FOURIER ANALYSIS AND THE UNCERTAINTY PRINCIPLE

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Abstract. Fourier analysis is not only a useful tool in mathematics, but has applications in other fields as well. Specifically, it can be used to analyze signals and solve partial differential equations, two areas which are important to physics and engineering. In this paper, I will give an introduction to Fourier series and Fourier transforms, and apply these techniques to prove the Uncertainty Principle.

Contents

1. L^p spaces and the Riemann-Lebesgue lemma

Since the concept of L^p spaces of functions will be used throughout the paper, here are a few standard theorems and a definition.

Definition 1.1. Let I be an interval of $\mathbf{R}$ and $0 < p < \infty$. Then $L^p(I)$ is the space consisting of all functions f such that $\int_I |f(x)|^p \, dx < \infty$, where dx refers to the Lebesgue measure. We define the *norm of* f to be

$$\|f\|_p = (\int_I |f(x)|^p \, dx)^{1/p};$$

for this norm to be well-defined we must view two functions as equivalent if they are equal almost everywhere.

Theorem 1.2. *$L^p(\boldsymbol{R})$ is a complete metric space for $1 \leq p < \infty$.*

Theorem 1.3. *(Hölder's Inequality) Let $1 < p, q < \infty$ be such that $1/p + 1/q = 1$. If $f \in L^p(I)$ and $g \in L^q(I)$, then $fg \in L^1(I)$ and $\int_I |f(t)g(t)| \, dt \leq \|f\|_p \|g\|_q$.*

A proof of both can be found in [2]. Finally, the following lemma is useful for several theorems later on.

Lemma 1.4. *(Riemann-Lebesgue) For $f \in L^1(I)$, $\lim_{\lambda \to \infty} \int_I f(u) \sin(\lambda u) du = 0$.*

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The lemma and its proof are found in Vretblad's *Fourier Analysis and Its Applications*. One first proves the statement for constant functions on an interval and then approximates a generic $f \in L^1(I)$ by means of step functions.

2. Fourier series

The seminal idea of Fourier series is to express a given periodic function f as a sum of terms $c_n e^{in\pi t/L}$, where $2L$ is the period of the function and $n \in \mathbb{Z}$. Each of these terms has a period of the form $2L/n$, a rational multiple of the period of the given function, and complex amplitude c_n. For motivation, suppose $f(t)$ is a periodic function which can be expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/L}. \tag{2.1}$$

To find the coefficients of this series, we multiply both sides by $e^{-im\pi t/L}$ and integrate. If we assume the series is convergent, we can integrate over each term and then sum the terms to obtain

$$\begin{aligned}\int_{-L}^{L} f(t)e^{-im\pi t/L}dt &= \int_{-L}^{L} \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/L} e^{-im\pi t/L} dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-L}^{L} c_n e^{in\pi t/L} e^{-im\pi t/L} dt.\end{aligned}$$

For $n \neq m$,

$$\int_{-L}^{L} c_n e^{i(n-m)\pi t/L} dt = \frac{c_n L e^{i(n-m)\pi t/L}}{i(n-m)\pi}\Big|_{-L}^{L} = \frac{2c_n L \sin(\pi(n-m))}{\pi(n-m)} = 0.$$

For $n = m$,

$$\int_{-L}^{L} c_m e^0 dt = 2Lc_m.$$

Thus, the sum reduces to one term, and

$$\int_{-L}^{L} f(t)e^{-im\pi t/L}dt = 2Lc_m.$$

Based on this example, we define the Fourier series of a periodic function, give properties of the coefficients, and later on, examine its convergence.

Definition 2.2. The *Fourier series of a function f with period* $2L$ is the (possibly divergent) series

$$\sum_{-\infty}^{\infty} c_n e^{in\pi t/L}$$

where $c_n = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\pi t/L}dt$.

Theorem 2.3. *If $f \in L^1([-L, L])$, then the sequence of coefficients c_n is bounded and tends to zero as $|n| \to \infty$.*

Proof. For the first statement,

$$|c_n| = \left| \frac{1}{2L} \int_{-L}^{L} f(t) e^{-in\pi t/L} dt \right| \leq \frac{1}{2L} \int_{-L}^{L} |f(t)| \left| e^{-in\pi t/L} \right| dt = \frac{1}{2L} \int_{-L}^{L} |f(t)| \, dt < \infty$$

because $f \in L^1([-L, L]$. For the second statement,

$$\begin{aligned} c_n &= \frac{1}{2L} \int_{-L}^{L} f(t)(\cos(n\pi t/L) + i \sin(n\pi t/L)) dt \\ &= \frac{1}{2L} \int_{-L}^{L} f(t) \cos(n\pi t/L) dt + i \frac{1}{2L} \int_{-L}^{L} \sin(n\pi t/L) dt. \end{aligned}$$

Applying Lemma 1.4 to the real and imaginary parts of this expression, both parts go to zero as $n \to 0$, so $|c_n| \to 0$ as well. □

Just given the definition, we do not know under what conditions the Fourier series will actually converge, and more specifically converge to $f(t)$. Before giving a convergence theorem, we will need some more definitions.

