DE RHAM COHOMOLOGY, CONNECTIONS, AND CHARACTERISTIC CLASSES

PATRICK GREENE

Abstract. The de Rham cohomology is a cohomology based on differential forms on a smooth manifold. It uses the exterior derivative as the boundary map to produce cohomology groups consisting of closed forms modulo exact forms. The existence of exact forms reflects 'niceness' of the topology, in that a potential for closed forms can often be constructed by integrating them over some submanifold, if the manifold has topological properties that allow this integral to be well defined. The failure of a closed form to be exact therefore indicates that the manifold has some global structure which prevents a potential from existing, such as holes or twists. Thus the de Rham groups are a way of understanding, via the tangent bundle, a manifold's global topology. We also discuss the concept of a connection, which is a way to define the total derivative of a vector field. If a manifold is equipped with an inner product, we can also use it to take the derivative in the direction of another vector field. We can then define a new connection which acts on the output of the first connection, giving us something analogous to the second derivative when we compose them. This composition is called the curvature tensor. It can be used either via the Gauss-Bonnet theorem, giving us a link with the Euler characteristic, or, as we will discuss, via characteristic classes, which are special classes in the de Rham cohomology groups arising from the action of invariant polynomials on the curvature tensor.

Contents

1. The Basics

The de Rham cohomology allows us to answer the question of when closed forms on a manifold are exact. It turns out that the de Rham cohomology is homotopy invariant, and in particular, invariant under homeomorphism. Thus knowing seemingly unrelated properties about existence of closed but not exact forms gives us information about the topology of a manifold. For example, we can compute the number of "holes" of various dimension in a manifold, which is related to results we would get by using the fundamental group to calculate the same thing.

Date: August 2009.

Let M be a smooth manifold, $\Lambda^n(M)$ the alternating n-tensor bundle, $A^n(M)$ the vector space of differential n-forms on M, and $d : A^n(M) \to A^{n+1}(M)$ the exterior derivative.

Definition 1.1. An n-form $\omega : M \to \Lambda^n(M)$ is closed if $d\omega = 0$.

Definition 1.2. An n-form ω is exact if $\omega = d\eta$ for some $\eta : M \to \Lambda^{n-1}(M)$.

Definition 1.3. An n-form is conservative if its integral over any n-dimensional submanifold without boundary is zero.

Every exact form is closed, since $d \circ d = 0$. However, closed forms need not be exact. For example, consider the 1-form on $\mathbb{R}^2 \setminus 0$ defined by:

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

Then

$$\begin{aligned} d\omega &= \frac{(dx \wedge dy - dy \wedge dx)x^2 + y^2 - (2xdx + 2ydy)(xdy - ydx)}{(x^2+y^2)^2} \\ &= \frac{2(x^2+y^2)dx \wedge dy - (2x^2 dx \wedge dy - 2y^2 dy \wedge dx)}{(x^2+y^2)^2} \\ &= 0 \end{aligned}$$

So, ω is closed. But writing ω in polar coordinates and integrating around a circle centered at 0 in $\mathbb{R}^2 \setminus 0$ gives $\int_{S^1} \omega = 2\pi$. Since exactness implies conservativity by Stokes Theorem, ω cannot be exact. Note that writing ω in polar coordinates actually gives us $\omega = d\theta$, but that θ is not a continuous function on $\mathbb{R}^2 \setminus 0$. However, if we restrict the domain to the open upper half plane, for example, then θ is a smooth, continuous function, and ω is exact. We will later prove that all closed forms are locally exact, which perhaps gives an indication of why the presence of closed but not exact forms might give clues to the global structure of the manifold. Also, it can be shown that any closed 1-form ω is exact in any star-shaped region, by defining $f(x) = \int_{\gamma_x} \omega$, where the region is star-shaped with respect to c and γ_x is the straight-line path from c to x. Thus, the example above with $\mathbb{R}^2 \setminus 0$ indicates that the failure of ω to be exact perhaps results from the hole at the origin.

Definition 1.4. The n^{th} de Rham cohomology group of M (which is really a vector space over $\mathbb{R}$) is denoted $H^n(M)$, and defined by

$$H^n(M) = \{\text{closed n-forms on M}\} \big/ \{\text{exact n-forms on M}\}$$

Note that the set of closed n-forms equals the kernel of $d : A^n(M) \to A^{n+1}(M)$, and the set of exact n-forms equals the image of $d : A^{n-1}(M) \to A^n(M)$, by definition, so the exterior derivative is the boundary map on the cochain complex of the differential n-form modules. The equivalence class of a form ω, denoted $[\omega]$ in $H^n(M)$, is called the cohomology class of ω.

Definition 1.5. Let $G : M \to N$ be a smooth map, $p \in M$, and $X_1, ...X_k \in T_pM$. The pullback of G, denoted G^*, is defined on any k-form ω by $G^*(\omega)(p)(X_1, ...X_k) = \omega_{G(p)}(G_*X_1, ..., G_*X_k)$, where G_* is the pushforward of G.

Proposition 1.6. *If $G : M \to N$ is any smooth map, G^* takes closed forms to closed forms and exact forms to exact forms, and thus descends to a linear map, still denoted $G^* : H^n(N) \to H^n(M)$, which we define in the obvious way: $G^*[\omega] = [G^*\omega]$.*

Proof. Let ω be an exact form. Then $G^*\omega = G^*d\eta = d(G^*\eta)$ for some η. Similarly, if ω is closed, then $d(G^*\omega) = G^*d(\omega) = G^*0 = 0$. Thus if $[\omega] = [\omega'], G^*[\omega] = G^*[\omega']$, so G^* descends to a linear map on the cohomology groups. □

Since pullbacks of diffeomorphisms are isomorphisms $A^n(N) \to A^n(M)$, we see immediately that the de Rham cohomology groups of diffeomorphic manifolds are isomorphic. However, we will now prove that even homotopy equivalent manifolds have the same de Rham cohomology. First though, we will state without proof the following important results:

Theorem 1.7 (Whitney Approximation on Manifolds)**.** *If $F : M \to N$ is a continuous map between smooth manifolds, then F is homotopic to a smooth map $\tilde{F} : M \to N$. If F is smooth on a closed subset $A \subset M$, then the homotopy can be taken relative to A.*

Theorem 1.8. *If $F, G : M \to N$ are homotopic smooth maps, then they are smoothly homotopic. If they are homotopic relative to some subset $A \subset M$, then they are smoothly homotopic relative to A.*

The key idea in the proof that homotopy equivalent manifolds have the same de Rham groups is to show that homotopic maps give the same induced maps between cohomology groups, for then we have the following:

Theorem 1.9. *Let $F : M \to N$ be a homotopy equivalence between M and N, with homotopy inverse $G : N \to M$. Suppose that, for any two maps A and B , $\tilde{A} \simeq \tilde{B} \Rightarrow \tilde{A}^* = \tilde{B}^*$ on the cohomology groups, where $\tilde{A}$ and $\tilde{B}$ are defined as in the Whitney approximation theorem above. Then $\tilde{F}^*$ is an isomorphism, and thus $H^n(N) = H^n(M)$ for all n.*

