GENERAL ABSTRACT NONSENSE

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ABSTRACT. In this paper, we seek to understand limits, a unifying notion that brings together the ideas of pullbacks, products, and equalizers. To do this, we will build up the basic framework of category theory, starting from the definition of a category. With this done, we will define pullbacks, products, and equalizers, and we will close this paper by showing two results: first, that having products and equalizers is equivalent to having pullbacks and a terminal object, and second, that having all finite limits is equivalent to having products and equalizers of all cardinalities.

CONTENTS

1. THE BASICS

1.1. Definitions. Category theory grew out of a generalization of abstract algebra. Thus, most of the definitions we will see are analogous to ones in algebra. We start by defining a category.

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Definition 1.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data

- A collection of objects A, B, C, denoted $\mathcal{C}_0$.
- A collection of arrows f, g, h, denoted $\mathcal{C}_1$
- For every arrow $f \in \mathcal{C}_1$, there exist objects $\mathrm{dom}(f), \mathrm{cod}(f) \in \mathcal{C}_0$, and we write $f : \mathrm{dom}(f) \to \mathrm{cod}(f)$.
- Given arrows $f : A \to B$, $g : B \to C$, i.e. $\mathrm{cod}(f) = \mathrm{dom}(g)$, there exists an arrow $g \circ f \in \mathcal{C}_1$ such that $g \circ f : A \to C$.
- For all objects A, there exists an identity arrow $\mathrm{Id}_A : A \to A$

These data are required to satisfy the following two properties:

Associativity: For all arrows $f : A \to B$, $g : B \to C$, and $h : C \to D$, $h \circ (g \circ f) = (h \circ g) \circ f$, i.e. the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}$$

Unit: For all arrows $f : A \to B$, $f \circ \mathrm{Id}_A = f = \mathrm{Id}_B \circ f$, i.e. the following commutes:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Id}_A"] \arrow[d, "f"'] \arrow[dr, "f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{Id}_B"'] & B
\end{tikzcd}$$

Remark 1.2. We denote the collection of all arrows from an object A to an object B in a category $\mathcal{C}$ as $\mathcal{C}(A, B)$.

Remark 1.3. Assuming that an object A has two identity arrows, by the unit property we have that $\mathrm{Id}'_A = \mathrm{Id}'_A \circ \mathrm{Id}_A = \mathrm{Id}_A$, which shows that the identity arrow must be unique.

Definition 1.4 (Isomorphism)**.** An arrow $f : A \to B$ is an *iso(morphism)* if there exists an arrow $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$.

In algebra, we have structure-preserving maps, like group homomorphisms for groups and linear transformations for vector spaces. Similarly, we have structure-preserving maps for categories.

Definition 1.5 (Functor)**.** A *functor* is a map $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ that sends objects to objects (i.e. $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$) and arrows to arrows (i.e. $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$) such that

- $F(f : A \to B) = Ff : FA \to FB$
- $F(g \circ f) = Fg \circ Ff$
- $F(\mathrm{Id}_A) = \mathrm{Id}_{FA}$

It's not hard to see that for each category there is a canonical identity functor, $\mathrm{Id}_{\mathcal{C}}$ which takes every arrow and object to itself and thus trivially satisfies the structure-preserving properties of a functor. It's helpful to seem some examples of categories and functors at this point.

Example 1.6.

(1) The category **Sets** is the category which has as its objects all sets and $\mathbf{Sets}(X, Y) = Y^X$. Composition of arrows is simply the composition of functions, and the identity arrow for any set is simply its identity function.
(2) The category **Grp** is the category which has as its objects all groups and $\mathbf{Grp}(G, H)$ is the collection of all group homomorphisms from G to H fora any two groups.
(3) Two ubiquitous functors in category theory are the identity functor, describes above, and the forgetful functor, U. Generally, the forgetful functor takes a category of structured sets and their functions (e.g homomorphisms) to **Sets** and sends each object in that category to its underlying set and each "homomorphism" to its underlying function of sets.
(4) The category **Cat** is the category where the objects are all locally small categories and the arrows are all the functors between them.
(5) Consider a monoid M. Then M is an single object category where the elements are arrows from the object to itself and the composition of arrows is the product of the corresponding elements, i.e. $mn = m \circ n$, and the unit element u_M is the identity arrow Id_*. Further, any monoid homomorphism $h : M \to N$ is a functor between their respective categories.
(6) Similarly, since a group is a monoid with inverse elements, then a group is equivalent to a single element category where every arrow is an isomorphism.
(7) A poset can also be considered a be considered a category where the elements of the poset are the objects of the category and there is a unique arrow $f : a \to b$ if $a \leq b$. This should not be confused with **Pos**, the category of posets where the objects are posets and the arrows are monotonic functions.
(8) Let an ordered pair (A, a) be called a pointed set, where A is a set and $a \in A$. Then we can consider a category of pointed sets, where the objects are pointed sets and an arrow $f : (A, a) \to (B, b)$ is a function $f : A \to B$ such that $f(a) = b$. We denote this category $\mathbf{Sets}_*$
(9) For a collection of sets, $\mathbf{Rel}_0$, define an arrow $f : A \to B$ as a relation from A to B, i.e. such that $f \subseteq A \times B$. The identity relation on a set A is $\mathrm{Id}_A = \{\langle a, a\rangle \in A \times A | a \in A\}$. Composition for relations $f \subset A \times B$ and $g \subseteq B \times C$ is defined as $g \circ f = \{\langle a, c\rangle \in A \times C | \exists\, b \in B$ such that $\langle a, b\rangle \in f$ and $\langle b, c\rangle \in g\}$. If we denote the collection of relations among sets in $\mathbf{Rel}_0$ as $\mathbf{Rel}_1$, then $\mathbf{Rel}_0$ and $\mathbf{Rel}_1$ form a category, **Rel**.

The notion of a category is rather flexible and can come in a wide variety of flavors. Despite the very abstract feel of most categories, it can be shown that any category is isomorphic to one in which the objects are sets and the arrows are functions.

Theorem 1.7. *Every category $\mathcal{C}$ is isomorphic to one in which the objects are sets and the arrows are functions*

The proof follows from something akin to the Cayley representation theorem of groups. For a given category $\mathcal{C}$, define a collection of sets $\overline{C} = \{$ arrows $f \in \mathcal{C}_1 | \mathrm{cod}(f) = C\}$ for every $C \in \mathcal{C}_0$. Then for all arrows $f : C \to D$, define a function

$\overline{f} : \overline{C} \to \overline{D}$ such that $\overline{f}(h) = f \circ h$. One can check that this collection of sets and functions is a category, where $\overline{\mathrm{Id}_C} = \mathrm{Id}_{\overline{C}}$.

1.2. **Constructions.** Having established what categories are and seen some examples of them in nature, we can discuss some categories we can construct from ones we already have.

Definition 1.8 (Dual Category)**.** For a given category $\mathcal{C}$, define the *dual category* of $\mathcal{C}$, denoted $\mathcal{C}^{\mathbf{op}}$, such that $\mathcal{C}_0 = \mathcal{C}_0^{\mathbf{op}}$ and $(\overline{f} : \overline{A} \to \overline{B}) \in \mathcal{C}_1^{\mathbf{op}}$ if and only if $(f : B \to A) \in \mathcal{C}_1$.

Intuitively, the dual category is exactly the same as $\mathcal{C}$, except all the arrows have been formally "turned around". What this means is that for arrows $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, $\overline{g \circ f} = \overline{f} \circ \overline{g}$ in $\mathcal{C}^{\mathbf{op}}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A & B \arrow[l, "\overline{f}"'] \\ & C \arrow[u, "\overline{g}"'] \arrow[ul, "\overline{f}\circ\overline{g}"] \end{tikzcd}$$

Definition 1.9 (Product Category)**.** For categories $\mathcal{C}$ and $\mathcal{D}$, we define $\mathcal{C} \times \mathcal{D}$, the *product category* of $\mathcal{C}$ and $\mathcal{D}$, as the category with the following data:

- $(\mathcal{C} \times \mathcal{D})_0 = \{(C, D) | C \in \mathcal{C}_0, D \in \mathcal{D}_0\}$
- $(\mathcal{C} \times \mathcal{D})_1 = \{(f, g) : (C, D) \to (C', D') | f \in \mathcal{C}(C, D), g \in \mathcal{D}(C', D')\}$
- composition is defined component-wise: $(f, g) \circ (f', g') = (f \circ f', g \circ g')$
- $\mathrm{Id}_{(C,D)} = (\mathrm{Id}_C, \mathrm{Id}_D)$

There are two canonical projection functors here:

$$\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$$

defined such that $\pi_1(C, D) = C$ and $\pi_1(f, g) = f$, with π_2 defined analogously.

