \in V$.

Definition 3.7. A directed or an undirected graph $G = (V, E)$ is k**-choosable** if G is f-choosable for the constant function $f(v) = k$.

Definition 3.8. For $v \in V$ a vertex of a directed graph $D = (V, E)$, the **indegree** of v, $d^-(v)$, is equal to $|\{(u, v) : (u, v) \in E\}|$, the number of directed edges to v. Similarly, the **outdegree** of v, $d^+(v)$, is equal to $|\{(v, u) : (v, u) \in E\}|$, the number of directed edges from v. For $H = (V(H), E(H))$, a subdigraph of D, and $v \in V(H)$, $d^-_H(v)$ and $d^+_H(v)$ denote the indegree and the outdegree of v as a vertex of H, respectively.

Definition 3.9. A subdigraph $H = (V(H), E(H))$ of a directed graph $D = (V, E)$ is **Eulerian** if $d^-_H(v) = d^+_H(v)$ for all $v \in V$. H is **even** if $|E(H)|$ is even, and, likewise, H is **odd** if $|E(H)|$ is odd. $EE(D)$ and $EO(D)$ denote the numbers of even and odd Eulerian subgraphs, respectively.

Lemma 3.10. *Every Eulerian subgraph is a union of edge-disjoint directed simple cycles.*

Proof. For each vertex of an Eulerian subgraph, its indegree is equal to its outdegree. Hence, one can inductively remove directed simple cycles from the subgraph. □

Definition 3.11. For an undirected graph $G = (V, E)$ where $V = \{v_1, ..., v_n\}$, its **graph polynomial** $f_G = f_G(x_1, .., x_n)$ is defined as follows:

$$f_G(x_1, ..., x_n) = \prod_{\{v_i, v_j\} \in E; i<j} (x_i - x_j)$$

Definition 3.12. For each directed edge $e = (v_i, v_j)$ oriented by an orientation $D : E \mapsto E'$ of an undirected graph $G = (V, E)$, its **weight** $w(e)$ is defined as follows:

$$\begin{aligned} w(e) &= x_i \text{ if } i < j \\ &= -x_i \text{ if } i > j \end{aligned}$$

The **weight** $w(D)$ of an orientation D is $\prod_{e \in E'} w(e)$.

Definition 3.13. For a directed edge $e = (v_i, v_j)$, e is **decreasing** if $i > j$. An orientation $D : E \mapsto E'$ of an undirected graph $G = (V, E)$ where $V = \{v_1, ..., v_n\}$ is called **even** if E' has an even number of decreasing directed edges. Otherwise, D is **odd**. For non-negative integers $d_1, ..., d_n$, $DE(d_1, ..., d_n)$ and $DO(d_1, ..., d_n)$ denote the sets of all even and odd orientations of G such that the outdegree of the vertex v_i is d_i for $1 \le i \le n$, respectively.

Lemma 3.14. *For an undirected graph $G = (V, E)$, let $\bar{D}$ be the set of all orientations of G. Then,*

$$f_G(x_1, ..., x_n) = \sum_{D \in \bar{D}} w(D) = \sum_{d_1, ..., d_n \ge 0} (|DE(d_1, ..., d_n)| - |DO(d_1, ..., d_n)|) \prod_{i=1}^{n} x_i^{d_i}$$

Proof. Each term in the expansion of $f_G(x_1, ..., x_n) = \prod_{\{v_i, v_j\} \in E; i<j} (x_i - x_j)$ corresponds to $w(D)$ for some orientation D of G by Definition 3.12. Thus, $f_G(x_1, ..., x_n) = \sum_{D \in \bar{D}} w(D)$. Moreover, for a directed edge $e = (v_i, v_j)$ for a fixed v_i, $w(e) = x_i$ if e is increasing and $w(e) = -x_i$ if e is decreasing. Then, for $D \in DE(d_1, ..., d_n)$, $w(D) = \prod_{i=1}^{n} x_i^{d_i}$, and for $D \in DE(d_1, ..., d_n)$, $w(D) = -\prod_{i=1}^{n} x_i^{d_i}$. Hence, $f_G(x_1, ..., x_n) = \sum_{d_1, ..., d_n \ge 0} (|DE(d_1, ..., d_n)| - |DO(d_1, ..., d_n)|) \prod_{i=1}^{n} x_i^{d_i}$. □

Definition 3.15. For $D_1, D_2 \in DE(d_1, ..., d_n) \cup DO(d_1, ..., d_n)$, $D_1 \oplus D_2$ denotes the set of oriented edges in D_1 whose orientation in D_2 is in the opposite direction.