Proof. By the Whitney approximation theorem, there exist smooth maps $\tilde{F}, \tilde{G}$ which are homotopic to F and G, respectively. Then $\tilde{F} \circ \tilde{G} \simeq F \circ G$ by the map $H = H_F \circ H_G$, where H_F is the homotopy between F and $\tilde{F}$ and similarly for H_G. Then $\tilde{F} \circ \tilde{G} \simeq Id_N$, so that $(\tilde{F} \circ \tilde{G})^* = Id_N^*$ by assumption. But $(\tilde{F} \circ \tilde{G})^* = \tilde{G}^* \circ \tilde{F}^*$ and $Id_N^* = Id_{H^n(N)}$. Similarly, $\tilde{G} \circ \tilde{F} \simeq Id_M \Rightarrow (\tilde{G} \circ \tilde{F})^* = Id_N^* = Id_{H^n(M)}$. Thus, $\tilde{F}^*$ is bijective and thus an isomorphism between $H^n(N)$ and $H^n(M)$ for all n. □

Now we show that homotopic maps give the same induced maps between cohomology groups. To do this, we need to show that, if $F \simeq G$, then for any closed form ω, $(F^* - G^*)\omega = d\eta$ for some n-1 form η. In other words, $(F^* - G^*)\omega$ is exact, and thus 0 in the quotient. To do this, we will construct a collection of linear maps so that for each n, there exists an $h : A^n(N) \to A^{n-1}(M)$ in the collection such that

$$d(h\omega) + h(d\omega) = (G^* - F^*)\omega$$

An h which satisfies this equation for all ω is called a homotopy operator, or a cochain homotopy. Since $d\omega$ is an $n+1$ form, h also needs to extend to a map $A^{n+1}(N) \to A^n(M)$. Note that this reduces to what we want in the case that ω

is closed. Following [Lee], we will present a special case of a homotopy operator first.

Definition 1.10. Let ω be an n-form on M and v a vector field on M. Then the contraction of v with ω, denoted $v \lrcorner \omega$, is an $(n-1)$-form defined at a point p by

$$v \lrcorner \omega(p)(X_1, ..., X_{n-1}) = \omega_p(v_p, X_1, ..., X_{n-1})$$

where $X_i \in T_pM$ for all i.

Lemma 1.11. *Let $i_t : M \to M \times I$ be the embedding $i_t(x) = (x, t)$. Then there exists a homotopy operator $h : A^n(M \times I) \to A^{n-1}(M)$ between i_0^* and i_1^*.*

As we saw in the definition above, one way to make an n-form ω on $M \times I$ into an n-1 form on $M \times I$ is to contract it with any vector field on $M \times I$. Since we then need to make it into a form on M, it makes sense to contract this n-form with $\frac{\partial}{\partial t}$ and then integrate over $t \in I$ to remove the dependence of ω on t. So, we define

$$h\omega = \int_0^1 (\frac{\partial}{\partial t} \lrcorner \omega) dt$$

where the action on vectors $X_1, ..., X_{n-1} \in T_pM$ is given by

$$h\omega(p)(X_1, ..., X_{n-1}) = \int_0^1 \omega_{(p,t)}(\frac{\partial}{\partial t}, X_1, ..., X_{n-1}) dt$$

Note that ω is either of the form $f(x, t)dt \wedge dx^I$, where I is a multi-index of length $p - 1$, or it has no dt component, and has the form $f(x, t)dx^J$, where J is a multi-index of length p.

The proof that h is indeed a homotopy operator then consists of expanding out the definition of $d(h\omega) - h(d\omega)$ and showing that it equals $(i_1^* - i_0^*)\omega$ for both cases. The proof can be found in [Lee].

Proposition 1.12. *If $F, G : M \to N$ are homotopic smooth maps, then $F^* = G^*$, where F^* and G^* are the induced maps on the cohomology classes.*

Proof. Let h and i_t be the homotopy operator and embedding, respectively, defined above. Let H be a smooth homotopy between F and G (which exists because F and G are smooth, by theorem 1.8), and H^* its pullback. Define a new map $g : A^n(N) \to A^{n-1}(M)$ by

$$g = h \circ H^* = h(H^*\omega) = \int_0^1 \frac{\partial}{\partial t} \lrcorner (H^*\omega) dt$$

We then have that

$$g(d\omega) + d(g\omega) = h(H^*d\omega) + d(h(H^*\omega))$$

Since pullbacks commute with d, we have

$$= hd(H^*\omega) + d(h(H^*\omega))$$

By definition of h being a homotopy operator between i_0^* and i_1^*,

$$\begin{aligned} &= i_1^*(H^*\omega) - i_0^*(H^*\omega) \\ &= (H \circ i_1)^*\omega - (H \circ i_0)^*\omega = (G^* - F^*)\omega \end{aligned}$$

So, when ω is closed and we descend to the quotient, we have

$$\begin{aligned}(F^* - G^*)[\omega] &= [F^*\omega - G^*\omega] \\ &= [g(d\omega) + d(g\omega)] \\ &= [d(g\omega)] = [0]\end{aligned}$$

□

2. Zigzag Lemma and Mayer-Vietoris

Now we will define a useful tool in actually computing de Rham cohomology groups, namely the Mayer-Vietoris Theorem.

Definition 2.1. Let A^* and B^* be cochain complexes (i.e. sequences of modules such that $d^2 = 0$, where d is the boundary map). A cochain map $F : A^* \to B^*$ is a collection of linear maps (which we'll also denote by F) $F : A^n \to B^n$ such that for each n, $d \circ F = F \circ d : A^n \to B^{n+1}$.

Since each F commutes with d and is linear, F induces a linear map F^* on the cohomology groups H^n (with the maps between individual cohomology groups also denoted by F^*, as above).

Definition 2.2. An exact sequence is one in which $\mathrm{im}(d : A^{n-1}(M) \to A^n(M)) = \ker(d : A^n \to A^{n+1})$.

Definition 2.3. A short exact sequence (SES) of complexes consists of three complexes, A^*, B^*, and C^*, along with maps $F^* : A^* \to B^*$ and $G^* : B^* \to C^*$ such that, for each n, the sequence

$$0 \to A^n \xrightarrow{F} B^n \xrightarrow{G} C^n \to 0 \tag{2.4}$$

is exact.

Lemma 2.5 (Zigzag Lemma)**.** *Given cochain complexes A^*, B^* and C^*, and cochain maps F^* and G^* as above, such that they form an SES of complexes, (i.e. are a SES for each n), then there exists a linear map δ such that*

$$\cdots H^{n-1}(C^*) \xrightarrow{\delta} H^n(A^*) \xrightarrow{F^*} H^n(B^*) \xrightarrow{G^*} H^n(C^*) \xrightarrow{\delta} H^{n+1}(A^*) \xrightarrow{F} H^{n+1}(B^*) \to \cdots$$

is an exact sequence.