Definition 1.10 (Arrow Cateogory)**.** Given a category $\mathcal{C}$, define $\mathcal{C}^{\to}$ such that:

- $\mathcal{C}_0^{\to} = \mathcal{C}_1$
- An arrow $g = \langle g_1, g_2 \rangle : (f : A \to B) \to (f' : A' \to B')$ in $\mathcal{C}^{\to}$ is a pair of arrows $g_1 : A \to A'$ and $g_2 : B \to B'$ such that $g_2 \circ f = f' \circ g_1$. Diagrammatically:

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g_2"'] & B \end{tikzcd}$$

- $\mathrm{Id}_f = \langle \mathrm{Id}_A, \mathrm{Id}_B \rangle$ for $f : A \to B$.
- For arrows $h = \langle h_1, h_2 \rangle$ and $g = \langle g_1, g_2 \rangle$ $g \circ h = \langle g_1 \circ h_1, g_2 \circ h_2 \rangle$, which Diagrammatically looks like:

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "f"'] & A' \arrow[r, "g_1"] \arrow[d, "f'"] & A'' \arrow[d, "f''"] \\ B \arrow[r, "h_2"] & B' \arrow[r, "g_2"] & B'' \end{tikzcd}$$

This construction has two canonical functors as well: **dom** and **cod**.

Definition 1.11 (Slice Category)**.** The *slice category* of a category $\mathcal{C}$ over an object C in $\mathcal{C}$, denoted $\mathcal{C}/C$, is given by the following data:

- $(\mathcal{C}/C)_0$ is the collection of arrows in $\mathcal{C}$ with $\mathrm{cod}(f) = C$.
- An arrow in $\mathcal{C}/C$, $f : (x : X \to C) \to (x' : X' \to C)$, is an arrow $f : X \to X'$ in $\mathcal{C}$ such that $x'f = x$, that is, the following commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "x"'] & X' \arrow[d, "x'"] \\ & C \end{tikzcd}$$

- The identity arrow for an object $x : X \to C$ is simply the arrow Id_X, which makes the above diagram commute trivially: $x\mathrm{Id}_X = x$.
- Let x, x', and x'' be objects in $\mathcal{C}/C$, and let $f \in \mathcal{C}/C(x, x')$ and $g \in \mathcal{C}/C(x', x'')$. Then we define composition so that the following commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "x"'] & X' \arrow[r, "g"] \arrow[d, "x'"] & X'' \arrow[dl, "x''"] \\ & C & \end{tikzcd}$$

  Thus, $g \circ f \in \mathcal{C}/C(x, x'')$ is the composite arrow $g \circ f : X \to X''$ in $\mathcal{C}$ such that $x''(gf) = x$.

1.3. **Duality.** The definition includes objects and arrows and four operations (dom, cod, Id_A, and $g \circ f$) that satisfy the following:

$$\mathrm{dom}(\mathrm{Id}_A) = \mathrm{cod}(\mathrm{Id}_A) = A$$

$$f \circ \mathrm{Id}_{\mathrm{dom}(f)} = f = \mathrm{Id}_{\mathrm{cod}(f)} \circ f$$

$$\mathrm{dom}(g \circ f) = \mathrm{dom}(f), \quad \mathrm{cod}(g \circ f) = \mathrm{cod}(g)$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Note that the operation "$g \circ f$" is only defined where $\mathrm{dom}(g) = \mathrm{cod}(f)$. So for any statement in category theory Σ, we can form the "dual statement" Σ^* by replacing $g \circ f$ with $f \circ g$, cod for dom, and dom for cod (i.e. inverting all the arrows and orders of composition. Σ^* will also be a well-formed sentence. Thus, supposing that a well-formed sentence Σ entails another well-formed sentence Δ, by inversion of arrows we find that Σ^* entails Δ^*. Knowing this, we can also see that the axioms of category theory are self-dual, i.e. $\mathbf{CT} = \mathbf{CT}^*$. Combining these two ideas, we arrive at what is known as the Duality Principle of category theory.

Proposition 1.12 (Duality)**.** *For any statement Σ in category theory that follows by the axioms of category theory, the dual of that statement Σ^*, also follows from the axioms.*

Proof. Suppose **CT** entails a statement Σ. Then as demonstrated above, $\mathbf{CT}^*$ entails Σ^*. But since $\mathbf{CT} = \mathbf{CT}^*$, then **CT** entails Σ^*. $\square$

Along these lines, since $\mathcal{C}^{\mathbf{op}}$ is $\mathcal{C}$ with inverted arrows, then any true statement Σ in $\mathcal{C}$ necessarily implies the truth of Σ^* in $\mathcal{C}^{\mathbf{op}}$. Combining this with the observation that $(\mathcal{C}^{\mathbf{op}})^{\mathbf{op}} = \mathcal{C}$, we arrive at a slightly more formal version of the duality principle.

Theorem 1.13 (Duality 2.0)**.** *If Σ is a well-formed sentence that holds in all categories $\mathcal{C}$, then so is/does Σ^*.*

Proof. Suppose Σ is a statement in the language of category theory that holds in all categories. Given a category $\mathcal{C}$, Σ necessarily holds in $\mathcal{C}^{\mathbf{op}}$ since it holds in all categories. This implies that Σ^* holds in $(\mathcal{C}^{\mathbf{op}})^{\mathbf{op}}$. Since $(\mathcal{C}^{\mathbf{op}})^{\mathbf{op}} = \mathcal{C}$, then Σ^* holds in $\mathcal{C}$. So Σ^* holds in all categories. $\square$

Many times we will come across pairs of notions which are dual to each other, and it will suffice to show something about just one of them because the dual statement follows from the duality principle. As an example of this, we can add to our list of constructed categories the coslice category of a category $\mathcal{C}$ under an object C, denoted $C/\mathcal{C}$. By reversing the arrows in the slice category $\mathcal{C}/C$, we see that the objects in $C/\mathcal{C}$ are the arrows in $\mathcal{C}$ whose domain is C. The rest follows starting from this and keeping the duality principle in mind.

Remark 1.14. It is common practice to use the prefix co- for the dual of an existing notion, e.g. coslice is the dual of slice. Along these lines, we will encounter coproducts, coequalizers, colimits, and pushouts.

1.4. **Abstract Structures.** There are several types of structure we can talk about in the context of a category.

Definition 1.15 (Monomorphism and Epimorphism)**.** An arrow $f : A \to B$ is a *mono(morphism)*, or is said to be monic, if for any set of parallel arrows $g, h : X \to A$ such that $fg = fh$, i.e. $X \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B$ commutes, we have that $g = h$.

An arrow $f : A \to B$ is an *epi(morphism)*, or is said to be epic, if for any set of parallel arrows $i, j : B \to Z$ such that $if = jf$, i.e. $A \xrightarrow{f} B \overset{i}{\underset{j}{\rightrightarrows}} Z$ commutes, we have that $i = j$.

Note that the notion of a monomorphism is dual to the notion of epimorphism. Despite their abstract nature, these structures do have some very concrete examples in nature.

Proposition 1.16. *A function $f : A \to B$ (arrow in* **Sets***) is monic if and only if it's injective.*

Proof. Suppose f is monic. Let a and a' be distinct points of A. And let $\{*\}$ be any singleton set. Then we have two functions $\overline{a}, \overline{a'} : \{*\} \to A$ such that $\overline{a}(*) = a$ and $\overline{a'}(*) = a'$. Since f is monic and since $\overline{a} \neq \overline{a'}$ then $f\overline{a} \neq f\overline{a'}$. Since $f(\overline{a}(*)) = f(a)$ and $f(\overline{a'}(*)) = f(a')$, then $f(a) \neq f(a')$.