Definition 2.4. A function f is *piecewise continuous on an interval* $[a, b]$ if f is continuous on [a,b] except possibly at a finite number of points, and, at those points, $\lim_{h \to 0} f(t+h)$ and $\lim_{h \to 0} f(t-h)$ for $h > 0$ exist. If f is piecewise continuous on all closed and bounded subintervals of $\mathbf{R}$, then f is *piecewise continuous on* $\boldsymbol{R}$.

Basically, a piecewise continuous function has only a finite number of finite jump discontinuities on a bounded interval.

Definition 2.5. If $s_n = \sum_{k=1}^{n} a_k$ is a series, let σ_N be the average of the partial sums given by

$$\sigma_N = \frac{1}{N} \sum_{k=1}^{N} s_k.$$

Then we say s_n sums to s *in the sense of Cesàro*, or s *is the Cesàro sum of* s_n if $\lim_{N \to \infty} \sigma_N = s$.

If a series is in the usual sense convergent to s, then it will converge in the Cesàro sum to s as well. However, series that are ordinarily divergent may converge in this new way. We now give the following definitions/lemmata.

Lemma 2.6. *The* Dirichlet kernel *is the sum* $D_N(u) := \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inu}$*; we have the identity*

$$D_N(u) = \frac{sin(N + \frac{1}{2})u}{2\pi sin(\frac{1}{2}u)}.$$

Lemma 2.7. *The* Fejèr kernel *is* $F_N(u) := \frac{1}{N+1} \sum_{n=0}^{N} D_n(u)$*; we also have that*

$$F_N(u) = \frac{1}{2\pi(N+1)} \frac{sin^2(\frac{1}{2}(N+1)u)}{sin^2(\frac{1}{2}u)}.$$

The proofs of these lemmata are formal. In each case one transforms the kernel into a sum from $n = 0$ to $n = 2N$ and then derives the identity by means of a geometric series. Next, we give several properties of functions in the sequence

$\frac{\pi}{L}F_N(u\pi/L)$. First, we can see from the form of the Fejèr kernel given by the lemma that

$$\frac{\pi}{L}F_N(u\pi/L) \geq 0. \tag{2.8}$$

Then, since

$$\begin{aligned}\int_{-L}^{L}\frac{\pi}{L}D_n(u\pi/L)du &= \int_{-L}^{L}\frac{1}{2L}\sum_{k=-n}^{n}e^{ik\pi u/L}du\\ &= \int_{-L}^{L}\frac{1}{2L}\sum_{k=-n}^{n}\cos(k\pi u/L)+i\sin(k\pi u/L)du\\ &= \int_{-L}^{L}\frac{1}{2L}(1+2\sum_{k=1}^{n}\cos(k\pi u/L))du = 1,\end{aligned}$$

we have that

$$\int_{-L}^{L}\frac{\pi}{L}F_N(u\pi/L)du = \frac{1}{N+1}\sum_{n=0}^{N}\int_{-L}^{L}\frac{\pi}{L}D_n(u\pi/L)du = 1. \tag{2.9}$$

Lastly, for any $\delta > 0$,

$$\begin{aligned}0 \leq \int_{\delta}^{L}\frac{\pi}{L}F_N(u\pi/L)du &= \frac{1}{2L(N+1)}\int_{\delta}^{L}\frac{\sin^2(\frac{1}{2}(N+1))\frac{u\pi}{L}}{\sin^2(\frac{1}{2}\frac{u\pi}{L})}\\ &\leq \frac{1}{2L(N+1)}\int_{\delta}^{L}\frac{1}{\sin^2(\frac{1}{2}\frac{u\pi}{L})} = \frac{\pi-\delta}{2\pi\sin^2(1/2\delta)}\frac{1}{N+1}.\end{aligned}$$

This last expression approaches zero as $N\to\infty$, so

$$\lim_{n\to\infty}\int_{\delta}^{L}\frac{\pi}{L}F_N(u\pi/L)du = 0 = \lim_{n\to\infty}\int_{-L}^{-\delta}\frac{\pi}{L}F_N(u\pi/L)du \tag{2.10}$$

by symmetry.