Proof. We will give a sketch of the proof which illustrates the main idea. Let c^n be a cohomology class in C^n. Since the map G into C^n is surjective by our SES hypothesis, there exists some $b^n \in B^n$ such that $G(b^n) = c^n$. Since G commutes with d, $G(db^n) = dG(b^n) = dc^n = 0$ since c^n is closed. Thus $db^n \in \mathrm{Ker}G = \mathrm{Im}F$. So, there exists $a^{n+1} \in A^{n+1}$ such that $F(a^{n+1}) = db^n$. This a^{n+1} is closed, since $F(da^{n+1}) = dF(a^{n+1}) = ddb^n = 0$ and the fact that F is injective imply that $da^{n+1} = 0$. Thus $[a^{n+1}]$ represents a cohomology class in $H^{n+1}(A^*)$. We can then define $\delta[c^n] = [a^{n+1}]$. In other words, a^{n+1} is any element of A^{n+1} for which there exists a corresponding $b^n \in B^n$ which satisfies the equations

$$Fa^{n+1} = db^n$$

and

$$Gb^n = c^n$$

It can be shown that this selection of $[a^{n+1}]$ is well-defined, makes δ linear, and makes the corresponding cohomology sequence exact. The details of the proof can be found in [Hatcher]. $\square$

Note that this lemma is true in general for any cohomology (or homology), where $H^n(X^*)$ is the kernel of the boundary operator modulo its image. In de Rham cohomology, A^*, B^* and C^* are sequences of vector spaces of differential n-forms on subspaces of the manifold, and our boundary operator is d.

Let M be a smooth manifold which is covered by the open sets U and V. Consider the inclusion maps $k : U \to M$, $l : V \to M$, $i : U \cap V \to U$ and $j : U \cap V \to V$. These maps induce cohomology maps, which we will denote k^*, l^*, etc., which are just the restrictions of classes in the cohomology groups to the subspaces U, V, etc.

Let $k^* \oplus l^* : A^n(M) \to A^n(U) \oplus A^n(V)$ be defined by $(k^* \oplus l^*)\omega = (k^*\omega, l^*\omega)$, and let $(i^* - j^*) : A^n(U) \oplus A^n(V) \to A^n(U \cap V)$ be defined by $(i^* - j^*)(\omega, \eta) = i^*\omega - j^*\eta$. Because restriction is linear, and pullbacks commute with d, these maps descend to linear maps on the cohomology groups, which we will write the same way.

Theorem 2.6 (Mayer-Vietoris)**.** *Let U, V be as above. Let $k^* \oplus l^* : H^n(M) \to H^n(U) \oplus H^n(V)$ and $(i^* - j^*) : H^n(U) \oplus H^n(V) \to H^n(U \cap V)$ be the maps defined above. Then for each n there exists a linear map $\delta : H^n(U \cap V) \to H^{n+1}(M)$ such that the sequence*

$$\cdots \xrightarrow{\delta} H^n(M) \stackrel{k^* \oplus l^*}{\to} H^n(U) \oplus H^n(V) \stackrel{i^* - j^*}{\to} H^n(U \cap V) \xrightarrow{\delta} H^{n+1}(M) \to \cdots$$

is exact.

Proof. Using the zigzag lemma, it suffices to show that, for each n, the sequence

$$0 \to A^n(M) \stackrel{k^* \oplus l^*}{\to} A^n(U) \oplus A^n(V) \stackrel{i^* - j^*}{\to} A^n(U \cap V) \to 0$$

is an exact sequence. Since k^* is just the restriction of a form to U, and l^* is the restriction to V, it follows that any form in the kernel of $k^* \oplus l^*$ must be 0 on both U and V, thus 0 on all of M. This proves exactness at $A^n(M)$. At the next step of the sequence, we note that restricting a form to U, then further restricting it to $U \cap V$, is the same as restricting it to V then $U \cap V$. Thus subtracting the resulting forms is always zero. On the other hand, any element ω which $i^* - j^*$ takes to 0 is the direct sum of a form on U and a form on V which agree on the intersection. But then this defines a single form on M, whose image under $k^* \oplus l^*$ is exactly ω. Thus we have exactness at $A^n(U) \oplus A^n(V)$. Finally, we need to show exactness at $A^n(U \cap V)$, which just means showing $i^* - j^*$ is surjective. Let $\{\phi, \psi\}$ be a partition of unity subordinate to $\{U, V\}$, respectively. Given a form ω on $U \cap V$, define a form η on U by

$$\eta = \begin{cases} \psi\omega & \text{on } U \cap V \\ 0 & \text{on } U \setminus \operatorname{supp}(\psi) \end{cases}$$

and a form η' on V by

$$\eta' = \begin{cases} \phi\omega & \text{on } U \cap V \\ 0 & \text{on } V \setminus \operatorname{supp}(\phi) \end{cases}$$

Then we have that $i^* - j^*(\eta \oplus -\eta') = \eta|_{U \cap V} + \eta'|_{U \cap V} = \phi\omega + \psi\omega = \omega$, as desired. $\square$

Using the description in the zigzag lemma and our construction in the last part of the Mayer-Vietoris theorem, we can define δ explicitly. By the proof of the zigzag lemma, we know that $\delta[\omega] = [\sigma]$ if and only if there exists $(\eta, \eta') \in A^n(U) \oplus A^n(V)$ such that $i^*\eta - j^*\eta' = \omega$ and $(k^* \oplus l^*)(\sigma) = (d\eta, d\eta')$. We can satisfy the first requirement by defining, for any partition of unity $\{\phi, \psi\}$ subordinate to $\{U, V\}$, η and η' exactly as we did in the proof of the Mayer Vietoris theorem. We can then see that $\omega = \eta|_{U\cap V} - \eta'|_{U\cap V}$ (remember that i^* and j^* are just restrictions to $U \cap V$). If we then extend $d\eta$ to all of M by having it be 0 outside of $U\cap V$, then $\sigma = d\eta$ satisfies the second equation. This is because on $U \cap V$, $d\eta|_{U\cap V} = d(\omega + \eta')|_{U\cap V} = d\eta'|_{U\cap V}$, where the last equality comes because ω is closed. So, restricting $d\eta$ to V is the same as restricting $d\eta'$ to V, since they agree on the intersection of U and V and are 0 elsewhere. Thus we can write explicitly that $\delta[\omega] = [d\eta]$.

3. Some Calculations

Note that the 0^{th} de Rham cohomology group is just the set of closed 0-forms. But closed 0-forms are just smooth functions such that $df = 0$, which implies that they are locally constant. If M is connected, then any such function must be globally constant. Hence:

Proposition 3.1. *If M is connected, then*

$$H^0(M) = \{\textit{the set of constant functions}\} \cong \mathbb{R}$$

Similarly, if M is a zero dimensional manifold (i.e. is just a set of discrete points), then $H^0(M) \cong \prod_{\alpha \in A} \mathbb{R}$ where the cardinality of A is the cardinality of M. All higher cohomology groups are trivial, since if ω is any differential n-form on a manifold of dimension $p < n$, the dimension of the tangent space at any point is p, thus any n vectors will be linearly dependent, making ω zero.

Proposition 3.2 (Poincare Lemma)**.** *If U is any contractible subset of $\mathbb{R}^n$, then $H^m(U) = 0$ for any $m \geq 1$.*

Proof. Any contractible subset of $\mathbb{R}^n$ is homotopy equivalent to a point, by definition. Since the de Rham cohomology groups are homotopy invariant, they are isomorphic to those of a point. We showed above that, in this case, $H^0(U) \cong \mathbb{R}$, and $H^m(U) = 0$ for all $m \geq 1$. □

Note that this tells us the cohomology for Euclidean space in general.