Lemma 3.16. *$D_1 \oplus D_2$ is an Eulerian subgraph of D_1.*

Proof. For each $v_i \in V$, let $d_1^+(v_i)$ and $d_2^+(v_i)$ denote the outdegree of v_i in the orientations D_1 and D_2, respectively. Similarly, let $d_1^-(v_i)$ and $d_2^-(v_i)$ denote the indegree of v_i in the orientations D_1 and D_2, respectively. Let $d_{1\oplus}^+(v_i)$ and $d_{2\oplus}^+(v_i)$ denote the numbers of directed edges from v_i that are the elements of $D_1 \oplus D_2$ in D_1 and D_2, respectively, and let $d_{1\ominus}^+(v_i)$ and $d_{2\ominus}^+(v_i)$ denote the numbers of directed edges from v_i that are not the elements of $D_1 \oplus D_2$ in D_1 and D_2, respectively. Similarly, let $d_{1\oplus}^-(v_i)$ and $d_{2\oplus}^-(v_i)$ denote the numbers of directed edges to v_i that are the elements of $D_1 \oplus D_2$ in D_1 and D_2, respectively, and let $d_{1\ominus}^-(v_i)$ and $d_{2\ominus}^-(v_i)$ denote the numbers of directed edges to v_i that are not the elements of $D_1 \oplus D_2$ in D_1 and D_2. Then,

$$\begin{aligned} d_1^+(v_i) &= d_{1\oplus}^+(v_i) + d_{1\ominus}^+(v_i) \\ d_1^-(v_i) &= d_{1\oplus}^-(v_i) + d_{1\ominus}^-(v_i) \\ d_2^+(v_i) &= d_{2\oplus}^+(v) + d_{2\ominus}^+(v_i) \\ d_2^-(v_i) &= d_{2\oplus}^-(v) + d_{2\ominus}^-(v_i) \end{aligned}$$

By Definition 3.15, $d_{1\oplus}^+(v) = d_{2\oplus}^-(v)$ and $d_{1\oplus}^-(v) = d_{2\oplus}^+(v)$. In addition, if an edge is not a member of $D_1 \oplus D_2$, it has the same orientation in D_1 and D_2. Thus,

$d^+_{1\ominus}(v) = d^+_{2\ominus}(v)$ and $d^-_{1\ominus}(v) = d^-_{2\ominus}(v)$. Since $d^+_1(v_i) = d_i = d^+_2(v_i)$,

$$d^+_{1\oplus}(v_i) + d^+_{1\ominus}(v_i) = d^+_{2\oplus}(v) + d^+_{2\ominus}(v_i)$$
$$d^+_{1\oplus}(v_i) + d^+_{1\ominus}(v_i) = d^-_{1\oplus}(v) + d^+_{2\ominus}(v_i)$$
$$d^+_{1\oplus}(v_i) = d^-_{1\oplus}(v)$$

Since $d^+_{1\oplus}(v_i)$ is the outdegree of v_i in the orientation $D_1 \oplus D_2$ and $d^-_{1\oplus}(v_i)$ is the indegree of v_i in the orientation $D_1 \oplus D_2$, $D_1 \oplus D_2$ is Eulerian. □

Lemma 3.17. *For a fixed sequence $d_1, ..., d_n$ and $D_1 \in DE(d_1, ..., d_n) \cup DO(d_1, ..., d_n)$, a map $T_{D_1} : DE(d_1, ..., d_n) \cup DO(d_1, ..., d_n) \mapsto EE(D_1) \cup EO(D_1)$ such that $T_{D_1}(D_2) = D_1 \oplus D_2$ is a bijection. Moreover, if D_1 is even, T maps even orientations to even Eulerian subgraphs and odd orientations to odd Eulerian subgraphs. Otherwise, it maps even orientations to odd Eulerian subgraphs and odd orientations to even Eulerian subgraphs. Thus,*

$$||DE(d_1, ..., d_n)| - |DO(d_1, ..., d_n)|| = |EE(D_1) - EO(D_2)|$$

Proof. For $A \in EE(D_1) \cup EO(D_1)$, let D_A be the orientation constructed by reversing the orientations for the edges of A in D_1. Since A is Eulerian, the outdegree of v_i in D_A is equal to that of D_1. Thus, $D_A \in DE(d_1, ..., d_n) \cup DO(d_1, ..., d_n)$ and $T_{D_1}(D_A) = A$. In addition, if $A \neq B$ for $A, B \in EE(D_1) \cup EO(D_1)$, D_A and D_B are clearly not equal. Thus, T_{D_1} is a bijection. For $D_1 \in DE(d_1, ..., d_n)$, if $D_2 \in DE(d_1, ..., d_n)$, the number of edges that are decreasing in D_1 and increasing in D_2 or vice-versa is even. Thus, $D_1 \oplus D_2$ is even. Other statements are proven by analogous arguments. □