Suppose f is 1-to-1 and $g, h : C \to A$ are parallel arrows into A such that $g \neq h$. Then there exists $c \in C$ such that $g(c) \neq h(c)$. Since f is 1-to-1, then $f(g(c)) \neq f(h(c))$, which implies that $fg \neq fh$. By contrapositive, $fg = fh$ implies $g = h$. $\square$

Proposition 1.17. *A function $f : A \to B$ is epic if and only if it's onto.*

Proof. Consider a surjective map $f : X \to Y$, and let $g, h : Y \to Z$ be parallel arrows such that $gf = hf$. Consider $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. By construction, $g(f(x)) = h(f(x))$, which means that $g(y) = h(y)$. Since this holds for any $y \in Y$, $g = h$.

Let $f : X \to Y$ be epic. Suppose f is not onto. Then $\mathrm{Range}(f) \subsetneq Y$. Let $g : Y \to \{0.1\}$ be defined such that $g(y) = 1$ if $y \in \mathrm{Range}(f)$ and $g(y) = 0$ otherwise. Further, define $h : Y \to \{0, 1\}$ such that for all y, $h(y) = 1$. Then $gf = hf$. However, $h \neq g$ since for all $y \notin \mathrm{Range}(f)$ $g(y) = 0$, and we know there is at least one such y. This contradicts f being epic. Thus, f is onto. □

Proposition 1.18. *In a fixed poset category P, every arrow $p \leq q$ is both monic and epic.*

Proof. By construction, a poset category has at most one arrow between any two objects. Thus, any parallel arrows must be equal. This makes all arrows trivially monic and epic. □

Proposition 1.19. *Every iso is both monic and epic.*

Proof. Let $f : X \to Y$ be an iso and suppose that $g : Y \to X$ is the arrow such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. Let $x, y : A \to X$ be parallel arrows such that $fx = fy$ and let $i, j : X \to B$ be parallel arrows such that $ig = jg$. Diagrammatically:

$$\begin{tikzcd}
A \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & X \arrow[r, "f"] \arrow[dr, "\mathrm{Id}_X"'] & Y \arrow[d, "g"] \\
& & X \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & B
\end{tikzcd}$$

If $fx = fy$ then $g(fx) = g(fy)$. Since $gf = \mathrm{Id}_X$ then $x = y$, so f is monic. Similarly, if $ig = jg$ then $(ig)f = (jg)f$. Since $gf = \mathrm{Id}_Y$ then $i = j$, so g is monic. Reversing the situation and applying the same logic will show that g is monic and f is epic. Thus, every iso is both monic and epic. □

It should be noted that arrows that are both monic and epic are not necessarily iso. Let $(\mathbb{R}, \leq)$ denote $\mathbb{R}$ with the standard order topology, let $(\mathbb{R}, \mathcal{F})$ denote $\mathbb{R}$ with the full topology, i.e. every subset of $\mathbb{R}$ is open, and consider both as objects in the category **Top** of topological spaces and continuous functions. Then for any topological space $(X, \mathcal{A})$, any function $f : (\mathbb{R}, \mathcal{F}) \to (X, \mathcal{A})$ is continuous since for any open subset $V \subseteq X$ $f^{-1}(V) \subset \mathbb{R}$ is necessarily open.

Consider the function $h : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \leq)$ defined such that for all $x \in \mathbb{R}$ $h(x) = x$. Since the underlying set function of this map is $|h| = \mathrm{Id}_{\mathbb{R}}$, it's fairly easy to check that its both monic, epic, and bijective. However it is not an isomorphism. In **Top**, an isomorphism is a continuous function with a continuous inverse, which is exactly a homeomorphism. The inverse arrow of the underlying set function is again $|g| = \mathrm{Id}_{\mathbb{R}}$, and since $|h| = \mathrm{Id}_{\mathbb{R}}$ is iso and isos are both monic and epic then $|g|$ is unique. However, $g : (\mathcal{R}, \leq) \to (\mathbb{R}, \mathcal{F})$ is not continuous since for any $x \in \mathbb{R}$, the preimage of the open set $\{x\}$ in $(\mathbb{R}, \mathcal{F})$, $g^{-1}(x) = x$, is the same single point set, which is not open under the order topology.

Definition 1.20 (Initial and Terminal Objects)**.** In a category $\mathcal{C}$, an object is an an initial object, denoted $\mathbf{0}$, if for all objects C there exists a unique arrow $!_C : \mathbf{0} \to \mathbf{C}$. An object is a terminal object, denoted $\mathbf{1}$, if for all objects C there exists a unique arrow $!_C : C \to \mathbf{1}$.

In **Sets**, the initial object is the empty set, and any singleton set is a terminal object. The uniqueness of the initial (terminal) object is addressed by the following proposition:

Proposition 1.21. *Initial (terminal) objects are unique up to isomorphism.*

Proof. Suppose a category $\mathcal{C}$ has two initial objects $\mathbf{0}$ and $\mathbf{0}'$. Then by the universal mapping property of initial objects, there exist unique arrows $! : \mathbf{0} \to \mathbf{0}'$ and $!' : \mathbf{0}' \to \mathbf{0}$. Thus, $! \circ !' : \mathbf{0}' \to \mathbf{0}'$ and $!' \circ ! : \mathbf{0} \to \mathbf{0}$. By the universal mapping property of initial objects, $! \circ !' = \mathrm{Id}_{\mathbf{0}'}$ and $!' \circ ! = \mathrm{Id}_{\mathbf{0}}$. Thus, $\mathbf{0}$ and $\mathbf{0}'$ are isomorphic. Since terminal objects are the dual notion of initial objects, an analogous proof with arrows reversed shows that terminal objects are unique up to isomorphism. $\square$

It is worth noting two things at this point. First, a category does not need an initial or terminal object. Take for example the poset category $(\mathbb{Z}, \leq)$ which has neither terminal nor initial object. Second, along those same duality lines, it's worth noting that the initial (terminal) object of a category $\mathcal{C}$ is the terminal (initial) object of $\mathcal{C}^{\mathbf{op}}$. Knowing this, the uniqueness up to isomorphism of terminal objects in any category $\mathcal{C}$ follows from the uniqueness up to isomorphism of the initial object in $\mathcal{C}^{\mathbf{op}}$.

In some categories, the initial and terminal object are the same

2. Studying Objects in a Category

2.1. **Elements and Arrows.** Arrows from terminal object provide some insight into the structure of the objects of the category. In **Sets**, the terminal object is any singleton set (which are clearly isomorphic to each other). Arrows $\overline{a} : \mathbf{1} \to A$, where $\overline{a}(\cdot) = a \in A$, are exactly the elements of the set A. In fact, we have in **Sets** an isomorphism $X \cong \mathbf{Sets}(\mathbf{1}, X)$ for any set X. Further, we can generalize this notion to any category with a terminal object.

Definition 2.1 (Points or Elements)**.** In a category $\mathcal{C}$ with a terminal object $\mathbf{1}$, arrows $\overline{a} : \mathbf{1} \to A$ are called the *elements* or *points* of A.

In **Sets**, the elements of A are enough to distinguish between parallel arrows.

Proposition 2.2. *If $f, g : A \to B$ are parallel arrows in* **Sets***, then $f = g$ if and only if for all elements $\overline{a} : \mathbf{1} \to A$ of A, $f\overline{a} = g\overline{a}$.*

Proof. The forward direction is trivial. For the reverse direction, suppose that $f \neq g$. Then there exists $a \in A$ such that $f(a) \neq g(a)$. Define $\overline{a} :! \to A$ by $\overline{a}(\cdot) = a$. Then $g\overline{a} \neq f\overline{a}$. We've shown that if $f \neq g$ then there exists an element of A such that $f\overline{a} \neq g\overline{a}$, so the result follows by contrapositive. $\square$

This argument also holds in **Pos**, but it does not hold in general. In **Grp**, the terminal object, the single element group that consists of just an identity element, is also initial. Thus, for all groups G, there exists only one arrow $g : \mathbf{1} \to G$, so groups have only one point. This tells us that our notion of element or point is not general enough to capture the information we would like to obtain. Thus, we must abstract further to generalized elements, which can be used to distinguish between arrows in any category..

Definition 2.3 (Generalized Elements)**.** Arbitrary arrows $x : X \to A$ (for any domain X) are regarded as *generalized or variable elements* of A.

Think of these arrows of things in A that are 'shaped" like an arbitrary X. In **Sets**, arrows from the terminal object point out the parts of set that look like the terminal object that is, they are exactly the elements, in the usual sense, of A. Similarly, arrows from for a two point set $\{1, 2\} \to A$ are exactly the subsets of A that have exactly two elements. A more illustrative example comes from looking at generalized elements in categories of structured sets.