Corollary 3.3 (All Closed Forms are Locally Exact)**.** *Any closed form of degree greater than or equal to 1 is exact on some open neighborhood around each point.*

Proof. At every point on a smooth manifold, there exists an open ball B around that point which is diffeomorphic to $\mathbb{R}^n$. Since $\mathbb{R}^n$ is star-shaped, it has trivial cohomology groups H^n for all $n \geq 1$. Thus so does B by homotopy invariance (and in particular, diffeomorphism invariance) of the de Rham cohomology groups. Thus all closed forms on B are exact. □

Upon seeing the previous lemma, the reader may wonder why, if all closed forms ω have some local potential η such that $d\eta = \omega$, it isn't always possible to simply use a partition of unity to patch together a global potential for ω. One reason for this is that if U and V are overlapping open sets with η and η' the respective local potentials for ω, they need not be equal in $U \cap V$. This is because the requirement that $d\eta = d\eta'$ only implies that $d(\eta - \eta') = 0$, in other words, that $\eta - \eta'$ is closed, not that $\eta = \eta'$. Thus, where they overlap, we have some potential $\phi\eta + \psi\eta'$ where we cannot just factor out η, and thus the exterior derivative of this patched-together potential need not be ω.

Next we show a connection between the first de Rham cohomology group and the fundamental group. The following lemma will be stated without proof (a proof can be found in [Lee]):

Lemma 3.4. *If M is a smooth manifold, ω is a closed 1-form, and γ_1 and γ_2 are smooth, path homotopic paths between any two points on M, then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{3.5}$$

Theorem 3.6. *If M is a simply-connected smooth manifold, then $H^1(M) = 0 = \pi_1(M)$.*

Proof. Since closed 1-forms are exact if and only if they're conservative, it suffices to show that the integral of a 1-form around any closed loop is zero. But since M is simply connected, any closed loop is path homotopic to the constant loop. Thus, by the previous lemma, the integral around any loop is just the constant integral, which is zero. Thus all closed 1-forms on M are conservative, which implies they are exact. □

Now we'll use Mayer-Vietoris to do a simple calculation of the cohomology groups of S^1 and S^2 as a stepping stone to more general calculations.

Example 3.7 (Cohomology of S^1)**.** Let U and V be open sets which cover everything except for the north pole, and everything except for the south pole, respectively. Then U and V form an open cover of S^1, so we can apply Mayer-Vietoris. We then have the following exact sequence, for some map δ:

$$0 \to H^0(S^1) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \xrightarrow{\delta}$$
$$\xrightarrow{\delta} H^1(S^1) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to 0$$

Since U and V are both punctured circles, they're homotopy equivalent (in fact homeomorphic) to $\mathbb{R}$. Since $\mathbb{R}$ is star-shaped, $H^0(\mathbb{R}) = H^0(U) = H^0(V) = \mathbb{R}$. Since S^1 is connected, $H^0(S^1) \cong \mathbb{R}$. Since $U \cap V$ is retracts to two discrete points, $H^0(U \cap V) \cong \mathbb{R} \oplus \mathbb{R}$. Since U and V are star-shaped, their first cohomology group is 0, and since $U \cap V$ retract to points, which have dimension 0, the first cohomology group of $U \cap V$ is also 0. The image of δ is equal to the kernel of the map from $H^1(S^1)$ to $H^1(U) \oplus H^1(V)$, but $H^1(U) \oplus H^1(V) = 0$, so δ is surjective. The kernel of δ is equal to the image of the map from $H^0(U) \oplus H^0(V)$ to $H^0(U \cap V)$. But recall that this map is just the subtraction map, and since the difference of two constant functions is a constant function, its image is the set of constant functions, which is isomorphic to $\mathbb{R}$. Thus $H^1(S^1) \cong \mathbb{R}$. So, we have that $H^0(S^1) = \mathbb{R}$ and $H^1(S^1) = \mathbb{R}$

Example 3.8 (Cohomology of S^2)**.** Now that we have S^1, we can solve for the cohomology groups of S^2, using the same methods as above. We will use as our cover two open sets U and V which cover everything except the north and south pole, respectively. Note that each of these open sets is just a punctured S^2, which is homeomorphic to $\mathbb{R}^2$, and that their intersection retracts to S^1. This gives us the following sequence:

$$(H^0(S^2) \cong \mathbb{R}) \to (H^0(U) \oplus H^0(V)) \cong (\mathbb{R} \oplus \mathbb{R}) \to (H^0(U \cap V \simeq S^1) \cong \mathbb{R}) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} (H^1(S^2) \cong 0) \to (H^1(U) \oplus H^1(V)) \cong 0 \to (H^1(S^1) \cong \mathbb{R}) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} (H^2(S^2) \cong \mathbb{R}) \to (H^2(U) \oplus H^2(V)) \cong 0 \to (H^2(S^1) \cong 0)$$

Now we can use induction to find the cohomology groups of S^n. Our calculations for S^2 tell us that we should use

$$H^p(S^m) = \begin{cases} \mathbb{R} & \text{if } p = 0, m \\ 0 & \text{otherwise} \end{cases}$$

as our inductive hypothesis for $m < n$.

Example 3.9 (Cohomology of S^n)**.** Suppose that we know the cohomology groups of S^m for $m < n$. Analogously with S^2, we'll use as our cover for S^n two open sets U and V which cover everything except the north and south pole, respectively. Note that each of these open sets is just a punctured S^n, which is homeomorphic to $\mathbb{R}^n$. Thus $H^p(U) = H^p(V) = 0$ for all $p > 0$, by the Poincare lemma. Also, $U \cap V$ is S^n with two holes in it, which is homeomorphic to $\mathbb{R}^n$ with one hole in it, which retracts onto S^{n-1}. Thus $H^p(U \cap V) \cong H^p(S^{n-1})$ for all $p \leq n$. So, we know all the cohomology groups of everything except for $H^p(S^n)$ itself, for p between 1 and n. But we can solve for those using methods similar to the S^1 case because our sequence is exact, and we know the images and kernels of all the maps in the sequence.

The map from $H^0(S^n)$ to $H^0(U) \oplus H^0(V)$ has trivial kernel and image isomorphic to $\mathbb{R}$. Thus $H^0(S^n) \cong \mathbb{R}$. The map δ from $H^0(S^{n-1})$ to $H^1(S^n)$ is surjective since the map from $H^1(S^2)$ to $H^1(U) \oplus H^1(V)$ is trivial (since $H^i(U) \oplus H^i(V) = 0$ for all $i > 0$ by the above). In fact, this is true for all the δ maps in our sequence, for the same reason. The kernel of this map is the image of the map which just the subtracts the two constant functions, so it has kernel isomorphic to $\mathbb{R}$. But that's exactly what $H^0(S^{n-1})$ is, by assumption, so it must be the trivial map. Thus $H^1(S^n) = 0$. For $1 < p < n$, the maps going into and out of $H^p(S^n)$ are all trivial, hence $H^p(S^n)$ is trivial. Finally, for $H^n(S^n)$, the δ map is again surjective, and since $H^{n-1}(S^{n-1}) \cong \mathbb{R}$ by assumption, and the image of the subtraction map into $H^{n-1}(S^{n-1})$ is just 0, the last δ map has trivial kernel and thus is an isomorphism. So, $H^n(S^n) \cong \mathbb{R}$, as we wished to prove.

