BROUWER'S FIXED POINT THEOREM

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ABSTRACT. In this paper we prove Brouwer's Fixed Point Theorem, which states that for any continuous transformation $f : D \to D$ of any figure topologically equivalent to the closed disk, there exists a point p such that $f(p) = p$.

CONTENTS

1. INTRODUCTION AND DEFINITION OF BASIC TERMS

This paper presents the proof of Brouwer's Fixed Point Theorem, which states that for any continuous transformation $f : D \to D$ of the closed disk $D = \{p \in \mathbb{R}^2 : ||p|| \leq 1\}$, there exists a point p such that $f(p) = p$. In fact, we prove that this is true for the *cell*, that is, any figure homeomorphic to $\mathcal{D}$. Before we prove this theorem, we present definitions.

Definition 1.1. Let f be a continuous transformation of a set D into itself. A *fixed point* of f is a point p such that $f(p) = p$.

Definition 1.2. If, for every continuous transformation $f : D \to D$, there exists a point p such that $f(p) = p$, then D has the *fixed point property.*

These definitions allow us to restate Brouwer's Fixed Point Theorem in a much simpler manner:

Theorem 1.3 (Brouwer's Fixed Point Theorem)**.** *Cells have the fixed point property.*

We want to show that for any continuous transformation $f : D \to D$ of any figure topologically equivalent to the closed disk, there exists a point p such that $f(p) = p$. In order to do so, we shall consider a vector field V which represents f. We will find a point p such that $V(p) = 0$, which we show is equivalent to finding a fixed point. To find this point p, we shall find arbitrarily small triangles whose vertices can be used to make three sequences which all have p as their limit point.

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Before we delve into the proof of Brouwer's Fixed Point Theorem, we will first need to prove Sperner's lemma, which proves the existence of arbitrarily small triangles with vertices whose vectors are in three different quadrants of the plane.

2. Constructing a Sperner Labeling and Proving Sperner's Lemma

As a model for our cell, we shall use a triangle since the triangle is homeomorphic to the disk. Specifically, we shall consider the triangle in Figure A. In order to understand and prove Sperner's lemma, we will first need to consider triangulations.

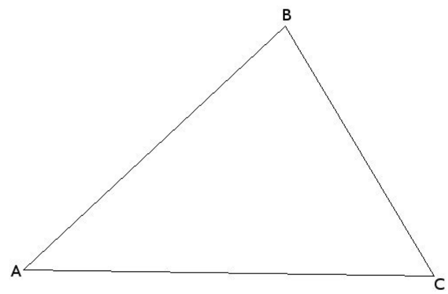

Figure A: The original triangle T

Definition 2.1. A *triangulation* is a division of a triangle T into a finite number of triangles such that each edge of a subtriangle on the boundary edge of T is the edge of just one triangle of the division, and each edge in the interior of T is an edge of exactly two triangles of the subdivision.

The *barycentric subdivision* is a triangulation of a triangle T. To construct the barycentric subdivision, first place a vertex at the center of the face of T. Then, draw a line from each of the original vertices of T through this new center point until the opposite edge is reached. This results in six subtriangles, each with smaller sides than the original triangle. Note that each triangle created by this subdivision can have the barycentric subdivision performed upon it since each is a triangle. Thus, we can achieve arbitrarily small triangles using this method. The barycentric subdivision of the triangle in Figure A can be observed in Figure B.

Consider a triangle T with vertices labelled A, B, and C. Take a triangulation of T. For each vertex of a subtriangle on an edge of the original triangle T, label that vertex with a letter from one of the two vertices which make up the edge in the original triangle. For each vertex in the interior of T, label it arbitrarily A, B, or C. This system of labeling is a *Sperner labeling*, as illustrated in Figure B.

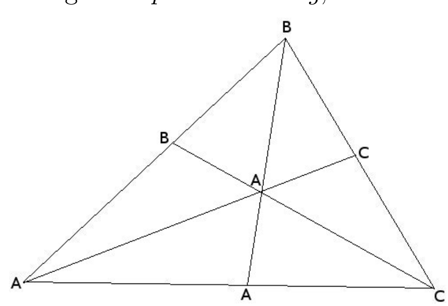

Figure B: The barycentric subdivision of T with a Sperner labeling

Definition 2.2. A *complete triangle* in a Sperner labeling of a triangulation is a subtriangle with vertices labeled A, B, and C, as illustrated by the shaded area in Figure C.