Example 2.4.

(1) Consider the three element group $G = \{u_G, g, g^{-1}\}$. Then arrows in **Grp** $G \to H$ from G to other groups H are exactly the subgroups of G that consist of an element, its inverse, and the identity element u_H.
(2) In **Pos**, consider the poset $P = \{1 \leq 2 \leq 3\}$. Then the arrows $p : P \to A$ are exactly the subsets of A consisting of three totally ordered elements.

The usefulness of generalized arrows comes from their ability to distinguish parallel arrows.

Proposition 2.5. *In any category $\mathcal{C}$, for all parallel arrows $f, g : C \to C'$, $f = g$ if and only if $fx = gx$ for all generalized elements $x : D \to C$.*

Proof. The forward direction is easy. For the reverse direction, if for all generalized elements $x : D \to C$ we have $gx = fx$, then we have $g \circ \mathrm{Id}_C = f \circ \mathrm{Id}_C$. Since $f \circ \mathrm{Id}_C = f$ and $g \circ \mathrm{Id}_C = g$, then $f = g$. □

Generalized elements are part of a class of tools used to elucidate the fine structure of a category. To do so, we need to loosen how strict our tools are. Isomorphic pairs of arrows are a very strong and useful tool for establishing relations among objects. However, isomorphisms require both compositions of the arrows to yield an identity. If we require only that one composition yield identity, then we have a more flexible notions: split monomorphism and split epimorphisms.

Proposition 2.6. *If $f : A \to B$ has a left inverse $g : B \to A$ (equivalently, g has a right inverse f) such that $gf = \mathrm{Id}_A$ then f is monic and g is epic.*

Proof. Since the statement that g is epic is dual to the one that f is monic, we'll omit its proof and claim that it follows from the duality principle. Consider parallel arrows $h, k : X \to A$ such that $fh = fk$. Then we have $g(fh) = g(hk)$ by composition. By associativity, $g(fh) = (gf)h = \mathrm{Id}_A \circ h = h$. Similarly, $g(fk) = k$. Thus $h = k$, so f is monic. □

Definition 2.7 (Split Mono(Epi))**.** A *split mono (epi)* is an arrow with a left (right) inverse. Given arrows $e : X \to A$ and $s : A \to X$ such that $es = \mathrm{Id}_A$, we say that s is a section or splitting of e and that e is a retraction of s. A is called a retract of X.

In a similar way that functors preserve isomorphisms, functors preserve split monos and epis.

Proposition 2.8. *In* **Sets**, *every mono splits.*

Proof. As shown previously, in **Sets** the monic arrows are exactly the injective functions. Thus all monos in **Sets** are bijective on their images. For every $x \in \mathrm{Im}(f)$ there's a unique $y \in A$ such that $f(y) = x$ by injectivity. So for $f : A \to B$ that's monic, define $g : B \to A$ by $g(b) = a$ if a is the unique element of A such that $f(a) = b$. Fix any $y \in A$ and define g on $b \in B \setminus \Im(f)$ by $g(b) = y$. Then $gf = \mathrm{Id}_A$, which is equivalent to saying that f splits. $\square$

Fact 2.9. The condition that every epi splits in **Sets** is the categorical equivalent of the axiom of choice.

To see this, consider an epi $e : E \to X$. As shown previously, s must be onto since it is epic. Thus, for all $x \in X$, there is a nonempty set $e_x := e^{-1}\{x\}$. A splitting of e is exactly a choice function on the family of sets $(e_x)_{x\in X}$; that is, a function $s : X \to E$ such that $es = \mathrm{Id}_X$. Conversely, given a family of nonempty sets e_x for every $x \in X$, taking $E = \{(x, y)|x \in X, y \in e_x\}$, we can define the function $e : E \to X$ such that $(x, y) \mapsto x$. A splitting of e would thus be a choice function on the family $(e_x)_{x\in X}$.

2.2. **Projective Objects.**

Projective objects are another tool that let us establish relations among the objects of a category by using something akin to a mapping property.

Definition 2.10 (Projective Objects)**.** An object P is said to be *projective* if for any epi $e : E \to X$ and for any arrow $f : P \to X$, there exists a not necessarily unique arrow $\overline{f} : P \to E$ such that $e \circ \overline{f} = f$. Diagrammatically:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, dashed, "\overline{f}"'] & X \\ & E \arrow[u, "e"'] \end{tikzcd}$$

We say that f lifts across e.

Let us consider what this rather abstract definition means in **Sets**.

Proposition 2.11. *The axiom of choice implies that all sets are projective in* **Sets**.

Proof. The axiom of choice is equivalent to saying that all epis split. So for a set A, consider the arrow $f : A \to B$ and the epi $e : E \to B$. Since epis split, there exists an arrow $s : B \to E$ such that $es = \mathrm{Id}_B$. It follows that $(es)f = \mathrm{Id}_B \circ f = f$. And by composition, we have that the following commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "fs"'] & B \arrow[d, shift left, "s"] \\ & E \arrow[u, shift left, "e"] \end{tikzcd}$$

Letting $\overline{f} = fs$, we've shown that A is projective. $\square$

Proposition 2.12. *In any category* $\mathcal{C}$, *the retract of a projective object is also projective.*

Proof. Let P be a projective object and let A be a retract of P, that is, there exists a split epi $e : P \to A$ with a right inverse $s : A \to P$ (which has to be monic). Consider any arrow $f : A \to B$ and any epi $i : E \to B$. Then $fe : P \to B$ by composition. Since P is projective, there exists $\overline{fe} : P \to E$ such that $i \circ \overline{fe} = fe$, and thus the following commutes:

$$\begin{tikzcd}
P \arrow[r, shift left, "e"] \arrow[drr, dashed, "\overline{fe}"'] & A \arrow[l, shift left, "s"] \arrow[r, "f"] & B \\
& & E \arrow[u, "i"]
\end{tikzcd}$$

By precomposition, we get that $(fe)s = i \circ \overline{fe} \circ s$. Since $(fe)s = f(es) = f \circ \mathrm{Id}_A = f$, then $f = i \circ \overline{fe} \circ s$ and the following commutes showing that A is projective:

$$\begin{tikzcd}
P \arrow[r, shift left, "e"] \arrow[drr, dashed, "\overline{fe}"'] & A \arrow[l, shift left, "s"] \arrow[r, "f"] \arrow[dr, dashed, "\overline{fe}s"] & B \\
& & E \arrow[u, "i"']
\end{tikzcd}$$

□

2.3. **Subobjects.** Our final piece of machinery in elucidating the fine structure of objects allows us to establish a sort of partial order among the objects by establishing certain objects as "subobjects" of each other.

Definition 2.13 (Subobject)**.** A *subobject* of an object X in a category $\mathcal{C}$ is a mono $m : M \to X$. Given subobjects m and m' of X, a morphism $f : m \to m'$ is an arrow such that the following commutes:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\
& X
\end{tikzcd}$$

Thus, we can talk about $\mathrm{Sub}_{\mathcal{C}}(X)$, the category of subobjects of X in $\mathcal{C}$.

Remark 2.14. Note that such an arrow f is an arrow in $\mathcal{C}/X$. In reality, $\mathrm{Sub}_{\mathcal{C}}(X)$ is a subcategory of $\mathcal{C}/X$, and so composition and identity are exactly the same as in $\mathcal{C}/X$.

Since m' is monic, there is at most one f from m to m'; thus $\mathrm{Sub}_{\mathcal{C}}(X)$ is a preorder category. We can use this to define an inclusion relation, where $m \leq m'$ if and only if there exists $f : m \to m'$. Two subobjects are then equivalent, denoted $m \equiv m'$, if and only if $m \leq m'$ and $m' \leq m$. This is equivalent to both triangles commuting:

$$\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[dr, "m"'] & M' \arrow[l, shift left, "f'"] \arrow[d, "m'"] \\
& X
\end{tikzcd}$$

It's worth noting that our diagram shows that $m = mf'f$. Since m is monic, this implies that $f'f = \mathrm{Id}_{M'}$. By the same argument, we see that $ff' = \mathrm{Id}_M$, which implies that $M \cong M'$. Thus, equivalent subobjects have isomorphic domains.