Now we'll compute the cohomology of the torus, and use that to compute the cohomology of any genus g surface. First we give two lemmas:

Lemma 3.10. *If M is a compact, connected, oriented, smooth n-manifold, $H^n(M)$ is 1-dimensional.*

The proof involves showing that the integration map $I : H^n(M) \to \mathbb{R}$ defined by $I([\omega]) = \int_M \omega$ is an isomorphism. The proof can be found in [Lee].

Lemma 3.11. *If M is a compact, connected, orientable, smooth manifold of dimension $n \geq 2$, and p is any point of M, then $H^n(M \setminus p) = 0$.*

Proof. We will prove this explicitly for $n = 2$ since that's all we need for the next example, but the proof generalizes to higher dimensions. Let U, V be an open cover of our surface, consisting of $M \setminus p$ and some neighborhood of p, respectively. First we will show that the map $\delta : H^1(U \cap V) \to H^2(M)$ is an isomorphism, then we will use Mayer-Vietoris to find $H^2(U)$. Note that $U \cap V$ is diffeomorphic to $\mathbb{R}^2 \setminus 0$, thus the pullback of the diffeomorphism gives an isomorphism on the space of k-forms for all k, and in particular the cohomology groups. So, instead of working with forms on $U \cap V$, we can simply use $\mathbb{R}^2 \setminus 0$ instead. Since $U \cap V$ is homotopy equivalent to S^1, we know that $H^1(U \cap V) \cong \mathbb{R}$. By lemma 3.10 above, $H^2(M) \cong \mathbb{R}$ as well. Since they're both 1-dimensional, it suffices to find some 1-form ω on $\mathbb{R}^2 \setminus 0$ such that $\delta([\omega]) \neq 0$. In the general case, we would begin by considering the form

$$\omega = \frac{1}{|x|^2} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge ... \wedge \hat{dx^i} \wedge ... \wedge dx^n$$

defined on $\mathbb{R}^n \setminus 0$, where the hat term is dropped. In the case $n = 2$, this reduces to

$$\omega = \frac{xdy - ydx}{(x^2 + y^2)}$$

which we showed at the beginning of this paper is closed but not exact, hence an element of $H^1(\mathbb{R}^2 \setminus 0)$. As shown earlier in our explicit description of δ, $\delta([\omega]) = [d\eta]$ extended by 0 to all of M, where we use the exact same characterization of η as before ($\eta = \psi\omega$ on $U \cap V$ and 0 on $U \setminus \text{supp}(\psi)$). We will identify the bump function ψ on M with its image on $\mathbb{R}^2$ under the diffeomorphism $V \mapsto \mathbb{R}^2$. Then we can say that $\psi(0) = 1$, and $\psi(x)$ approaches 0 as x approaches ∞. Then

$$d\eta = d(\psi\omega) = d\psi \wedge \omega$$

since ω is closed.

$$= (\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy) \wedge (\frac{1}{|x|^2})(xdy - ydx)$$
$$= \frac{1}{|x|^2} (\frac{\partial \psi}{\partial x} x + \frac{\partial \psi}{\partial y} y) dx \wedge dy$$

This form is defined on all of $\mathbb{R}^2 \setminus 0$, and thus all of U (since it's just 0 in $U \setminus U \cap V$. We extend it to include 0, and thus all of V, by defining it to be 0 at 0. This makes $d\eta$ into a smooth 2-form on all of $\mathbb{R}^2$. One can check that it is smooth at 0 because, after taking any number of partial derivatives of

$$\frac{\frac{\partial \psi}{\partial x} x + \frac{\partial \psi}{\partial y} y}{x^2 + y^2}$$

l'Hospital's rule and the fact that ψ is a bump function whose partials of all orders vanish at 0 give that the limit at 0 as approached from any direction is 0 (i.e., if we approach from the direction v, apply l'Hospital's rule twice using $\frac{\partial}{\partial v}$ as the derivative).

$d\eta$ is closed, since $d \circ d = 0$. It is not exact since any potential for $d\eta$ must be of the form $\eta + \tau$ for some closed form τ. This is because, if β is any potential for $d\eta$, then $d\beta = d\eta \Rightarrow d(\eta - \beta) = 0 \Rightarrow \eta - \beta = \tau$ for some closed form τ. But since $\psi(0) = 1$, $\eta = \psi\omega$ approaches ∞ as $x \longrightarrow 0$. Thus there cannot exist any smooth potential for $d\eta$ defined at 0, which implies there is no smooth potential defined at

$p \in M$, and thus $d\eta$ is not exact. So, $\delta([\omega]) = [d\eta] \neq 0 \Rightarrow \delta$ is an isomorphism.

Now we can use Mayer-Vietoris to complete the proof. The last line of the exact sequence for M is:

$$H^1(U \cap V) \xrightarrow{\delta} H^2(M) \to H^2(U) \oplus H^2(V) \to H^2(U \cap V)$$

Which we know is isomorphic to:

$$\mathbb{R} \xrightarrow{\delta} \mathbb{R} \to H^2(U) \oplus 0 \to 0$$

Since δ is an isomorphism, its image is $\mathbb{R}$, so the direct sum map has kernel $\mathbb{R}$, thus its image is 0. That means the subtraction map into 0 has trivial kernel, so it is an injection. But that implies $H^2(U)$ must be 0. □

Proposition 3.12. *The de Rham cohomology groups of the torus are given by:*

$$H^i(T^2) = \begin{cases} \mathbb{R} & \text{if } i = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & \text{if } i = 1 \end{cases}$$

Proof. Imagine dunking a donut vertically into a cup of coffee, so that a little more than half of it is submerged. The submerged part is what each of our open sets U, V will look like (except on opposite sides of the torus, so that they overlap in the middle). Then $U \cap V$ retracts to two disjoint copies of S^1, which has cohomology groups that are just the direct sum of the groups for S^1, which we know. Also, both U and V retract to S^1, so they both have cohomology groups the same as those for S^1. Given that, we see that $H^0(T^2) \cong \mathbb{R}$ since the torus is path connected. We also have $H^1(T^2) \cong \mathbb{R} \oplus \mathbb{R}$ since δ has kernel isomorphic to $\mathbb{R}$ (because as usual the subtraction map $(H^0(U) \oplus H^0(V)) \to (H^0(U \cap V)$ has image $\mathbb{R}$), and thus (by the rank-nullity theorem) has image isomorphic to $\mathbb{R}$. This implies that the kernel of the direct sum map from $H^1(T^2)$ to $H^1(U) \oplus H^1(V)$ is isomorphic to $\mathbb{R}$. By lemma 3.10, $H^2(T^2) \cong \mathbb{R}$, thus the surjective map δ going into $H^2(T^2)$ has kernel $\mathbb{R}$, again by rank-nullity. This implies that the image of the subtraction map into $H^1(U \cap V)$ is $\mathbb{R}$, and thus its kernel is $\mathbb{R}$, since its domain is $\mathbb{R}^2$. Exactness tells us that the image of the direct sum map is then also $\mathbb{R}$. But the only way the image and kernel of the direct sum map could be $\mathbb{R}$ is if $H^1(T^2) \cong \mathbb{R} \oplus \mathbb{R}$. □