Theorem 2.11. *If f has period $2L$, is piecewise continuous on $\boldsymbol{R}$, and is continuous at t, then $\lim_{N\to\infty}\sigma_N(t) = f(t)$, where $\sigma_N(t) = \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n}c_k e^{ik\pi t/L}$.*

Proof. (This is similar to a proof in Vretblad's Fourier Analysis and Its Applications.) We start with the partial sum of the Fourier series

$$\begin{aligned}s_n(t) &= \sum_{k=-n}^{n}c_k e^{ik\pi t/L} = \sum_{k=-n}^{n}(\frac{1}{2L}\int_{-L}^{L}f(u)e^{-ik\pi u/L}du)e^{ik\pi t/L}\\ &= \int_{-L}^{L}\sum_{k=-n}^{n}(\frac{1}{2L}f(u)e^{ik\pi(t-u)/L}du) = \frac{\pi}{L}\int_{-L}^{L}f(u)D_n((t-u)\pi/L)du\\ &= \frac{1}{2L}\int_{-L}^{L}f(u)\frac{\sin((n+\frac{1}{2})\frac{(t-u)\pi}{L})}{\sin(\frac{1}{2}\frac{(t-u)\pi}{L})}du = -\frac{1}{2L}\int_{t+L}^{t-L}f(t-u)\frac{\sin((n+\frac{1}{2})\frac{u\pi}{L})}{\sin(\frac{1}{2}\frac{u\pi}{L})}du\\ &= \frac{\pi}{L}\int_{t-L}^{t+L}f(t-u)D_n(u\pi/L)du = \frac{\pi}{L}\int_{-L}^{L}f(t-u)D_n(u\pi/L)du.\end{aligned}$$

Here, we substitute u for $t-u$ and take the integral from $-L$ to L instead since the function is periodic. Now, we find

$$\begin{aligned}\sigma_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} s_n(t) &= \frac{\pi}{L}\frac{1}{N+1}\sum_{n=0}^{N}\int_{-L}^{L} f(t-u)D_n(u\pi/L)du \\ &= \frac{\pi}{L}\int_{-L}^{L} f(t-u)F_N(u\pi/L)du.\end{aligned}$$

Now, there exists $\delta > 0$ such that when $|u| < \delta$, $|f(t-u) - f(t)| < \epsilon$, since f is continuous at t. Also, because f is piecewise continuous on the closed and bounded interval $[t-L, t+L]$, we can take the intervals on which it is continuous and add in the right and left hand limits as values at the respective endpoints to obtain a closed and bounded interval on which f is continuous. Because such an interval is compact, f takes a maximum value, and so, without the assignment of value at the endpoints, f remains bounded on each smaller interval. Since a finite number of these smaller intervals make up the entire interval, f is bounded on the whole interval, $|f(t-u)| < M$ for $-L < u < L$. From these observations and properties 2.8-2.10, we can find the limit of $\sigma_N(t) - f(t)$ as $N \to \infty$. We start with

$$\begin{aligned}\frac{\pi}{L}\int_{-L}^{L} f(t-u)F_N(u\pi/L)du - f(t) = &\frac{\pi}{L}\int_{-L}^{L} f(t-u)F_N(u\pi/L)du \\ &-f(t)\frac{\pi}{L}\int_{-L}^{L} F_N(u\pi/L)du.\end{aligned}$$

Taking the absolute value gives

$$\begin{aligned}&\left|\frac{\pi}{L}\int_{-L}^{L} F_N(u\pi/L)(f(t-u) - f(t))du\right| \le \frac{\pi}{L}\int_{-L}^{L} F_N(u\pi/L)\,|f(t-u) - f(t)|\,du \\ &= \frac{\pi}{L}\int_{-\delta}^{\delta} F_N(u\pi/L)\,|f(t-u) - f(t)|\,du + \frac{\pi}{L}\int_{\delta<|u|<L} F_N(u\pi/L)\,|f(t-u) - f(t)|\,du.\end{aligned}$$

For the first integral, since $|(t-u) - t| = |u| < \delta$, we see $\epsilon\frac{\pi}{L}\int_{-\delta}^{\delta} F_N(u\pi/L)du = \epsilon$. In the second integral, because $|u| < L$, we get $M\frac{\pi}{L}\int_{\delta<|u|<L} F_N(u\pi/L)du$, which goes to zero as $N \to \infty$. So the entire integral goes to zero, meaning

$$\lim_{N\to\infty} \sigma_N(t) = \lim_{N\to\infty} \frac{\pi}{L}\int_{-L}^{L} f(t-u)F_N(u\pi/L)du = f(t).$$

□

In particular, if a function is continuous and its Fourier series is convergent in the ordinary sense, we have the following.

Theorem 2.12. *If f is continuous, $F(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/L}$ denotes its (formal) Fourier series, and $\sum_{n=-\infty}^{\infty} |c_n|$ is convergent, then F converges to f everywhere.*

Proof. $f(t)$ is continuous, so $\lim_{N\to\infty} \sigma_N(t) = f(t)$ for all t. Since

$$\left|e^{in\pi t/L}\right| = |\cos(\pi nt/L) + i\sin(\pi nt/L)| = 1,$$

we get $\left|\sum_{n=-\infty}^{\infty} c_n e^{in\pi t/L}\right| \leq \sum_{n=-\infty}^{\infty} |c_n| < \infty$. Thus, the Fourier series converges normally, and its sum must be equal to the Fejer sum $f(t)$. $\square$

So in fact, for certain functions the Fourier series $\sum_{-\infty}^{\infty} c_n e^{in\pi t/L} = f(t)$.