Proposition 2.15. *Let m and m' be subobjects of X. Then an arrow $f : m \to m'$ is also monic, so f is a subobject of M'.*

Proof. Consider subobjects m and m' of X such that $m \leq m'$, that is, there exists $f : m \to m'$ such that $m'f = m$. Suppose there are parallel arrows $i, j : Z \to M$ such that $fi = fj$. Then by composition, $m'fi = m'fj$. Since $m'f = m$, then $mi = mj$. Finally, since m is monic, then $i = j$. Thus, f is monic and is a subobject of M' by composition. This can be read off the following diagram:

$$\begin{tikzcd} Z \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\ & & X \end{tikzcd}$$

□

For generalized elements $z : Z \to X$ of X, we can define a local membership relation, $z \in_X m$, between generalized elements and subobjects $m : M \to X$ by the following: $z \in_X m$ if and only if there exists $f : Z \to M$ such that

$$\begin{tikzcd} Z \arrow[r, dashed, "f"] \arrow[dr, "z"'] & M \arrow[d, "m"] \\ & X \end{tikzcd}$$

commutes. Since m is monic, then f must be unique.

Fact 2.16. It is an entirely surprising result that $\mathrm{Sub}_{\mathbf{Sets}}(X) = P(X)$. Despite the appearance of what appears to be a terrible pun, you can see the truth of this statement by seeing that any monic function into a set X is an injective "inclusion" arrow. Consider any two subobjects $z : Z \to X$ and $z' : Z' \to X$. Then $z \leq z'$ means that there's a monic (injective) arrow $f : Z \to Z'$ that commutes appropriately. If we have both $z \leq z'$ and $z' \leq z$, then by our definition we have $z \equiv z'$. But this allows us to see just how clever the definition of this relation is, because if we have $z \leq z'$ and $z' \leq z$, then we have injective arrows $f : Z \to Z'$ and $f' : Z' \to Z$, which by the Schroeder-Bernstein Theorem gives us that $|Z| = |Z'|$, which is equivalent to saying that the respective sets are isomorphic in **Sets**

3. The Three Faces of Limits

In this section, we will introduce the notions of products, pullbacks, and equalizers, which we will go on to show later as being specific examples of a more abstract notion: limits.

3.1. **Products.** We know from set theory that we can take the cartesian products of sets: for any sets A and B, let $A \times B = \{(a, b) | a \in A, b \in B\}$. And from this product, we have two obvious projection functions, $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, where $(a, b) \xmapsto{\pi_1} a$ and $(a, b) \xmapsto{\pi_2} b$. Thus, for any $c \in A \times B$, we have $c = (\pi_1 c, \pi_2 c)$. This is equivalent to the following diagram commuting:

$$\begin{tikzcd} & 1 \arrow[dl, "\overline{a}"'] \arrow[d, dashed, "(a,b)"] \arrow[dr, "\overline{b}"] & \\ A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \end{tikzcd}$$

This example works with elements of A and B. But if we abstract to generalized elements, we arrive at a definition of products in any category.

Definition 3.1 (Products)**.** In any category $\mathcal{C}$, a *product diagram* for the objects A and B consists of an object P and arrows $p_1 : P \to A$ and $p_2 : P \to B$, $A \xleftarrow{p_1} P \xrightarrow{p_2} B$, that satisfies the following universal mapping property:

Given any diagram of the form $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there exists a unique arrow $u : X \to P$ making the following diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \end{tikzcd}$$

That is, $x_1 = p_1 u$ and $x_2 = p_2 u$.

Example 3.2. In **Sets**, the cartesian product $A \times B$ of sets A and B along with the standard projection functions π_1 and π_2 satisfies the universal mapping property of products.

Proof. Consider a set X and functions $f_1 : X \to A$ and $f_2 : X \to B$. Then define the function $f = (f_1, f_2) : X \to A \times B$ such that $p \overset{f}{\longmapsto} (f_1(p), f_2(p))$. Then clearly $f_1 = \pi_1 f$ and $f_2 = \pi_2 f$. Suppose there exists $u = (u_1, u_2) : X \to A \times B$ that also commutes appropriately. Then $f_1 = \pi_1 u = u_1$, and a similar argument shows that $f_2 = u_2$. Thus, $u = (f_1, f_2) = f$. $\square$

Proposition 3.3. *Products are unique up to isomorphism.*

Proof. Suppose for objects A and B you have two product diagrams: $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ and $A \xleftarrow{q_1} Q \xrightarrow{q_2} B$. Then by universal mapping property, there exists $i : P \to Q$ such that $p_1 = q_1 i$ and $p_2 = q_2 i$. Analogously, there exists $j : Q \to P$ such that $q_1 = p_1 j$ and $q_2 = p_2 j$. Then by precomposition, $q_1 i = p_1 j i$ and $q_2 i = p_2 j i$. Since $p_1 = q_1 i$ and $p_2 = q_2 i$, then we have $p_1 = p_1 j i$ and $p_2 = p_2 j i$. A similar precomposition argument will shows that $q_1 = q_1 i j$ and $q_2 = q_2 i j$. Diagrammatically, the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, dashed, shift right, "i"'] & \\ A & & B \\ & Q \arrow[ul, "q_1"] \arrow[ur, "q_2"'] \arrow[uu, dashed, shift right, "j"'] & \end{tikzcd}$$

Since $p_1 = p_1 j i$ and $p_2 = p_2 j i$, it follows from the uniqueness of identity arrows that $ji = \mathrm{Id}_Q$, and analogously that $ij = \mathrm{Id}_P$. Thus, $P \cong Q$. $\square$

Proposition 3.4. *$A \times B \cong B \times A$.*

Proof. Follows from previous proposition. $\square$

As a general rule, given X, x_1, and x_2 as in the above diagram we write $\langle x_1, x_2\rangle$ for $u : X \to A \times B$. We can also talk about arrows out of products, which can be written essentially as functions of two variables, and as such cannot be reduced. For example, $\langle f_1, f_2\rangle : A \times B \to X$.

Remark 3.5. As a general rule, products of structured sets can be constructed as products of the underlying sets with operations defined component wise. For example, the product of two groups G and H, $G \times H$, has as its underlying set the cartesian product of the underlying sets of G and H. Further,

$$\langle g, h\rangle \cdot \langle g', h'\rangle = \langle gg', hh'\rangle$$

$$u_{G\times H} = \langle u_G, u_H\rangle$$
$$\langle a, b\rangle^{-1} = \langle a^{-1}, b^{-1}\rangle$$

We wont go into detail again about the construction of the product category $\mathcal{C} \times \mathcal{D}$, though one can check that it satisfies the universal mapping property required by the definition.

Definition 3.6 (Binary Products)**.** A category $\mathcal{C}$ is said to *have binary products* if for every pair of objects A and B there exists an object $A \times B$ and a pair of projection arrows $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ that satisfies the universal mapping property of product diagrams.

Let's examine then what an arrow from a product to a product looks like. Consider two arrows $f : A \to B$ and $f' : A' \to B'$ in a category $\mathcal{C}$ with binary products. Suppose that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"'] \arrow[r, "q_2"] & B'
\end{tikzcd}$$

Then there exists a map which we denote $f \times f' : A \times A' \to B \times B'$ such that we say that both squares below commute:

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f\times f'"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"'] \arrow[r, "q_2"] & B'
\end{tikzcd}$$

Remark 3.7. It should be noted that a category that has binary products will necessarily have a terminal object. Imagine taking the "nullary" product, that is, a product of no objects. This product clearly has no projection arrows since there are no codomains to define those arrows on. Let us denote this product P. Thus for any other object C, we trivially have no arrows to the nonexistent objects that we used to define this product, so by the universal mapping property of products there exists a unique arrow $! : C \to P$ that makes all these nonexistent arrows commute trivially. It's clear then that P must be a terminal object since every element will trivially have arrows to the nonexistent objects that we took the product of to define P and so all objects must have a unique arrow to P.

Proposition 3.8. $C \times 1 \cong C$.