Corollary 3.18. *Let D be an orientation of an undirected graph $G = (V, E)$ where $V = \{v_1, ...v_n\}$. For $1 \leq i \leq n$, let $d_i = d^+_D(v_i)$ the outdegree of v_i in D. then the absolute value of the coefficient of the monomial $\prod_{i=1}^n x_i^{d_i}$ in the expansion of $f_G = f_G(x_1, ..., x_n)$ is $|EE(D) - EO(D)|$. Particularly, if $EE(D) \neq EO(D)$, then the coefficient is not zero.*

Proof. This follows from Lemma 3.14 and Lemma 3.17. □

Theorem 3.19. *Let $D = (V, E)$ be a directed graph where $V = \{v_1, ..., v_n\}$. Let $f : V \mapsto \mathbb{N}$ be a function such that $f(v_i) = d^+_D(v_i) + 1$ for $1 \leq i \leq n$. If $EE(D) \neq EO(D)$, then D is f-choosable.*

Proof. The degree of f_G is $\sum_{1 \leq i \leq n} d^+_D(v_i)$. In addition, since $EE(D) \neq EO(D)$, the coefficient of $\prod_{1 \leq i \leq n} x_i^{d^+_D(v_i)}$ is not zero by Corollary 3.18. Hence, by Corollary 1.4, there exists an n-tuple $(c_1, ..., c_n) \in S_1 \times ... \times S_n$ such that $f_G(c_1, ..., c_n) \neq 0$. Thus, by letting $c(v_i) = c_i$ for $1 \leq i \leq n$, D is f-choosable. □

Corollary 3.20. *Let $G = (V, E)$ be an undirected graph where $V = \{v_1, ..., v_n\}$. If G has an orientation D satisfying $EE(D) \neq EO(D)$ in which the maximum outdegree is d, then G is $(d+1)$-colorable. Particularly, if the maximum outdegree is d and D contains no odd directed simple cycle, then G is $(d+1)$-colorable.*

Proof. Let $S(i) = \{1, ..., d+1\}$ for $1 \leq i \leq n$. By Theorem 3.19, G is $(d+1)$-colorable. By Lemma 3.10, an odd Eulerian subgraph ought to contain at least one odd directed simple cycle. Then, $EO(D) = 0$, and since $\emptyset \in EE(D)$, $EE(D) \geq 1$. Hence, $EE(D) \neq EO(D)$, and G is $(d+1)$-colorable. □

Definition 3.21. For a directed or an undirected graph $G = (V, E)$, an **independent set** is a set of vertices $I \subset V$ such that for each pair $v_i, v_j \in I$, $(v_i, v_j) \notin E$.

Corollary 3.22. *Let $G = (V, E)$ be an undirected graph where $V = \{v_1, ..., v_n\}$. If G has an orientation D satisfying $EE(D) \neq EO(D)$ in which the maximum outdegree is d, then G has an independent set of size at least $\lceil n/(d+1) \rceil$. Particularly, if the maximum outdegree is d and D contains no odd directed simple cycle, then G has an independent set of size at least $\lceil n/(d+1) \rceil$.*

Proof. By the Pigeonhole Principle and Corollary 3.20, there exists an $s \in \{1, ...d + 1\}$ such that $|\{v_i \in V : c(v_i) = s\}| \geq \lceil n/(d+1) \rceil$. Such subset of V is clearly independent. □

Corollary 3.23. *Let $G = (V, E)$ be an undirected graph where $V = \{v_1, ..., v_n\}$. Suppose G has an orientation D satisfying $EE(D) \neq EO(D)$ and let $d_1 \geq ... \geq d_n$ be the ordered sequence of outdegrees of the vertices in D. Then, for every k, $n > k \geq 0$, G has an independent set of at least $\lceil (n-k)/(d_{k+1} + 1) \rceil$.*

Proof. Without the loss of generality, assume that $d_D^+(v_i) = d_i$. Let $S(i) = \{1, ..., d_i + 1\}$ for $1 \leq i \leq n$. By Theorem 3.19, there exists a proper coloring $c : V \mapsto \mathbb{Z}$ such that $c(v_i) \in S(i)$ for $1 \leq i \leq n$. For each k such that $0 \leq k < n$, $c(v_{k+1}), ..., c(v_n) \in \{1, ..., d_{k+1} + 1\}$. Thus, by the Pigeonhole Principle, there exists an independent set of the size at least $\lceil (n-k)/(d_{k+1} + 1) \rceil$. □

Definition 3.24. For an undirected graph $G = (V, E)$, $L(G) = max(|E(H)|/|V(H)|)$, where $H = (V(H), E(H))$ ranges over all subgraphs $H \subset G$.