Proposition 3.13. *The de Rham cohomology groups of the punctured torus are given by:*

$$H^i(\tilde{T}^2) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ \mathbb{R}^2 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Proof. Start with a non-punctured torus. Let U be the open set covering the whole torus minus a point (it doesn't matter where, since all we care about is homotopy equivalence), and V a small open ball over the point not contained in U. Then since we know $H^k(V \cong \mathbb{R}^2)$, $H^k(T^2)$, and $H^k(U \cap V \simeq S^1)$ for all k, we know every term in the resulting Mayer-Vietoris sequence except for the $H^k(U)$ terms. But since U is connected, we know $H^0(U) \cong \mathbb{R}$ and by lemma 3.11 above, $H^2(U) = 0$. Thus we only need to find $H^1(U)$ to know all the cohomology groups of U, the punctured torus. The subtraction map from $H^0(U) \oplus H^0(V)$ to $H^0(S^1)$ has image isomorphic to R, so that the δ map into $H^1(T^2)$ is trivial. Then the following direct sum map is an injection, which tells us $H^1(U)$ has dimension at least 2. But the subtraction map going into $H^1(S^1)$ then has kernel isomorphic to $\mathbb{R}^2$, and image isomorphic to either 0 or $\mathbb{R}$ (since $H^1(U \cap V)$ is 1-dimensional). So, $H^1(U)$ can have dimension at most 3. But if it were 3, then the subtraction map must be surjective, which implies that the following δ map into $H^2(T^2)$ is trivial, which means that the following direct sum map into $0 \oplus 0$ must be an injection. But we know $H^2(T^2) \cong \mathbb{R}$, so this is impossible. Thus, $H^1(U) \cong \mathbb{R}^2$. □

Proposition 3.14. *The de Rham cohomology groups of the genus-g surface are given by:*

$$H^i(\Sigma_g) = \begin{cases} \mathbb{R} & \text{if } i = 0, 2 \\ \mathbb{R}^{2g} & \text{if } i = 1 \end{cases}$$

Proof. We saw with the torus that the cohomology groups were $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}$. Using the groups we calculated for the punctured torus above, it can be shown that the cohomology groups for the genus-2 surface are $\mathbb{R}$, $\mathbb{R}^4$, and $\mathbb{R}$. The cohomology groups of any genus-g surface Σ_g can now be calculated by using the open sets which separate it into spaces homotopy equivalent to a punctured torus and a punctured genus-(g-1) surface (denoted $\tilde{\Sigma}_{g-1}$), with the intersection homotopy equivalent to S^1. To find a general formula then, we'll use induction on g. The examples above suggest that the pattern for the de Rham groups of a genus-g surface is $\mathbb{R}$, $\mathbb{R}^{2g}$, $\mathbb{R}$ (for H^0, H^1 and H^2, respectively). Assuming this is true for all $p < g$, we will use Mayer-Vietoris with a covering similar to the one for the punctured torus to show that the de Rham groups for the punctured genus-(g-1) surface must then be $\mathbb{R}$, $\mathbb{R}^{2(g-1)}$, and 0.

We know that $H^0(\tilde{\Sigma}_{g-1}) \cong \mathbb{R}$ since it is connected, and that $H^2(\tilde{\Sigma}_{g-1}) = 0$ by lemma 3.11. Since the δ map into $H^2(\Sigma_{g-1})$ is an isomorphism (from the proof of lemma 3.11), the image of the map into $H^1(U \cap V)$ is trivial, and thus its kernel is $H^1(\tilde{\Sigma}_{g-1}) \oplus (H^1(V) = 0)$. This implies the direct sum map into $H^1(\tilde{\Sigma}_{g-1})$ is surjective. Since the subtraction map into $H^0(S^1)$ has image isomorphic to $\mathbb{R}$ as usual, the δ map into $H^1(\Sigma_{g-1})$ has trivial image, and thus the direct sum map is injective as well. This tells us $H^1(\tilde{\Sigma}_{g-1}) \cong H^1(\Sigma_{g-1})$, which is isomorphic to $\mathbb{R}^{2(g-1)}$ by assumption.

By prop. 3.1 and lemma 3.10, $H^0(\Sigma_g) \cong \mathbb{R}$ and $H^2(\Sigma_g) \cong \mathbb{R}$. Our cohomology sequence for Σ_g looks like

$$H^i(\Sigma_g) \to H^i(\tilde{T^2}) \oplus H^i(\tilde{\Sigma}_{g-1}) \to H^i(S^1)$$

for $i = 0, 1, 2$, where our open sets are those mentioned above. By our previous work, we know all of these cohomology groups for all i, except for $H^1(\Sigma_g)$. Note that the δ map into $H^1(\Sigma_g)$ is trivial, making the direct sum map into $(H^1(\tilde{T^2}) \oplus H^1(\tilde{\Sigma}_{g-1})) \cong \mathbb{R}^2 \oplus \mathbb{R}^{2(g-1)} \cong \mathbb{R}^{2g}$ injective. Thus the image of this map is isomorphic to $H^1(\Sigma_g)$. Its image is equal to the kernel of the subtraction map into $H^1(S^1)$. Since $H^1(S^1) \cong \mathbb{R}$, by the rank-nullity theorem the kernel can only be either $\mathbb{R}^{2g}$ or $\mathbb{R}^{2g-1}$. Suppose it is $\mathbb{R}^{2g-1}$. Then the subtraction map's image is isomorphic to $\mathbb{R}$, so the following δ map into $H^2(\Sigma_g)$ is trivial. This implies that the next direct sum map into $(H^2(\tilde{T^2}) \oplus H^2(\tilde{\Sigma}_{g-1})) = 0$ is injective, but this is impossible since $H^2(\Sigma_g) \cong \mathbb{R}$. So, we must have that $H^1(\Sigma_g) \cong \mathbb{R}^{2g}$. This completes the induction.

□

4. Connections and Characteristic Classes

Now we'll take a look at certain special equivalence classes in the de Rham cohomology groups which come from invariant polynomials, called characteristic classes. These will tell us various things, though it is beyond the scope of this paper to say what those things might be.

Let TM be the tangent bundle over a manifold M, and let T^*M denote the complexified cotangent bundle (i.e. the tensor product of the usual cotangent bundle with $\mathbb{C}$, defined pointwise). Then the complex tensor product $T^*M \otimes TM$ is also a vector bundle over M.

Definition 4.1. Let $C^\infty(T^*M \otimes TM)$ denote the vector space of smooth sections of this bundle. A connection

$$\nabla : C^\infty(TM) \to C^\infty(T^*M \otimes TM)$$

is a $\mathbb{C}$-linear map which satisfies the Leibnitz formula:

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for every smooth function $f : M \to \mathbb{C}$ and vector field s. $\nabla(s)$ is called the covariant derivative of s.