Proof. Consider the product diagram $C \xleftarrow{p_1} C \times 1 \xrightarrow{p_2} 1$, where we necessarily have that $p_2 = !_{C\times 1}$, the unique arrow from $C \times 1$ to the terminal object. Then for the pair of arrows $\mathrm{Id}_C : C \to C$ and $!_C : C \to 1$, there exists a unique arrow $u : C \to C \times 1$ such that $\mathrm{Id}_C = p_1 u$ and $!_C = !_{C\times 1} u$. That is, the following commutes:

$$
\begin{tikzcd}
& C \arrow[dl, "\mathrm{Id}_C"'] \arrow[d, dashed, "u"] \arrow[dr, "!_C"] & \\
C & C \times 1 \arrow[l, "p_1"'] \arrow[r, "!_{C\times 1}"] & 1
\end{tikzcd}
$$

By composition, we have $!_{C\times 1} u p_1 : C \times 1 \to 1$. By uniqueness, $!_{C\times 1} u p_1 = !_{C\times 1}$. Thus, $up_1 = \mathrm{Id}_{C\times 1}$. Since we have $up_1 = \mathrm{Id}_{C\times 1}$ and $p_1 u = \mathrm{Id}_C$, the result follows. □

A similar, but much less exciting, proof also shows us that products are associative up to isomorphism.

3.2. **Equalizers.** In **Sets**, given any two functions $f, g : A \to B$, we can consider the subset $E = \{a \in A | f(a) = g(a)\}$. Then defining an inclusion function $e : E \to A$, we have made it so that $fe = ge$ by restricting f and g to the subset of A on which they are equal. It is worth noting that an inclusion function is monic and thus that E is a subobject of A, since we will want our generalization to retain these properties.

Definition 3.9 (Equalizer)**.** In any category $\mathcal{C}$, given parallel arrows $f, g : A \to B$, an *equalizer* of f and g consists of an object E and an arrow $e : E \to A$ such that $fe = ge$, that is, the following commutes:

$$
\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}
$$

Further, they satisfy the following universal mapping property:

Given any $z : Z \to A$ such that $fz = gz$, there exists a unique arrow $u : Z \to E$ such that $eu = z$. Diagrammatically, the following commutes:

$$
\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & &
\end{tikzcd}
$$

As promised, e is monic, which is equivalent to saying that it (or E) is a subobject of A.

Proposition 3.10. *Equalizer arrows are monic.*

Proof. Given the equalizer diagram $E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$, suppose we have parallel arrows $z_1, z_2 : Z \to E$ such that $ez_1 = ez_2$. By precomposition, $fez_1 = gez_1 : Z \to A$. Thus, by the universal mapping property of equalizers, there exists a unique arrow $u : Z \to E$ such that $eu = ez_1$. Thus, $u = z_1$. But since $ez_1 = ez_2$, then $u = z_2$. Therefore, $z_1 = z_2$, so e is monic. □

3.3. **Pullbacks.** Much like an equalizer, pullbacks attempt to "equalize" arrows through the use of generalized elements. In the case of equalizers, the two arrows share both a domain and a codomain. In the case of pullbacks, they only share a codomain.

Definition 3.11 (Pullback)**.** Given a corner of arrows $f : A \to C$ and $g : B \to C$, a *pullback* of f and g is a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$ such that $fp_1 = gp_2$, and are universal in this property:

Given any pair of arrows $z_1 : Z \to A$ and $z_2 : Z \to B$ such that $fz_1 = gz_2$, there exists a unique arrow $u : Z \to P$ such that $z_1 = p_1u$ and $z_2 = p_2u$. Diagrammatically,

$$\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Remark 3.12. We usually denote the pullback object using product notation, as in $A \times_C B$.

In a manner almost identical to all the previous examples with universal mapping properties, we can demonstrate that pullbacks are unique up to isomorphism. Thus, we will omit the proof and just state the result.

Proposition 3.13. *Pullbacks are unique up to isomorphism.*

As stated earlier, pullbacks and equalizers are very similar notions in motivation. It is probably due to the cleverness of Saunders Mac Lane that the notions themselves are also very closely related. The following lemma begins to establish this relationship.

Lemma 3.14.

$$\begin{tikzcd}
E \arrow[dr, dashed, "e"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \times B \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

In a category $\mathcal{C}$ with products and equalizers, given a corner of arrows $f : A \to C$ and $g : B \to C$, consider the diagram where $e : E \to A \times B$ is an equalizer for $f\pi_1$ and $g\pi_2$. If we define $p_1 = \pi_1 e$ and $p_2 = \pi_2 e$, then E, p_1, p_2 is a pullback of f and g.

Conversely, if E, p_1, p_2 is given as a pullback of f and g, then the arrow e defined as $e := (p_1, p_2) : E \to A \times B$ is an equalizer for $f\pi_1$ and $g\pi_2$.

We will quickly sketch both parts of the proof. For the former, suppose we have arrows $z_1 : Z \to A$ and $z_2 : Z \to B$ such that $fz_1 = gz_2$. This defines an arrow $(z_1, z_2) : Z \to A \times B$. Since E and e form an equalizer, there's a unique arrow

$u : Z \to E$ such that $eu = z$ by the universal mapping property of equalizers. Componentwise, this gives us $ep_1 = z_1$ and $ep_2 = z_2$, which yields the universal mapping property of pullbacks.

For the second part, given the pullback, define $e = (p_1, p_2) : E \to A \times B$. Supposing that there exists $z : Z \to A \times B$ such that $z(f\pi_1) = z(g\pi_2)$, we see that such an arrow into a product is the same as two arrows $z_1 : Z \to A$ and $z_2 : Z \to B$. Further, we get that $z_1 f = z_2 g$ since $z\pi_i = z_i$ by definition. By the universal mapping propertyof pullbacks, we get a unique arrow $u : Z \to E$ such that componentwise we get $z_i = p_i u$, which when put together gives us $z = eu$ since $e = (p_1, p_2)$ by construction. This satisfies the universal mapping propertyof equalizers.

This leads us to a very important corollary, which will be used later to collect pullbacks, equalizers, and products under the single banner of limits. Before we do that, we must first tie together what we have so far.

Corollary 3.15. *If a category $\mathcal{C}$ has binary products and equalizers then it has pullbacks.*

Proof. Consider a corner of arrows $f : A \to C$ and $g : B \to C$. Since $\mathcal{C}$ has products, then there exists an object $A \times B$ with the canonical projection arrows π_1 and π_2. Since $\mathcal{C}$ has equalizers, let $e : E \to A \times B$ be an equalizer of $f\pi_1$ and $g\pi_2$. By previous lemma, if we define $p_1 = \pi_1 e$ and $p_2 = \pi_2 e$, then E, p_1, p_2 is a pullback of f and g. $\square$

3.4. **Equivalences.** Having defined products, pullbacks, and equalizers, we can now prove our first big theorem showing the equivalence of these notions.

Theorem 3.16. *A category $\mathcal{C}$ has binary products and equalizers if and only if it has pullbacks and a terminal object.*

Proof. Let $\mathcal{C}$ be a category with pullbacks and a terminal object.

The forward direction of this proof is exactly the corollary from the previous section. The existence of the terminal object follows from the definition if having binary products. Thus, we will restrict the given proof here to the reverse direction.

Consider objects A and B. Then there exist unique arrows $!_A : A \to 1$ and $!_B : B \to 1$. Since we have pullbacks, there exist arrows $p_1 : A \times_1 B \to A$ and $p_2 : A \times_1 B \to B$ such that $p_1 !_A = p_2 !_A$ and are universal with this property. Suppose we have the diagram $A \xleftarrow{z_1} Z \xrightarrow{z_2} B$. By uniqueness of arrows to the terminal object, $!_A z_1 = !_Z = !_B z_2$. Diagrammatically, we have that the following commutes:

$$\begin{tikzcd}
Z \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& A \times_1 B \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "!_B"] \\
& A \arrow[r, "!_A"'] & 1
\end{tikzcd}$$

By the universal mapping propertyof pullbacks, there exists a unique arrow $u : Z \to A \times_1 B$ such that $z_1 = p_1 u$ and $z_2 = p_2 u$. This means that $A \xleftarrow{p_1} A \times_1 B \xrightarrow{p_2} B$

is a product of A and B because it satisfies the universal mapping propertyof products.