3. Fourier transforms

Fourier series are limited in that they only apply to periodic functions. We want to find how to represent a function as the period $2L \to \infty$. First, we have an intuitive deduction, starting with the Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi t/L} = \sum_{n=-\infty}^{\infty} (\frac{1}{2L}\int_{-L}^{L} f(u)e^{-in\pi u/L})e^{in\pi t/L}.$$

Since frequency is 2π divided by the period, and the period corresponding to n is $2L/n$, the frequency is $w_n = n\pi/L$. Also, the spacing between n is $\Delta n = 1$. The sum can then be rewritten

$$\sum_{n=-\infty}^{\infty} (\frac{1}{2L}\int_{-L}^{L} f(u)e^{-i\omega_n u})e^{i\omega_n t}\Delta n.$$

Since $\Delta\omega_n = \Delta n\pi/L = \pi/L = \Delta\omega$ and $n \to \pm\infty$ implies $\omega_n \to \pm\infty$, we can replace the sum with a sum over ω_n and obtain

$$\sum_{\omega_n=-\infty}^{\infty} (\frac{1}{2\pi}\int_{-L}^{L} f(u)e^{-i\omega_n u})e^{i\omega_n t}\Delta\omega.$$

Because the goal is to find what happens for when $L \to \infty$, we now take this limit, which means $\pi/L = \Delta\omega \to 0$. Substituting $\omega_n = \omega$, since ω is now a viewed as a continous variable, the expression above looks like a Riemann sum in the variable ω. Taking $\Delta\omega \to 0$ gives

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}(\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du)e^{i\omega t}d\omega.$$

This leads to the following definition and properties.

Definition 3.1. If $f \in L^1(\mathbf{R})$, the Fourier transform of $f(t)$ is $\hat{f}(\omega) = \int_{\mathbf{R}} f(t)e^{-i\omega t}dt$.

Theorem 3.2. *If $f \in L^1(\boldsymbol{R})$, then $\left|\hat{f}(\omega)\right| \leq \int_{\boldsymbol{R}} |f(t)|\,dt$, $\hat{f}$ is continuous on $\boldsymbol{R}$, and $\lim_{\omega\pm\infty} \hat{f}(\omega) = 0$.*

Proof. The proofs for the first and third statement are similar to the proofs for Theorem 2.3. For the second statement, we have

$$\begin{aligned}\left|\hat{f}(\omega+h) - \hat{f}(\omega)\right| &= \left|\int_{\mathbf{R}} f(t)e^{-i(\omega+h)t}dt - \int_{R} f(t)e^{-i\omega t}dt\right| = \left|\int_{\mathbf{R}} f(t)e^{-i\omega t}(e^{-iht-1})dt\right| \\ &= \left|\int_{\mathbf{R}} f(t)\frac{e^{-i(\omega-h/2)t}}{e^{ih/2t}}(e^{-iht/2} - e^{iht/2})dt\right| \\ &= \left|\int_{\mathbf{R}} f(t)\frac{e^{-i(\omega-h/2)t}}{e^{ih/2t}}2i\sin(ht/2)dt\right| \leq \int_{\mathbf{R}} |f(t)|\,|2\sin(ht/2)|\,dt.\end{aligned}$$

Because $f \in L^1(\mathbf{R})$, for every $\epsilon > 0$, we can find a finite interval $[a, b]$ such that $\int_a^{\infty} |f(t)|\,dt + \int_{-\infty}^{b} |f(t)|\,dt < \epsilon$. We can also find $g(t)$, a step function approximating

$f(t)$ on [a,b], such that $\int_a^b |f(t) - g(t)|\, dt < \epsilon$, since f is integrable. From the integral above, we have

$$\int_{\mathbf{R}} |f(t)|\, |2\sin(ht/2)|\, dt \leq 2(\int_a^b |f(t) - g(t)|\, |\sin(ht/2)|\, dt + \int_a^b |g(t)|\, |\sin(ht/2)|\, dt$$
$$+ \int_a^\infty |f(t)|\, |\sin(ht/2)|\, dt + \int_{-\infty}^b |f(t)|\, |\sin(ht/2)|\, dt).$$

We see that

$$\int_a^b |f(t) - g(t)|\, |\sin(ht/2)|\, dt \leq \int_a^b |f(t) - g(t)|\, dt < \epsilon, \tag{3.3}$$

and

$$\int_a^\infty |f(t)|\, |\sin(ht/2)|\, dt + \int_{-\infty}^b |f(t)|\, |\sin(ht/2)|\, dt < \epsilon \tag{3.4}$$