Lemma 4.2. *We can define ∇'s action on any section by its action on a local frame $s_1, ...s_n$ by the formula*

$$\nabla \sum (f_i s_i) = \sum \nabla(f_i s_i)$$

where nabla obeys the Leibnitz rule on each $f_i s_i$

Proof. We just need to show that ∇ still obeys the Liebnitz rule on fs. In other words, if we write $s = \sum g_i s_i$, then $\nabla(fs) = \nabla \sum (f g_i s_i)$, where we use the first definition on the left, and the second definition on the right. Expanding the right side, we get:

$$\begin{aligned} \nabla \sum (f g_i s_i) &= \sum d(f g_i) \otimes s_i + f g_i \nabla(s_i) \\ &= \sum \big((g_i df + f dg_i) \otimes s_i + \sum f g_i \nabla(s_i) \big) \end{aligned}$$

We can distribute the s_i in the first part of the sum, and since by definition $\sum g_i \nabla(s_i) = \nabla(s) - \sum dg_i \otimes s_i$, we can rewrite the second part, giving:

$$\begin{aligned} &= df \otimes \sum g_i s_i + f \sum (dg_i \otimes s_i) + f\nabla(s) - f \sum dg_i \otimes s_i \\ &= df \otimes s + f\nabla(s) \\ &= \nabla(fs) \end{aligned}$$

□

Note that ∇ decreases supports: if a section s vanishes in some region, then $\nabla(s)$ vanishes as well. It is a theorem of Peetre that any operator $C^\infty(A) \to C^\infty(B)$ which decreases supports (also called a local operator) can be written locally as a finite linear combination of partial derivatives with coefficients in $C^\infty(B)$ (see [Milnor]). In addition to the fact that it satisfies the Liebnitz formula, this gives some justification as to why ∇ can be thought of as the total derivative of a section.

Since ∇ is a linear map, by the above lemma we have the following:

Lemma 4.3. *Let U be some small open set on M such that the tangent bundle over U is trivial. Then there exists a local frame $s_1, ..., s_n$, defined on U. ∇ restricted to sections on U is then uniquely determined by $\nabla(s_1), ..., \nabla(s_n)$. We can then write ∇ as an $n \times n$ matrix of 1- forms $[\omega_{ij}]$, where the multiplication is the tensor product over $\mathbb{C}$.*

This gives us another way to see that ∇ is essentially a derivative. Define $\nabla_{x^i}(fs) = \nabla(fs)x^i$, where x^i is the i^{th} section in some local frame. Then we have that $\nabla_{x^i}(fs) = \frac{\partial f}{\partial x^i} dx^i \otimes s + f\nabla_{x^i}(s)$, which we can think of as the directional derivative of the section s in the direction of the constant vector field x^i.

Note that this conception of ∇ as a matrix is a bit unusual in that ∇ is linear over $\mathbb{C}$, but the entries in the matrix are not elements of the field; instead they are 1-forms. Fortunately we can avoid having two multipications by noting that the tensor product of an element of $\mathbb{C}$ with a section is just the usual scalar multiplication. Given a basis $s_1, ...s_n$, a connection whose matrix is just the zero matrix is called a flat connection. In this case, the Leibnitz formula reduces to $\nabla \sum(f_i s_i) = \sum df_i \otimes s_i$. Note that, because connections are not linear over $C^\infty(M)$, the matrix of a flat connection is not necessarily zero in other bases.

Definition 4.4. Given a connection ∇, define a $\mathbb{C}$-linear map $\hat{\nabla} : C^\infty(T^*M \otimes TM) \to C^\infty(\Lambda^2(M) \otimes TM)$ by the Leibnitz rule

$$\hat{\nabla}(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s)$$

for any 1-form ω and any $s \in C^\infty(TM)$.

Lemma 4.5. *As with ∇, we can define $\hat{\nabla}$'s action on any section by defining its action on a local frame $s_1, ...s_n$ by the formula*

$$\hat{\nabla} \sum \omega_i \otimes s_i = \sum \hat{\nabla}(\omega_i \otimes s_i)$$

where $\hat{\nabla}$ acts on each $\omega_i \otimes s_i$ by the Leibnitz rule.

The proof just involves writing out the definition and gathering terms, and follows from the second definition of ∇ above. From this we also get that

$$\hat{\nabla}(f(\omega \otimes s)) = df \wedge (\omega \otimes s) + f\hat{\nabla}(\omega \otimes s). \tag{4.6}$$

Definition 4.7. Given some connection ∇, let $K = \hat{\nabla} \circ \nabla : C^\infty(TM) \to C^\infty(\Lambda^2(M) \otimes TM)$. Then K is called the curvature tensor of ∇.

Lemma 4.8. *The value of $K(s)$ at a point x depends only on $s(x)$, and not the values s takes at nearby points. Thus if $s(x) = s'(x)$, then $K(s)(x) = K(s')(x)$. We can therefore think of K as a section of the complex vector bundle $Hom(TM, \Lambda^2(M) \otimes TM)$.*

Proof. Note that

$$\begin{aligned}
\hat{\nabla} \circ \nabla(fs) &= \hat{\nabla}(df \otimes s + f\nabla(s)) \\
&= d(df) \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f\hat{\nabla}(\nabla(s)) \\
&= f\hat{\nabla}(\nabla(s))
\end{aligned}$$

where the last two terms come from equation 4.6 above. This tells us that K is linear over $C^\infty(M)$. So, if $s(x) = s'(x)$, then we can write $s - s' = f_1 s_1 + ... + f_n s_n$, where $s_1, ...s_n$ is any local basis and $f_i(x) = 0$ for all i. But then

$$K(s - s') = K(s) - K(s') = \sum f_i K(s_i)$$

which is just 0 when evaluated at x. Sections, in general, assign to each point x on the manifold an element of the fiber over x, and every such assignment is completely determined by x. So, we can think of K as a section of the bundle $\mathrm{Hom}(TM, \Lambda^2(M) \otimes TM)$ by defining $K(s(x))$ as the element K_x of the Hom bundle evaluated at $s(x)$, thus giving us an element of $\Lambda^2(M) \otimes TM$. □

Since K is linear over $\mathbb{C}$, we can express it in terms of a matrix, just as we did for ∇. In fact, since K is also linear over $C^\infty(M)$, its matrix representation is more natural than that of ∇, in that we get $K(fs_i)$ directly from evaluating the matrix of K at fs_i, without having to add on an extra term afterwards like the $df \otimes s_i$ that we had to for ∇.