For parallel arrows $f, g : A \to B$, consider the corner of arrows $\langle f, g \rangle : A \to B \times B$ and $\langle \mathrm{Id}_B, \mathrm{Id}_B \rangle : B \to B \times B$. Since we have pullbacks, there exist arrows $e : E \to A$ and $h : E \to B$ such that $\langle f, g \rangle e = \langle \mathrm{Id}_B, \mathrm{Id}_B \rangle h$. This implies that $\langle fe, ge \rangle = \langle h\mathrm{Id}_B, h\mathrm{Id}_B \rangle = \langle h, h \rangle$. Thus, $fe = h = ge$. To show that E and e are universal with this property, suppose we have an arrows $z_1 : Z \to A$ and $z_2 : Z \to B$ such that $\langle f, g \rangle z_1 = \langle \mathrm{Id}_B, \mathrm{Id}_B \rangle z_2$. This gives us by the same argument as before that $fz_1 = z_2 = gz_1$. By the universal mapping propertyof pullbacks, there exists a unique arrow $u : Z \to E$ such that $z_1 = eu$ and $z_2 = hu$. From this we see that this satisfies the universal mapping property of equalizers: for an arrow $z_1 : Z \to A$ such that $fz_1 = gz_1$ there exists unique $u : Z \to E$ such that $z_1 = eu$. We can see this from the following commutative diagram:

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& E \arrow[r, "h"'] \arrow[d, "e"] & B \arrow[d, "\langle \mathrm{Id}_B, \mathrm{Id}_B \rangle"] \\
& A \arrow[r, "\langle f, g \rangle"'] & B \times B
\end{tikzcd}
$$

□

4. Limits

In this final section, we will build up the framework for limits and then prove our main theorem: that having limits is equivalent to having products and equalizers.

4.1. **Basic Definitions.** Here, we will present the definitions relevant to limits.

Definition 4.1 (Type-$\mathcal{J}$ Diagram)**.** A *type-$\mathcal{J}$ diagram* in a category $\mathcal{C}$ is a functor $D : \mathcal{J} \to \mathcal{C}$.

Remark 4.2. Objects in the index category $\mathcal{J}$ will be denoted by lower case letters $i, j, k, \ldots$, and their images under D will be denoted respectively as $D_i, D_j, D_k, \ldots$.

Definition 4.3 (Cone)**.** A *cone* to a diagram D consists of some object $C \in \mathcal{C}_0$ and a family of arrows $(c_j : C \to D_j) \in \mathcal{C}_1$ for each $j \in \mathcal{J}_1$ such that for all arrows $(\alpha : i \to j) \in \mathcal{J}_1$ we have that $c_j = D\alpha \circ c_i$. Equivalently, the following commutes for all such α:

$$
\begin{tikzcd}
C \arrow[r, "c_j"] \arrow[d, "c_i"'] & D_j \\
D_i \arrow[ur, "D\alpha"'] &
\end{tikzcd}
$$

Definition 4.4 (Morphism of Cones)**.** A *morphism of cones* $\vartheta : (C, c_j) \to (C', c'_j)$ is an an arrow $\vartheta : C \to C'$ in $\mathcal{C}$ such that for all $j \in \mathcal{J}_0$ we have that $c_j = c'_j \circ \vartheta$.

We can now talk about the category of cones to a diagram D, $\mathbf{Cone}(D)$. Checking that this is a category is somewhat tedious and rather trivial, so the proof will be omitted. Having this, we can finally give a definition for limits.

Definition 4.5 (Limit)**.** A *limit* of a diagram $D : \mathcal{J} \to \mathcal{C}$ is a terminal object in **Cone**(D). We call it a *finite limit* if $\mathcal{J}$ is a finite index category. We denote the limit (L_D, p_j).

Remark 4.6. It is worth clarifying to prevent confusion that the limit of a diagram has the following universal mapping property as terminal objects of **Cone**(D):

Given any cone (C, c_j) to D, there exists a unique morphism of cones $u : (C, c_j) \to (L_D, p_j)$ such that for all $j \in \mathcal{J}_0$ we have that $c_j = p_j u$, i.e. the following commutes:

$$\begin{tikzcd} C \arrow[r, "c_j"] \arrow[d, dashed, "u"'] & D_j \\ L_D \arrow[ur, "p_j"'] & \end{tikzcd}$$

4.2. **Plateau.** Since the notion of a limit is somewhat abstract, it helps to clarify it with a few examples. Specifically, we will show how products, pullbacks, equalizers, and terminal objects are each special cases of limits.

Example 4.7 (Products as Limits)**.** Take $\mathcal{J} = \{1, 2\}$ to be the discrete category on two elements with no nonidentity arrows. A diagram $D : \mathcal{J} \to \mathcal{C}$ is a pair of objects $D_1, D_2 \in \mathcal{C}_0$. A cone to D is an object $C \in \mathcal{C}_0$ with a pair of arrows: $D_1 \xleftarrow{c_1} C \xrightarrow{c_2} D_2$. Let $D_1 \xleftarrow{p_1} L_D \xrightarrow{p_2} D_2$ be the terminal cone to D. Then for any other cone (C, c_j), there exists a unique morphism of cones $u : (C, c_j) \to (L_D, p_j)$ such that the following commutes:

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[d, dashed, "u"] \arrow[dr, "c_2"] & \\ D_1 & L_D \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_2 \end{tikzcd}$$

Thus, the limit exactly satisfies the universal mapping property of products.

Example 4.8 (Equalizers as Limits)**.** Take $\mathcal{J}$ such that $\mathcal{J}_0 = \{1, 2\}$ and the only nonidentity arrows are the parallel arrows $\alpha, \beta : 1 \to 2$. A type$\mathcal{J}$ diagram then looks like:

$$\begin{tikzcd} D_1 \arrow[r, shift left, "D\alpha"] \arrow[r, shift right, "D\beta"'] & D_2 \end{tikzcd}$$

Then any cone to D looks like

$$\begin{tikzcd} D_1 \arrow[r, shift left, "D\alpha"] \arrow[r, shift right, "D\beta"'] & D_2 \\ C \arrow[u, "c_1"] \arrow[ur, "c_2"'] & \end{tikzcd}$$

where both triangles commute. Note that this means that $D\alpha \circ c_1 = c_2 = D\beta \circ c_1$, beginning to look like an equalizer. Thus, when we take the limit, we get the necessary universal property. The limit then is the terminal cone (L_D, p_j) with the property that for any other cone (C, c_j) there exists a unique morphism of cones

$u : (C, c_j) \to (L_D, p_j)$ such that the following commutes:

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "D\alpha"] \arrow[r, shift right, "D\beta"'] & D_2 \\
C \arrow[u, "c_1"] \arrow[ur, "c_2"] \arrow[r, dashed, "u"'] & L_D \arrow[u, "p_2"'] \arrow[ul, "p_1"']
\end{tikzcd}$$

There is an arrow $p_1 : L_D \to D_1$ such that $D\alpha \circ p_1 = D\beta \circ p_1$ since both must equal p_2. Further, for any arrow $c_1 : C \to D_1$ such that $D\alpha \circ c_1 = D\beta \circ c_1$ there exists a unique $u : C \to L_D$ such that $c_1 = p_1 u$. Thus the limit satisfies the universal mapping propertyof equalizers.

Example 4.9 (Pullbacks as Limits)**.** Take $\mathcal{J}$ to be the finite category with objects $\mathcal{J}_0 = \{i, j, k\}$ and with the only nonidentity arrows being the corner of arrows $\alpha : i \to k$ and $\beta : j \to k$. Then the type-$\mathcal{J}$ diagram (first figure) and any cone to it (second figure) look like the following:

$$\begin{tikzcd}
 & D_j \arrow[d, "D\beta"] \\
D_i \arrow[r, "D\alpha"] & D_k
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "c_j"] \arrow[d, "c_i"'] \arrow[dr, "c_k"] & D_j \arrow[d, "D\beta"] \\
D_i \arrow[r, "D\alpha"] & D_k
\end{tikzcd}$$

Note from the figure of the cone to D that we have that $D\beta \circ c_j = c_k = D\alpha \circ c_i$ by commutation. So let's consider a limit (L_D, p_j) of the diagram.

$$\begin{tikzcd}
L_D \arrow[r, "p_j"] \arrow[d, "p_i"'] \arrow[dr, "p_k"] & D_j \arrow[d, "D\beta"] \\
D_i \arrow[r, "D\alpha"] & D_k
\end{tikzcd}$$

Then for any other cone (C, c_j), there exists a unique $u : (C, c_j) \to (L_D, p_j)$ such that the following commutes:

$$\begin{tikzcd}
C \arrow[dr, dashed, "u"] \arrow[drr, bend left, "c_j"] \arrow[ddr, bend right, "c_i"'] \arrow[ddrr, bend right, "c_k"] & & \\
 & L_D \arrow[r, "p_j"] \arrow[d, "p_i"'] \arrow[dr, "p_k"] & D_j \arrow[d, "D\beta"] \\
 & D_i \arrow[r, "D\alpha"'] & D_k
\end{tikzcd}$$

Thus for any pair of arrows $c_i : C \to D_i$ and $c_j : C \to D_j$ such that $D\alpha \circ c_i = D\beta \circ c_j$, which follows from both equaling c_k, there exists a unique arrow $u : C \to L_D$ by the universal mapping property of limits such that $c_i = p_i u$ and $cj = p_j u$. Thus, the limit satisfies the universal mapping propertyof pullbacks and is a pullback of α and β.