Also, since g is a step function, we can take M such that $|g(t)| \leq M$ on $[a, b]$. We get $2\int_a^b |g(t)|\, |\sin(ht/2)|\, dt < 2M \int_a^b |ht/2|\, dt = M \int_a^b |ht|\, dt$, and

$$\lim_{h\to 0} 2\int_a^b |g(t)|\, |\sin(ht/2)|\, dt = 0 \tag{3.5}$$

Thus, $\lim_{h\to 0} \left|\hat{f}(\omega + h) - \hat{f}(\omega)\right| = 0$. □

The transform function is similar to finding the Fourier coefficients as both involve integrating $f(t)e^{-i\omega t}$ to find the component of frequency ω. The difference is that whereas before we could find a sum of components of discrete frequencies which converged to f, in this case f has components in the entire continuous spectrum of frequencies, so we find a function corresponding to f, which gives these components. As with the Fourier series, we now look for a way to get back our original function $f(t)$ from $\hat{f}(\omega)$.

Theorem 3.6. *If $f \in L^1(\boldsymbol{R})$, is continuous, and has a piecewise continuous derivative, then $f(t) = \lim_{A\to\infty} \frac{1}{2\pi}\int_{-A}^A \hat{f}(\omega)e^{i\omega t}d\omega$ for all t.*

Proof. (This is based on a proof in Vretblad's Fourier Analysis and Its Applications.) Similarly to the case of Fourier series, we first consider the finite integral $\frac{1}{2\pi}\int_{-A}^A \hat{f}(\omega)e^{it_0\omega}d\omega$ as a function of positive real number A

$$\begin{aligned} s(t_0, A) &= \frac{1}{2\pi}\int_{-A}^A \hat{f}(\omega)e^{it_0\omega}d\omega = \frac{1}{2\pi}\int_{-A}^A (\int_{-\infty}^\infty f(t)e^{-i\omega t}dt)e^{it_0\omega}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\frac{e^{iA(t_0-t)} - e^{-iA(t_0-t)}}{i(t_0-t)}dt = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\frac{\sin(A(t_0-t))}{t_0-t}dt \\ &= \frac{1}{\pi}\int_{-\infty}^\infty f(t_0-u)\frac{\sin(Au)}{u}du. \end{aligned}$$

Here, we have switched the order of integration and substituted $u = t_0 - t$. We next use a form of the Dirichlet integral

$$\int_0^\infty \frac{\sin(Au)}{u}du = \frac{\pi}{2}.$$

This integral is found by using the function $f(B, A) = \int_0^\infty e^{-Bu} \frac{\sin(Au)}{u} du$. Differentiating this with respect to B under the integral and writing $\sin(Au)$ as the imaginary part of e^{iAu}, we get $\Im(\frac{df}{dB}) = \Im(\int_0^\infty -e^{-Bu} e^{iAu} du)$, an integral of an exponential which can then be evaluated. Computing the integral on the right, simplifying, and taking the imaginary part gives an expression for $\frac{df}{dB}$. Finally, integrating this expression with respect to B and setting $B = 0$ gives the equation above. Knowing the Dirichlet integral, we obtain

$$\begin{aligned}
\frac{2}{\pi}\int_0^\infty f(t_0-u)\frac{\sin(Au)}{u}du - f(t_0) &= \frac{2}{\pi}\int_0^\infty (f(t_0-u)-f(t_0))\frac{\sin(Au)}{u}du \\
&= \frac{2}{\pi}\int_0^X (f(t_0-u)-f(t_0))\frac{\sin(Au)}{u}du \\
&+ \frac{2}{\pi}\int_X^\infty f(t_0-u)\frac{\sin(Au)}{u}du \\
&- \frac{2}{\pi}\int_X^\infty f(t_0)\frac{\sin(Au)}{u}du
\end{aligned}$$

where for any given $\epsilon > 0$, we choose X such that $\frac{2}{\pi}\int_X^\infty |f(t_0-u)|\, du < \epsilon$. This is possible because $\frac{2}{\pi}\int_{-\infty}^\infty |f(t_0-u)|\, du < \infty$, so f must go to zero at $\pm\infty$. Then, looking back at the second integral in the three integral sum, we get

$$\left|\frac{2}{\pi}\int_X^\infty f(t_0-u)\frac{\sin(Au)}{u}du\right| \leq \frac{2}{\pi}\int_X^\infty |f(t_0-u)|\, du < \epsilon. \tag{3.7}$$

The first inequality is because for $X > 1$, we have that $\left|\frac{\sin(Au)}{u}\right| \leq 1$ for $X < u < \infty$. The third term becomes

$$\left|-\frac{2}{\pi}\int_X^\infty f(t_0)\frac{\sin(Au)}{u}du\right| \leq \frac{2}{\pi}|f(t_0)|\int_{AX}^\infty \left|\frac{\sin(v)}{v}\right| dv \leq \frac{2}{\pi}|f(t_0)|\int_{AX}^\infty \left|\frac{1}{v}\right| dv$$

when we substitute $v = Au$. This last integral approaches zero as $A \to \infty$. Thus,

$$\lim_{A\to\infty}\left|-\frac{2}{\pi}\int_X^\infty f(t_0)\frac{\sin(Au)}{u}du\right| = 0. \tag{3.8}$$