Definition 3.25. A **matching** M in a graph $G = (V, E)$ is a subset $M \subset E$ such that no two edges in M are incident on the same vertex, i.e. if $(w, x), (y, z) \in M$, then w, x, y, z are distinct. A **maximum matching** of G is a matching of a maximum size.

Definition 3.26. For M a matching in a graph $G = (V, E)$, a vertex $v \in V$ is **M-saturated** if there exists an edge in M incident on v. Otherwise, v is **M-unsaturated**.

Definition 3.27. For M a matching in a graph $G = (V, E)$, an **M-alternating path** is a path in G whose edges are alternately in M and outside of M. An M-alternating path whose end vertices are M-unsaturated is called an **M-augmenting path**.

Lemma 3.28. *If M is a maximum matching of a graph $G = (V, E)$, there can be no M-augmenting paths in G.*

Proof. Assume the contradiction that there exists P an M-augmenting path in G. Let $M' = M \cup (P \cap M^c) \setminus (P \cap M)$. Since P is an M-augmenting path, none of the vertices in P is an endpoint of the edges in $M \setminus P$. Thus, M' is a valid matching in G. Since $|M'| = |M| + 1$, M is not a maximum matching, which leads to a contradiction. □

Definition 3.29. A graph $G = (V, E)$ is a **bipartite graph** if $V = V_1 \cup V_2$ where V_1 and V_2 are disjoint independent sets of vertices. Such bipartite graph is denoted by $G = (V_1, V_2, E)$.

Definition 3.30. For a bipartite graph $G = (V_1, V_2, E)$, a matching M in G is called a **complete matching** if M saturates all the vertices in V_1.

Definition 3.31. For a bipartite graph $G = (V_1, V_2, E)$ and a subset of vertices $S \subset V_1$, the **neighborhood** $N(S)$ is

$$N(S) = \{v \in V_2 : \exists u \in S_1, (u, v) \in E\}$$

Lemma 3.32. *(Hall's Theorem) For a bipartite graph $G = (V_1, V_2, E)$ where $|V_1| \leq |V_2|$, G has a complete matching if and only if $|S| \leq |N(S)|$ for all $S \subset V_1$.*

Proof. If G has a complete matching M, for each $S \subset V_1$, every vertex $v \in S$ has a matching vertex in V_2 by M. Thus, $|S| \leq |N(S)|$.
Conversely, if $|S| \leq |N(S)|$ for all $S \subset V_1$, assume the contradiction that G has no complete matching. Let M be a maximum matching in G. Since M is not complete, there exists s an M-unsaturated vertex in V_1. Let Z be the set of vertices in G that are reachable from s by M-alternating paths. Let $S = Z \cap V_1$ and $T = Z \cap V_2$. Since there exist no M-augmenting paths in G by Lemma 3.28, every vertex in T has a matching vertex in $S \setminus \{s\}$ by M, and every vertex in $S \setminus \{s\}$ has a matching vertex in T by M. Hence, $|T| = |S| - 1$. Moreover, $T = N(S)$. Thus, $|S| > |N(S)| = |T| = |S| - 1$, which leads to a contradiction. □

Lemma 3.33. *An undirected graph $G = (V, E)$ has an orientation D in which the maximum outdegree is d if and only if $L(G) \leq d$.*

Proof. If there exists such an orientation D, then, for each subgraph $H \subset G$,

$$|E(H)| = \sum_{v \in V(H)} d_H^+(v) \leq \sum_{v \in V(H)} d_D^+(v) \leq d|V(H)|$$

Hence, $|E(H)|/|V(H)| \leq d$ for each $H \subset G$, and $L(G) \leq d$.
Conversely, suppose $L(G) \leq d$. Let F be a bipartite graph on the classes of vertices, $A = E$ and B, a union of d disjoint copies $V_1, ..., V_n$ of V. Each $e = (u, v) \in E = A$ is joined by edges in F to the d copies of u and the d copies of v. For $E' \subset E$ a set of edges of a subgraph H of G whose vertices are the endpoints of the edges in E', in F, $|N(E')| = d|V(H)|$. By Definition 3.24, $|E'|/|V(H)| \leq L(G) \leq d$. Hence, $|E'| \leq d|V(H)| = |N(E')|$. By *Hall's theorem*, F has a complete matching M. By orienting each edge in E from its matching vertex by M, the resulting orientation D has the maximum outdegree d. □