Proposition 4.9. *Let $s_1, ...s_n$ be a local frame, with $\nabla(s_i) = \sum \omega_{ji} \otimes s_j$, where the entries of the matrix $[\omega_{ij}]$ are written in row-column notation. Then we can write*

$$K(s_i) = \sum \Omega_{ji} \otimes s_j \tag{4.10}$$

where

$$\Omega_{ji} = d\omega_{ji} - \sum_\alpha \omega_{\alpha i} \wedge \omega_{j\alpha} \tag{4.11}$$

This equation for Ω_{ji} is different by a sign from what [Milnor] has, and even if we use his convention that $\nabla(s_i) = \sum \omega_{ij} \otimes s_j$ rather than $\sum \omega_{ji} \otimes s_j$, the order of the indices is still different. [Milnor] is probably right, but he does not provide a proof, and neither I nor my mentor could see anything wrong with my version. So, here is the justification for what we have written above:

$$\hat{\nabla} \circ \nabla(s_i) = \hat{\nabla} \sum \omega_{ji} \otimes s_j$$

Since $\hat{\nabla}$ is linear, for simplicity fix j=k. Then

$$\hat{\nabla}(\omega_{ki} \otimes s_k) = d\omega_{ki} \otimes s_k - \omega_{ki} \wedge \nabla(s_k)$$

Writing out $\nabla(s_k) = \sum \omega_{jk} \otimes s_j$ gives

$$= d\omega_{ki} \otimes s_k - (\omega_{ki} \wedge \omega_{1k} \otimes s_1 + \omega_{ki} \wedge \omega_{2k} \otimes s_2 + ... + \omega_{ki} \wedge \omega_{nk} \otimes s_n)$$

Now, summing over all k (and thus turning k back into j), we get

$$= \sum_j (d\omega_{ji} - \sum_\alpha \omega_{\alpha i} \wedge \omega_{j\alpha}) \otimes s_j$$

$$\Rightarrow \Omega_{ji} = d\omega_{ji} - \sum_\alpha \omega_{\alpha i} \wedge \omega_{j\alpha}$$

Since $K(s_i)$ gives us the i^{th} column of the matrix for K.

Equivalently, we can write $\Omega_{ij} = d\omega_{ij} - \sum_{\alpha} \omega_{\alpha j} \wedge \omega_{i\alpha}$.

We finish this section by defining characteristic classes and showing that they are actually independent of the connection ∇.

Definition 4.12. Let $\mathbb{A}$ be an algebra, and $M_n(\mathbb{A})$ the algebra of $n \times n$ matrices with entries in $\mathbb{A}$. An invariant polynomial $P : M_n(\mathbb{A}) \to \mathbb{A}$ is a polynomial function on the entries of a matrix, which has the property

$$P(AB) = P(BA)$$

for any matrices $A, B \in M_n(\mathbb{A})$.

Examples of such polynomials are the determinant and trace. In fact, all invariant polynomials on the entries of $n \times n$ matrices are just polynomial functions of the first n elementary symmetric functions. The reason we are interested in invariant polynomials is because they are invariant under a change of basis; i.e. $P(SAS^{-1}) = P(A)$, so the output of the polynomial gives information about the underlying linear map. We can apply these polynomials to our curvature matrix $[\Omega_{ij}]$ to give us elements in the algebra of differential forms. Because of their invariance, given such a polynomial P, each curvature tensor K has a unique polynomial $P([\Omega_{ij}])$ assigned to it in every local frame. Since every point has a neighborhood in which a local frame exists, we can then patch these locally defined forms together to get a global form, which we call an exterior form. This form is denoted $P(K)$.

One can show that all exterior forms are closed (see [Milnor]), thus they are elements of the de Rham cohomology. We will now show that the exterior form $P(K)$ does not depend on the choice of connection ∇, even though K itself does. Thus given some curvature tensor K, the invariant polynomial P determines a cohomology class, called the characteristic class of P.

Definition 4.13. Given a map $g : M \to N$ the induced pullback map g^* on connections is the unique map such that $g^*\nabla(g^*(s)) = (g^* \circ \nabla)(s)$ for any section s, where, from left to right, the first g^* is the one we're defining, the pullback on connections, the second g^* is the pullback on sections of TM, and the third g^* (just to the right of the equal sign) is the pullback on sections of $T^*M \otimes TM$ (which is defined by pulling back the cotangent bundle sections and tangent bundle sections separately and tensoring them together). Thus, if $\nabla(s_i) = \sum \omega_{ji} \otimes s_j$, then

$$g^*\nabla(g^*s) = \sum g^*(\omega_{ji} \otimes s_j) = \sum g^*\omega_{ji} \otimes g^*s_j$$

Lemma 4.14. *Characteristic classes 'commute' with pullbacks:*

$$g^*P(K_\nabla) = P(K_{g^*\nabla})$$

where the g^* on the left side is the induced map on the cohomology classes, and the g^* on the right is the induced map on connections.

Proof. For the sake of clarity, some pullbacks by g will simply be marked with a prime; i.e. $g^*\nabla = \nabla'$, $g^*s = s'$, etc. On the right side of the equation above, we have, for some s,

$$\begin{aligned} K_{\nabla'}(\sigma') &= \hat{\nabla}(\omega' \otimes s') \\ &= d\omega' \otimes s' + \omega' \wedge \nabla'(s) \end{aligned}$$

And on the left side we have:

$$\begin{aligned} g^*K_\nabla(s) &= g^*\hat{\nabla}(\omega \otimes s) \\ &= g^*(d\omega \otimes s + w \wedge \nabla(s)) \end{aligned}$$

Since pullbacks of forms commute with d,

$$\begin{aligned} &= dg^*w \otimes s' + \omega' \wedge \nabla'(s) \\ &= d\omega' \otimes s' + \omega' \wedge \nabla'(s) \end{aligned}$$

□

Theorem 4.15. *The cohomology class $P(K_\nabla)$ is independent of the connection ∇.*

Proof. Let ∇_0 and ∇_1 be any two connections on a manifold M. Let $\pi : M\times\mathbb{R} \to M$ be the projection which takes (x,t) to x, and let $T'M$ denote the induced bundle π^*TM over $M \times \mathbb{R}$. Define

$$\nabla = t\nabla_1' + (1-t)\nabla_o'$$

, where ∇_0' and ∇_1' are the induced connections over $M \times \mathbb{R}$. Consider the map $i_\epsilon : M \to M \times \mathbb{R}$ which takes $x \to (x, \epsilon)$, where ϵ is 0 or 1. Define the induced map i_ϵ' on ∇ by evaluating t at ϵ, then pulling back to M. Thus

$$i_0'(\nabla) = i_0'\nabla_0' = \nabla_0$$

since for fixed $t = 0$, π and i_0 are inverses. The map i_ϵ induces a map on the cohomology groups, denoted i_ϵ^*, which has the property that

$$i_\epsilon^*(P(K_\nabla)) = P(K_{\nabla_\epsilon})$$

by the previous lemma. But since i_0 is homotopic to i_1 by the map $H(x,t) = i_t(x)$ (i.e. the identity on $M \times \mathbb{R}$), the cohomology classes $i_0^*(P(K_\nabla)) = P(K_{\nabla_0})$ and $i_1^*(P(K_\nabla)) = P(K_{\nabla_1})$ must be equal, by Prop. 1.12. □

5. Acknowledgements

Thank you to my graduate mentors Jim Fowler and Strom Borman. It was a great pleasure learning and working with them this summer, and I appreciate all the time and effort that they put into meeting with me and answering the many questions I had.

References

[1] [Hatcher] Allen Hatcher. Algebraic Topology. Cambridge University Press, Cambridge, UK, 2002.
[2] [Lee] John Lee. Introduction to Smooth Manifolds. Springer-Verlag, New York, NY, 2003.
[3] [Milnor] John Milnor and James Stasheff. Characteristic Classes. Princeton University Press, Princeton, NJ, 1974.