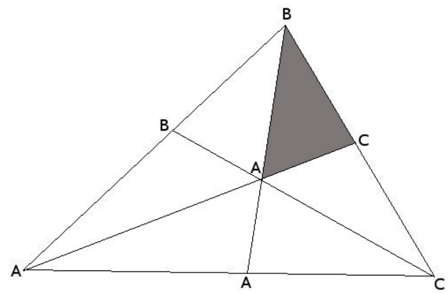

Figure C: A Sperner labeling of T with complete triangles shaded in

Lemma 2.3 (Sperner's Lemma)**.** *At least one subtriangle in a Sperner labeling receives all three labels: A, B, and C.*

We will actually show that there are an odd number of complete subtriangles in a Sperner labeling. This statement implies the statement of the lemma since the number of triangles must be odd and positive. The smallest positive odd number is one, so there must be at least one subtriangle which receives all three labels.

Proof. Let T be a triangle with a Sperner labeling. To determine the number of edges with vertices AB, we will first consider a single edge. Then, we will expand this process in order to find the complete subtriangles of the triangle T. Consider the edge AB of the original triangle T. Note that all the vertices of the triangulation on this edge are labeled A or B.

Let b be the number of line segments with endpoints A and B and let a be the number of line segments with endpoints that are only labeled A. Each segment counted in a has two vertices labeled A and each segment in b has one vertex labeled A, so combined we get $2a+b$ vertices labeled A. Note that we have double-counted the number of A's in the interior of the original edge AB, while we have only counted the A vertex of the original triangle once. Let c be the number of these interior vertices labeled A. Then,

$$2a + b = 2c + 1.$$

Since $2c + 1$ is odd but $2a$ is even, we see that b must be odd.

Now, we will take this process and expand it to the original triangle T. Let E be the set of complete triangles and e be the cardinality of E. Let F be the set of triangles with vertices labeled A, B, and A or B, A, and B and f be the cardinality of F. Then there are two AB edges for each triangle in F and one AB edge for each triangle in E. Together, E and F contain all the edges labeled AB. Consider the sum $2f + e$.

Let g be the number of interior edges labeled AB and h be the number of edges with vertices labeled AB on the edges of the original triangle T. The sum $2f + e$ counts all edges labeled AB on the edges of the original triangle T once and all interior edges labeled AB twice since each interior edge belongs to two triangles. Thus,

$$2f + e = 2g + h.$$

Note here that $h = b$ since both are the number of edges labeled AB on an edge of T. Since b is odd, as shown above, h is also odd. Therefore, e must be odd, since $2g + h$ is odd, and $2f$ is even. Hence, the number of complete triangles is odd. This completes our proof of Sperner's Lemma. $\square$

3. Proof of Brouwer's Fixed Point Theorem

To show Brouwer's Fixed Point Theorem, we must show that for any continuous transformation $f : D \to D$ of a cell, there exists a point p such that $f(p) = p$. In order to do so, we shall construct a vector field V which represents f. We will find a point p such that $V(p) = 0$, which is equivalent to finding a fixed point since $V(P) = 0$ means that the point p does not change position in any way. To find this point p, we show there exist arbitrarily small triangles with vertices whose vectors are in three different quadrants of the plane. Then, we can create sequences of these vertices that will all share a limit point, which will be p.

Proof. Let $\mathcal{D}$ be a cell. Since any cell is homeomorphic to the triangle illustrated in Figure A, we assume that $\mathcal{D}$ is a triangle.

Let f be a continuous transformation of $\mathcal{D}$ into itself and let V be the vector field defined by $V(p) = f(p) - p$ for each point $p \in \mathcal{D}$. In terms of our vector field V, we want to find a point q such that $V(q) = 0$. This will be a fixed point for f.

For the purposes of this proof, we will consider vectors to be of three forms: south, northeast, and northwest. Consider the vector $V(p)$ for any $p \in \mathcal{D}$ with its tail at the origin. If the vector is below the x-axis it is considered south. If it is in the first quadrant, it is a northeast vector. Finally, if the vector is in the second quadrant, it is considered northwest. To ensure that every non-zero vector points one of these directions, we define a vector on the positive x-axis or the positive y-axis to be northeast and vectors on the negative x-axis to be northwest. For simplification, label northeast vectors A, northwest vectors C, and south vectors B.