Lastly, in the first integral, the expression $\lim_{u\to 0} \frac{f(t_0-u)-f(t)}{-u} = f\prime(t_0)$ is bounded because $f\prime$ is, and $\frac{f(t_0-u)-f(t)}{-u}$ is continuous and bounded on (0,X] because f is. This means $\int_0^X \left|\frac{f(t_0-u)-f(t)}{-u}\right| du < \infty$ and applying Lemma 1.4,

$$\lim_{A\to\infty}\frac{1}{2\pi}\int_0^X \left(\frac{f(t_0-u)-f(t)}{-u}\right)\sin(Au)du = 0. \tag{3.9}$$

Thus, from statements 3.7-3.9, we get $\lim_{A\to\infty}\frac{2}{\pi}\int_0^\infty f(t_0-u)\frac{\sin(Au)}{u}du - f(t_0) = 0$, or

$$\lim_{A\to\infty}\frac{1}{\pi}\int_0^\infty f(t_0-u)\frac{\sin(Au)}{u}du = \frac{f(t_0)}{2} = \lim_{A\to\infty}\frac{1}{\pi}\int_{-\infty}^0 f(t_0-u)\frac{\sin(Au)}{u}du,$$

where the last equality holds by the same arguments, just changing the limits of integration. Then,

$$\lim_{A\to\infty} s(t_0, A) = \lim_{A\to\infty}\frac{1}{\pi}\int_{-\infty}^\infty f(t_0-u)\frac{\sin(Au)}{u}du = f(t_0).$$

□

In particular, if $\hat{f} \in L^1(\mathbf{R})$, then $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega$. The next important result is the Plancherel theorem. First are some helpful theorems.

Theorem 3.10. *(Fubini) Assume that $f : \boldsymbol{R} \times \boldsymbol{R} \to \overline{\boldsymbol{R}}$ is measurable and $E \times F$ is a measurable set on $\boldsymbol{R}$. If f is nonnegative on $E \times F$, then $\int_{E\times F} f(x,y)dxdy = \int_E dx \int_F f(x,y)dy = \int_F dy \int_E f(x,y)dx$. If f is integrable on $E \times F$, the function $x \mapsto f(x,y)$ is integrable for almost every y, the function $y \mapsto f(x,y)$ is integrable for almost every x, and the three integrals above are finite and equal. f is integrable if and only if $\int_E dx \int_F |f(x,y)|\, dy$ or $\int_F dy \int_E |f(x,y)|\, dx$ is finite.*

The theorem gives the conditions under which the the order of integration of a double integral may be switched. The proof of it is too long to give here, but we now apply it to get the following.

Theorem 3.11. *If f and g are two functions in $L^1(\boldsymbol{R})$, then $f\hat{g}$ and $\hat{f}g$ are in $L^1(\boldsymbol{R})$ and $\int_{-\infty}^{\infty} f(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx$.*

Proof. From Theorem 3.2, which says $|\hat{g}(t)| < M$ for some $0 < M < \infty$, we get that $\int_{\mathbf{R}} |f(t)\hat{g}(t)|\, dt \leq M \int_{\mathbf{R}} |f(t)|\, dt < \infty$. By the same reasoning, $\hat{f}g \in L^1(\mathbf{R})$. For the second part, we simply switch the order of integration since $f\hat{g}$ is integrable to get

$$\int_{-\infty}^{\infty} f(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} f(t)(\int_{-\infty}^{\infty} e^{-ixt}g(x)dx)dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ixt}f(t)g(x)dtdx$$
$$= \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx.$$

$\square$

Finally, here is the main theorem.

Theorem 3.12. *(Plancherel) For f in $L^1(\boldsymbol{R})$, $\int_{\boldsymbol{R}} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{\boldsymbol{R}} \left|\hat{f}(\omega)\right|^2 d\omega$.*

Proof. Since $|f(t)|^2 = f(t)\overline{f(t)}$, taking $\hat{g}(t) = \overline{f(t)} \in L^1(\mathbf{R})$ and $g(\omega) = \frac{1}{2\pi}\int_{\mathbf{R}} \overline{f(t)}e^{i\omega t}dt$ in the previous theorem, gives

$$\int_{\mathbf{R}} |f(t)|^2\, dt = \int_{\mathbf{R}} \hat{f}(\omega)(\int_{\mathbf{R}} \overline{f(t)}\frac{e^{i\omega t}}{2\pi}dt)dw = \frac{1}{2\pi}\int_{\mathbf{R}} f(\omega)\overline{\hat{f}(\omega)}d\omega.$$