Theorem 3.34. *Every bipartite graph $G = (V_1, V_2, E)$ is $(\lceil L(G) \rceil + 1)$-choosable.*

Proof. For $d = \lceil L(G) \rceil$, there exists an orientation D of G in which the maximum outdegree is at most $\lceil L(G) \rceil$. Since bipartite graphs have no odd cycles, $EE(D) \neq EO(D)$, and by Theorem 3.19, G is $(\lceil L(G) \rceil + 1)$-choosable. □

Remark 3.35. The assumption that G is bipartite is necessary.

Proof. For $G = K_n$ a complete graph on n vertices, $L(G) = \frac{n-1}{2}$, but G is clearly not k-choosable for $k < n$. □

Remark 3.36. For every k, there exists a bipartite graph G such that $L(G) \leq k$ and G is not k-choosable. Hence, Theorem 3.34 is sharp.

Proof. Let G be a complete bipartite graph on the classes of vertices, A and B, where $|A| = k^k$ and $|B| = k$. For H the induced graph on $A' \cup B'$ where $A' \in A$ and $B' \in B$, $|E(H)| = \sum_{a \in A'} d_H(a) \leq k|A'| \leq k|V(H)|$. Thus, $L(G) \leq k$. For $B = \{b_1, ..., b_n\}$, let $S(b_i) = \{(i-1)k+1, (i-1)k+2, ..., ik\}$ for each $1 \leq i \leq n$. For $A = \{a_{i_1,...,i_k} : 1 \leq i_j \leq k \text{ for } 1 \leq j \leq k\}$, let

$$S(a_{i_1,...,i_k}) = \{i_1, k+i_2, ..., (k-1)k + i_k\}$$

Suppose that there exists a proper coloring $c : A \cup B \mapsto \mathbb{Z}$ such that $c(v) \in S(v)$ for all $v \in A \cup B$. Then, there exists an k-tuple $(c_1, ..., c_k)$ such that $1 \leq c_1, ..., c_k \leq k$ and $c(b_i) = (i-1)k + c_i$ for $1 \leq i \leq k$. However, $a_{c_1,...,c_k}$ has no value in $S(a_{i_1,...,i_k})$ which is distint from the colors of its neighbors. Hence, c is not a proper coloring, which leads to a contradiction. □

4. Cube Covering by Hyperplanes

Theorem 4.1. *Let $H_1, ..., H_m$ be a family of hyperplanes in R^n that cover all the vertices of the unit cube $\{0,1\}^n$ but one. Then, $m \geq n$.*

Proof. Without the loss of generality, assume that the uncovered vertex is $\mathbf{0} = (0, ..., 0)$. For each $1 \leq i \leq m$, H_i is defined by the equation $a_i \cdot x = b_i$ where $a_i = (a_{i,1}, ..., a_{i,n})$ and $x = (x_1, ..., x_n)$. Since H_i does not cover the origin for each $1 \leq i \leq m$, $b_i \neq 0$ for $1 \leq i \leq m$. Assume the contradiction that $m < n$, and consider the polynomial

$$F(x) = (-1)^{n+m} \prod_{j=1}^{m} b_j \prod_{i=1}^{n} (x_i - 1) - \prod_{i=1}^{m} [(a_i, x) - b_i]$$

The degree of F is $n = \sum_{i=1}^{n} 1$, and the coefficient of $\prod_{i=1}^{n} x_i$ is $(-1)^{n+m} \prod_{j=1}^{m} b_j \neq 0$. Let $S_i = \{0,1\}$ for all $1 \leq i \leq n$. Since $|S_i| = 2 > 1$ for all $1 \leq i \leq n$, by Corollary 1.4, there exists $c = (c_1, ..., c_n) \in \{0,1\}^n$ such that $F(c) \neq 0$. Since

$$F(\mathbf{0}) = (-1)^{n+m} \prod_{j=1}^{m} b_j (-1)^n - \prod_{i=1}^{m} (-b_i) = (-1)^m \prod_{j=1}^{m} b_j - (-1)^m \prod_{i=1}^{m} (b_i) = 0,$$

$c \neq \mathbf{0}$. Since c is covered by some H_i, $(a_i, x) - b_i = 0$ for some i. Then, $F(c) = 0 - 0 = 0$, which leads to a contradiction. □

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