First, note that the labeling of the vertices, as in the example in Figure A, corresponds to the general direction of the vectors on these points. For example, the vertex A must have a vector going northeast (unless the vector is the zero vector, in which case we are done) since it must be moving somewhere, and $f(A)$ must also stay in the triangle, so $V(A)$ cannot go south or northwest. Now, we take a barycentric subdivision of $\mathcal{D}$. Each vertex has A, B, or C associated with it from the vector field, so this is a Sperner labeling of the triangle. The vertices on the edges of the original triangle also have the correct vector labeling to produce a Sperner labeling. Take, for example, the edge AB on the original triangle. Any vertex on this edge cannot be going northeast (which is the C labeling) since that would move the point out of the triangle. Therefore, the vector must be northwest or south, so it will be labeled A or B, respectively.

Now, by applying Sperner's lemma, we know there exists at least one complete subtriangle S. Barycentric subdivide S and, as above, use the vectors of the vertices created to label the vertices A, B, and C. We can again use Sperner's lemma to find a new complete triangle S' in the subdivision. Note that S' is smaller than S since it is in the barycentric subdivision of S. S' is also labeled A, B, and C, so we can repeat this process on S'. Thus, we can find arbitrarily small complete triangles since we can repeat this process to get subtriangles that are of arbitrarily small side length. We will now show that there is a point q in $\mathcal{D}$ where $V(q) = 0$, which will complete our proof of Brouwer's Fixed Point Theorem.

$\mathcal{D}$ contains a sequence $\mathcal{T}$ of complete triangles with vertices P_n, Q_n, and R_n labeled A, B, and C respectively. For this sequence $\mathcal{T}$, we include triangles of the same subdivision only once. Thus, $\mathcal{T}$ is infinite and can be ordered such that each triangle in the sequence is smaller than the preceding one and is part of the subdivision of the previous one. Since the triangles of $\mathcal{T}$ get arbitrarily small, the length of the sides of these triangles tends towards zero. Let $\mathcal{P} = \{P_n\}$, $\mathcal{Q} = \{Q_n\}$, and $\mathcal{R} = \{R_n\}$. Note that since the cell $\mathcal{D}$ is compact, all sequences in $\mathcal{D}$ have limit points. Let q be the limit point of $\mathcal{P}$. As n approaches infinity, the sides of the triangles defined by P_n, Q_n, and R_n become arbitrarily close to each other, so q is also a limit point of $\mathcal{Q}$ and $\mathcal{R}$.

By the continuity of f, the vector $V(q)$ is a limit point of the sequence of vectors $V(\mathcal{P})$ as well as the sequences $V(\mathcal{Q})$ and $V(\mathcal{R})$. Next, we shall examine the closure of all northeast vectors, northwest vectors, and all south vectors, so as to determine where $V(q)$ is.

First, we will show that the closure of all northeast vectors is the first quadrant and the positive y-axis and positive x-axis. Assume this is not true. Then, there exists a point p and sequence $\mathcal{S}$ such that p is not in the first quadrant nor on the y-axis nor on the x-axis but every point of $\mathcal{S}$ is, and p is a limit point of $\mathcal{S}$. A similar argument shows that y cannot be negative either. Let the coordinates of p be (x, y). Since p is not in the first quadrant, either x or y is negative. Assume $x < 0$. Let B be a ball in the plane with radius $(-x)/2$ and centered at (x, y). All points in B have a negative x coordinate, thus no point of $\mathcal{S}$ is in this ball, so p is not a limit point of $\mathcal{S}$. But, this is a contradiction, since we assumed p is a limit point of $\mathcal{S}$. So, x cannot be negative. A similar proof shows that y cannot be negative either. Thus, the closure of the set of all northeast vectors is the set of all northeast vectors and the positive x- and y-axis.

Similarly, the closure of the set of northwest vectors is the set of northwest vectors and the positive y-axis and the negative x-axis. Also, the closure of the set of south vectors is the set of south vectors and the entire x-axis. Given these closures, the only intersections of the closure of all south vectors, all northeast vectors, and all northwest vectors is $\{0\}$. Since $V(q)$ is a limit point of $V(\mathcal{P})$, $V(q)$ is in the closure of the set of northwest vectors. Similarly, $V(q)$ is in the closure of the set of northeast vectors and the set of south vectors. Therefore, $V(q)$ is in the intersection of these three closures. So, $V(q)$ must be zero. Now, we have found a point q such that $V(q)$ is zero, so $q = f(q)$. Hence, q is a fixed point. This completes our proof of Brouwer's Fixed Point Theorem. $\square$

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References

[1] Michael Henle. A Combinatorial Introduction To Topology. Dover Publications Inc. 1979.