The last equality follows because $\overline{\hat{f}(\omega)} = \overline{\int_{\mathbf{R}} f(t)e^{-i\omega t}dt} = \int_{\mathbf{R}} \overline{f(t)e^{-i\omega t}}dt$. $\square$

Up until this point, we have discussed Fourier transforms in the space L^1 because the defining integral $\int_{\mathbf{R}} f(t)e^{-i\omega t}dt \leq \int_{\mathbf{R}} |f(t)|dt < \infty$ for $f \in L^1$. However, there are many reasons for expanding the concept to functions in L^2, which are not necessarily in L^1. In particular, for this paper, the density functions used in the next section exist on the space L^2. In extending Fourier transforms to L^2, the main problem is that the definition of $\hat{f}(\omega)$ might not converge. Thus, we instead construct a sequence f_n such that $f_n, \hat{f}_n \in L^1$ and $f_n \to f$ in the L^2 norm. Applying Plancherel,

$$\left\|\hat{f}_n - \hat{f}_m\right\|^2 = \int_{\mathbf{R}} \left|\hat{f}_n(\omega) - \hat{f}_m(\omega)\right|^2 d\omega = 2\pi\int_{\mathbf{R}} |f_n(t) - f_m(t)|^2\, dt = 2\pi \left\|f_n(t) - f_m(t)\right\|^2.$$

Since f_n is a Cauchy sequence in L^2, as $n, m \to \infty$, $2\pi \|f_n(t) - f_m(t)\|^2 \to 0$. This implies that $\hat{f}_n$ is a Cauchy sequence, and also convergent in L^2, since L^2 is complete. We claim that the limit does not depend on the sequence f_n but only on f and define it to be $\hat{f}$. A more detailed explanation of the extension to L^2 can be found in Gasquet's Fourier Analysis and Applications. We note that the properties, including Plancherel, still hold in L^2.

4. The Uncertainty Principle

The idea of the Uncertainty Principle is that it is impossible for a function to both vanish outside some finite interval and have only frequency components smaller than some constant. We will use this to examine the relationship between the probability density functions for position and momentum. Because of this, instead of using variables t and ω for time and frequency, we write $f(x)$ to represent a function of position, and $\hat{f}(\xi)$ for the transform. The following definitions quantify the spread of the function.

Definition 4.1. For f such that f, xf, and $\xi\hat{f}$ in L^2, the *dispersion of* f is

$$\Delta f = \frac{\int_{\mathbf{R}} x^2 |f(x)|^2 \, dx}{\int_{\mathbf{R}} |f(x)|^2 \, dx}.$$

Since in the definitions of dispersion x^2 is in the numerator, Δf will be larger if f is more spread out, and the same for $\Delta\hat{f}$. The point is that Δf can only be made small only if $\Delta\hat{f}$ is made large, as stated in the next theorem.

Theorem 4.2. *For f such that f, xf, and $\xi\hat{f}$ are in L^2, $\Delta f \Delta\hat{f} \geq 1/4$.*

Proof. We begin by finding $\hat{f'}$. Integrating by parts gives

$$\hat{f'}(\xi) = \int_{\mathbf{R}} f'(x) e^{-i\xi x} dx = e^{i\xi x} f(x)|_{-\infty}^{\infty} - \int_{\mathbf{R}} f(x)(-i\xi e^{-i\xi x}) dx.$$

Since $f \in L^1$, $\lim_{x\to\pm\infty} f(x) = 0$ and

$$\hat{f'}(\xi) = i\xi\hat{f}(\xi) \tag{4.3}$$

Now, we consider the expression

$$\int_a^b x\overline{f(x)}f'(x)dx = x|f(x)|^2|_a^b - \int_a^b f(x)(\overline{f(x)} + x\overline{f'}(x))dx$$

by integrating by parts. Because $f(x)\overline{f(x)} = |f(x)|^2$,

$$\int_a^b |f(x)|^2 \, dx = x|f(x)|^2 - \int_a^b xf(x)\overline{f'(x)}dx - \int_a^b x\overline{f(x)}f'(x)dx.$$

The integral on the left is real, so we can take the real part of the expression on the right without changing the equality. Since

$$\Re(f(x)\overline{f'(x)}) = \Re(f(x))\Re(f'(x)) + \Im(f(x))\Im(f'(x)) = \Re(\overline{f(x)}f'(x)),$$

we have

$$\int_a^b |f(x)|^2 \, dx = x|f(x)|^2|_a^b - 2\Re(\int_a^b \overline{xf(x)}f'(x)dx).$$

Taking $a \to -\infty$, and $b \to \infty$, the first term on the right approaches zero. If $\lim_{x\to\infty} x|f(x)|^2$ were not zero, $f(x)$ would have to decrease slower than $x^{1/2}$.