Example 4.10 (Terminal Objects as Limits)**.** Take $\mathcal{J}$ to be the empty category. Then a cone to a type-$\mathcal{J}$ diagram D consists of just a single object in C. Let L_D be the limit of this diagram. Then for every cone to D, which is every object C in $\mathcal{C}$, there exists a unique arrow $u : C \to L_D$, which makes the limit satisfy the universal mapping propertyof terminal objects.

4.3. **Climax.** Here we state and prove the theorem we've been working towards. One final definition is missing.

Definition 4.11 (All Finite Limits)**.** A category $\mathcal{C}$ has all finite limits if every finite diagram $D : \mathcal{J} \to \mathcal{C}$ has a limit in $\mathcal{C}$.

And the theorem itself:

Theorem 4.12. *A category has all finite limits if and only if it has all binary products and equalizers.*

We should note three things things at this point. As we showed earlier, having all binary products and equalizers is equivalent to having pullbacks and a terminal object, so a quick corollary of this theorem would be

Corollary 4.13. *A category has all finite limits if and only if it has pullbacks and a terminal object.*

Second, courtesy of the duality principle, once this theorem is proven we will have its dual statement and the dual of the previous corollary:

Corollary 4.14. *A category has all finite colimits if and only if it has all binary coproducts and coequalizers.*

Corollary 4.15. *A category has all finite colimits if and only if it has pushouts and an initial object.*

Here, a colimit is the dual notion of a limit. To sketch it out, the colimit is the initial object is the category of cocones (dual of cones) to a diagram D. Third, since we've already shown how using finite limits we can construct products, equalizers, pullbacks, and terminal objects, we will omit the forward direction of the proof of the theorem and restrict ourselves to the reverse direction.

Proof. Consider a finite diagram $D : \mathcal{J} \to \mathcal{C}$, and consider the products

$$\prod_{i \in \mathcal{J}_0} D_i \qquad \prod_{(\alpha : i \to j) \in \mathcal{J}_1} D_j$$

Define two arrows $\phi, \psi : \prod_i D_i \to \prod_\alpha D_j$ by taking taking their composites with the projections π_α from the second product to be

$$\pi_\alpha \circ \phi = \phi_\alpha = \pi_{\mathrm{cod}(\alpha)}$$
$$\pi_\alpha \circ \psi = \psi_\alpha = D\alpha \circ \pi_{\mathrm{dom}(\alpha)}$$

or equivalently, the following diagrams commute:

$$\begin{tikzcd}
\prod_{i \in \mathcal{J}_0} D_i \arrow[r, "\psi"] \arrow[d, "\pi_{\mathrm{dom}(\alpha)}"'] \arrow[dr, dashed, "\psi_\alpha"] & \prod_{(\alpha : i \to j) \in \mathcal{J}_1} D_j \arrow[d, "\pi_\alpha"] \\
D_{\mathrm{dom}(\alpha)} \arrow[r, "D\alpha"'] & D_{\mathrm{cod}(\alpha)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_{i \in \mathcal{J}_0} D_i \arrow[r, "\phi"] \arrow[dr, dashed, "\phi_\alpha = \pi_{\mathrm{cod}(\alpha)}"'] & \prod_{(\alpha : i \to j) \in \mathcal{J}_1} D_j \arrow[d, "\pi_\alpha"] \\
 & D_{\mathrm{cod}(\alpha)}
\end{tikzcd}$$

Then, we take the equalizer

$$\begin{tikzcd}
E \arrow[r, "e"] & \prod_{i \in \mathcal{J}_0} D_i \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \prod_{(\alpha : i \to j) \in \mathcal{J}_1} D_j
\end{tikzcd}$$

If for all $j \in \mathcal{J}_0$ we define $e_j = \pi_j e$, we want to show that (E, e_j) is the limit of the diagram D.

Letting $c : C \to \prod_i D_i$, we can write that $c = \langle c_i \rangle$ where $c_i = \pi_i c$ for all $i \in \mathcal{J}_0$, noting that (C, c_i) is a cone to D if and only if $\phi c = \psi c$.

To see this, suppose we have a cone, then for all $(\alpha : i \to j) \in \mathcal{J}_1$ we should have $D\alpha \circ c_i = c_j$. Since $c_i = \pi_i c$ and $c_j = \pi_j c$, and since $i = \text{cod}(\alpha)$ and $j = \text{cod}(\alpha)$, then we have $D\alpha\pi_{\text{dom}(\alpha)}c = \pi_{\text{cod}(\alpha)}c$, which is the same as $\psi_\alpha c = \phi_\alpha$ for all α. Conversely, suppose that we have $\psi c = \phi c$. Then for all any α, we have $D\alpha\pi_{\text{dom}(\alpha)}c = \pi_{\text{cod}(\alpha)}c$. We know that $c_{\text{dom}(\alpha)} = \pi_{\text{dom}(\alpha)}c$ and $c_{\text{cod}(\alpha)} = \pi_{\text{cod}(\alpha)}c$, so it follows that for any α $D\alpha \circ c_{\text{dom}(\alpha)} = c_{\text{cod}(\alpha)}$, which satisfies the definition of cones.

Since $\psi e = \phi e$ by construction, (E, e_j) is a cone. Suppose we have another cone (C, c_j). Then as shown above, this gives us that $\psi c = \phi c$. Then by the universal mapping property of equalizers, there exists a unique arrow $u : C \to E$ such that $c = eu$. Then by composition, we have $\pi_i c = \pi_i eu$, which gives us $c_i = e_i u$ for all $i \in \mathcal{J}_0$, which is exactly a morphism of cones. Since for all cones, we have a unique morphism of cones to (E, e_j), then it is the limit. □

4.4. **Resolution.** To close, we will discuss one final aspect of limits, specifically their preservation and creation under functors.

Definition 4.16 (Preserving Limits and Continuous Functors)**.** A functor $F\mathcal{C} \to \mathcal{D}$ is said to *preserve limits* of type-$\mathcal{J}$ if whenever we have a limit (L_D, p_j) of a diagram $D : \mathcal{J} \to \mathcal{C}$, the cone (FL_D, Fp_j) is a limit for the diagram $FD : \mathcal{J} \to \mathcal{D}$. Symbolically, $F(L_D, p_j) = (L_{FD}, Fp_j)$. We say that such a functor is *continuous*.

Definition 4.17 (Representable Functor)**.** In a locally small category $\mathcal{C}$ and for any fixed object $A \in \mathcal{C}_0$, we define the *representable functor* of A, denoted $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Sets}$, to be the functor defined such that for any object B, $B \longmapsto \mathcal{C}(A, B)$, and for any arrow $f : X \to Y$, $f \longmapsto \mathcal{C}(A, f) : \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$.

Proposition 4.18. *Representable functors are continuous.*

We will omit the proof, but we will mention that it suffices to show that representable functors preserve products and equalizers. Finally, let us recall the forgetful functor $U : \mathcal{C} \to \mathbf{Sets}$. We will close this paper with one final proposition, which we will also leave unproven.

Definition 4.19 (Creating Limits)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *create limits of type-$\mathcal{J}$* if for all diagrams $D : \mathcal{J} \to \mathcal{C}$ and any type-$\mathcal{J}$ limit $(L, p_j : L \to FC_j)$ in $\mathcal{D}$, there exists a unique cone $(\overline{L}, \overline{p_j} : \overline{L} \to C_j)$ to D in $\mathcal{C}$ with $F\overline{L} = L$ and $F\overline{p_j} = p_j$ such that $(\overline{L}, \overline{p_j} : \overline{L} \to C_j)$ is a type-$\mathcal{J}$ limit in $\mathcal{C}$.

Proposition 4.20. *The forget functor U :* $\mathbf{Grp} \to \mathbf{Sets}$ *creates all limits.*

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References

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