This is not possible because $\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$, meaning f decreases as fast as $1/x$. Thus, we get

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = -2\Re(\int_{-\infty}^{\infty} \overline{xf(x)} f'(x) dx)$$

Applying Theorem 1.3 for $p = q = 2$,

(4.4)
$$(\int_{-\infty}^{\infty} |f(x)|^2 \, dx)^2 \leq 4(\int_{-\infty}^{\infty} \left|\overline{xf(x)} f'(x)\right| dx)^2 \leq 4 \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx \int_{-\infty}^{\infty} |f'(x)|^2 \, dx.$$

Plancherel and equation 4.3 give

$$\int_{-\infty}^{\infty} |f'(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\hat{f}'(\xi)\right|^2 d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^2 \left|\hat{f}(\xi)\right|^2 d\xi. \tag{4.5}$$

Substituting this back into equation 4.4 and applying Plancherel to the left side,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 \, dx \int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi \leq \frac{4}{2\pi} \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx \int_{-\infty}^{\infty} \xi^2 \left|\hat{f}(\xi)\right|^2 d\xi.$$

Reorganizing the terms gives the theorem. □

An application of this theorem is the wave function which describes the position of a particle. We begin with the Schrodinger equation.

$$i\hbar \frac{\delta \psi}{\delta t} = \frac{-\hbar^2}{2m} \frac{\delta^2 \psi}{\delta x^2} + v(x)\psi(x,t) \tag{4.6}$$

Assuming the particle is in free space, when $v(x) = 0$, the solution to this differential equation is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbf{R}} \phi(p) e^{i(px - Et)/\hbar} dp \tag{4.7}$$

where ψ, ϕ are in L^2, and $\|\psi\| = 1$ and $\|\phi\| = 1$. To see this, we take $\|\psi\| = 1$ and define

$$\tilde{\psi}(p) = \frac{1}{2\pi\hbar} \hat{\psi}(p/\hbar) \tag{4.8}$$

By the Inversion theorem, we have

$$\psi(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \psi(p/\hbar) e^{i\frac{p}{\hbar}x} d(p/\hbar) = \frac{1}{2\pi\hbar} \int_{\mathbf{R}} \sqrt{2\pi\hbar}\tilde{\psi}(p) e^{i\frac{p}{\hbar}x} dp \tag{4.9}$$

Comparing this equation with 4.7, we take $\tilde{\psi}(p) = \phi(p) e^{-iEt/\hbar}$ and

$$|\phi(p)|^2 = \left|\tilde{\psi}(p)\right| = \frac{1}{2\pi\hbar} \left|\hat{\psi}(p/\hbar)\right|^2 \tag{4.10}$$

Applying Plancherel theorem,

$$\|\phi\| = (\frac{1}{2\pi\hbar} \int_{\mathbf{R}} \left|\hat{\psi}(p/\hbar)\right|^2 dp)^{1/2} = (\frac{1}{2\pi} \int_{\mathbf{R}} \left|\hat{\psi}(p/\hbar)\right|^2 d(p/\hbar))^{1/2} = (\int_{\mathbf{R}} |\psi(x)|^2 \, dx)^{1/2} = 1$$

Thus, $|\psi(x,t)|^2$ is interpreted as the probability density function for position, meaning that the probability that the particle is in any interval $[a,b]$ is $\int_a^b |\psi(x,t)|^2 \, dx$. $|\phi(p)|^2$ then represents the probability density for momentum. Using Theorem 4.2, we find that the dispersions of ψ and ϕ cannot both be small. Using equation 4.10, we have

$$\Delta\hat{\psi} = \frac{\int_{\mathbf{R}} (p/\hbar)^2 \left|\hat{\psi}(p/\hbar)\right|^2 d(p/\hbar)}{\int_{\mathbf{R}} \left|\hat{\psi}(p/\hbar)\right|^2 d(p/\hbar)} = \frac{\frac{2\pi\hbar}{\hbar}\int_{\mathbf{R}} (p/\hbar)^2 \left|\phi(p)\right|^2 dp}{\frac{2\pi\hbar}{\hbar}\int_{\mathbf{R}} \left|\phi(p)\right|^2 dp} = \frac{1}{\hbar^2}\Delta\phi. \tag{4.11}$$

Since $\Delta\psi\Delta\hat{\psi} \geq 1/4$,

$$\Delta\psi\Delta\phi \geq \hbar^2/4 \tag{4.12}$$

This means that the spread of probability distributions of position and momentum are inversely related. So, if we want to know the position of a particle with greater certainty, we have to give up some certainty in knowing the momentum.

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References

[1] Gerald B. Folland. Fourier Analysis and Its Applications. Wadsworth, Inc. 1992.
[2] C. Gasquet. Fourier Analysis and Applications. Springer-Verlag New York, Inc, 1999.
[3] Walter A. Strauss. Partial Differential Equations. John Wiley & Sons, Inc. 1992.
[4] Anders Vretblad. Fourier Analysis and Its Applications. Springer-Verlag New York, Inc. 2003.