DIFFERENTIAL TOPOLOGY AND THE POINCARÉ-HOPF THEOREM

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1. Introduction

In this paper we approach the topology of smooth manifolds using differential tools, as opposed to algebraic ones such as homology or the fundamental group. The main result is the Poincaré-Hopf index theorem, which states the sum of the indices of a vector field with finitely many zeros on a smooth compact oriented boundaryless manifold is equal to the Euler characteristic of the manifold. To lead up to this theorem, we will look at smooth maps between manifolds and study intersection numbers, fixed points, and transversality.

If $f : X \longrightarrow Y$ is a smooth map of oriented manifolds with Z a submanifold of Y, the intersection number of f with Z, denoted $I(f, Z)$ is the number of points in $f^{-1}(Z)$ counted with signs ± 1 depending on the way the map f behaves locally with respect to the orientations on X, Y, and Z. As a simple example, let $f : S^1 \longrightarrow \mathbb{R}^2$ be a simple closed curve and let Z be the unit circle as a submanifold of $\mathbb{R}^2$. The intersection of f with Z is counted positively at a point of intersection if the positive tangent vector to f and the positively oriented tangent vector to Z together form a positively oriented basis for the tangent space of $\mathbb{R}^2$. Consequently, whenever f travels from outside the unit circle to inside the unit circle, the intersection is positive, and whenever f travels from inside the unit circle to outside the unit circle, the intersection is negative (or vice versa, depending on the chosen orientations).

In order to ensure that the set $f^{-1}(Z)$ is finite, we will have to assume that $\dim X + \dim Z = \dim Y$ and that f is transversal to Z. We say f is transversal to Z if for all $x \in f^{-1}(Z)$, $\mathrm{im} df_x + T_{f(x)}Z = T_{f(x)}Y$, where the $+$ denotes the span of two subspaces. Transversality intuitively means that the tangent plane to f and the tangent plane to Z at given point in $\mathrm{im} f \cap Z$ is not contained in any hyperplane in the ambient tangent space.

We will study fixed points and Lefschetz theory as a way of proving the Poincaré-Hopf theorem. The proof of the Poincaré-Hopf theorem consists of two stages. First we show that the global Lefschetz number of a smooth map is equal to the sum of its local Lefschetz numbers, and provide a concrete way to compute these local Lefschetz numbers as the degrees of maps defined on local spheres.

The second part of the proof involves vector fields. We will show that the degree of a vector field is equal to the global Lefschetz number of its flow. Rather than using integral curves or solutions to differential equations, we will construct a more rudimentary deformation of the of the identity that is only tangent to the vector field at time zero but that will still suffice. Since a deformation of the identity is by definition homotopic to the identity and since intersection number is homotopy

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invariant, this will show that the sum of the indices of the vector field is indeed equal to the self-intersection number of the diagonal of X with itself, which is the Euler characteristic.

Finally, we will use the Poincaré-Hopf Theorem to provide an intuitive way of computing the Euler characteristic of smooth orientable compact 2-manifolds (surfaces of genus g), and to prove the theorem that any vector field on an even-dimensional sphere has a zero.

This paper is based on my reading of *Differential Topology*, by Guillemin and Pollack [1], and many of the proofs and the overall order of presentation are based on this text.

2. Preliminaries

Definition 2.1. A manifold X is a locally Euclidean, Hausdorff, second-countable, topological space.

Definition 2.2. A *smooth manifold* of dimension n is a manifold X together with an open cover $\{U_\alpha\}$ and homeomorphisms $\phi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^n$ such that for all α, β, the map $\phi_\alpha \circ \phi_\beta^{-1}$ is C^∞ on its domain. The pairs (U_α, ϕ_α) are called charts and the maps ϕ_α are called coordinate maps.

All manifolds in this paper will be smooth. If X is a manifold and $x \in X$, the *tangent space* to X at x is denoted T_xX and is a vector space of the same dimension as X. If X is a manifold of dimension n and Y a manifold of dimension m, we say that a map $f : X \longrightarrow Y$ is *smooth* if for any coordinate maps ϕ of X and ψ of Y the composition $\psi \circ f \circ \phi^{-1}$ is a C^∞ map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ on its domain. The map $f : X \longrightarrow Y$ is a *diffeomorphism* if it is bijective and both f and f^{-1} are smooth. The *differential* of f at a point $x \in X$ is denoted df_x and is a linear map $df_x : T_xX \longrightarrow T_{f(x)}Y$ between the tangent space of x at X and the tangent space of Y at y. The map f is said to be an *immersion* if for all $x \in X$, df_x is injective. The map f is said to be a *submersion* if for all $x \in X$, df_x is surjective. We say that a manifold $Z \subset Y$ is a *submanifold* of Y if its inclusion map $i : Z \hookrightarrow Y$ is a smooth, injective immersion. The *codimension* of Z in Y is the dimension of Y minus the dimension of Z.

Theorem 2.3. *(Inverse Function Theorem) Let $f : X \longrightarrow Y$ be a smooth map. The differential df_x is an isomorphism if and only if there exists an open neighborhood U of x such that $f : U \longrightarrow f(U)$ is a diffeomorphism.*

Definition 2.4. (Regular values) We say that $y \in Y$ is a *regular value* of $f : X \longrightarrow Y$ if for all $x \in f^{-1}(y)$, df_x is surjective. We say that $y \in Y$ is a *critical value* if it is not a regular value.

Theorem 2.5. *(Regular Submanifold Theorem) If $y \in Y$ is a regular value of the smooth map $f : X \longrightarrow Y$, then $f^{-1}(y)$ is a smooth submanifold of Y of dimension equal to* $\dim X - \dim Y$.

This theorem can be proved using the Implicit Function Theorem to construct coordinate maps for $f^{-1}(Y)$. For a proof, see page 21 of [1]. The Regular Submanifold Theorem is the basis for all the results in this paper. One simple application is to show that if $f : X \longrightarrow Y$ is a map between n-manifolds and $y \in Y$ a regular value, then $f^{-1}(y)$ is a 0-manifold, hence a discrete set of points. Thus if X is compact, $f^{-1}(y)$ is finite.

Theorem 2.6. *(Sard's theorem) If $f : X \longrightarrow Y$ is a smooth map, then almost all $y \in Y$ are regular values of f, i.e., the set of critical values has measure* 0.

Sard's theorem will be used routinely to ensure the existence of regular values. See [1] for a proof.

Let V be an n-dimensional vector space. If $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are two ordered bases of V. Define a linear map A by setting $A(v_i) = w_i$ for $i = 1, \dots, n$. We say that the bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are equivalently oriented if $\det A > 0$. This determines an equivalence relation on the set of ordered bases of V having two equivalence classes. An *orientation* on V is the choice to call one equivalence class positive and the other negative.

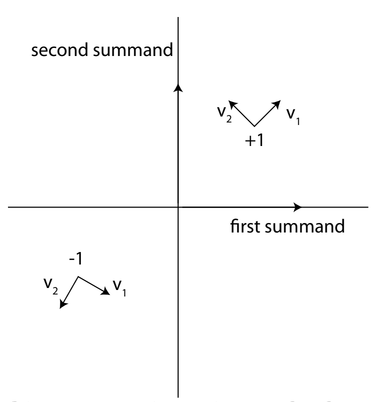
Example 2.7. The *standard orientation* on $\mathbb{R}^n$ is the equivalence class containing the standard ordered basis $e_1, \dots, e_n$.

Definition 2.8. A smooth manifold X *orientable* if there is a smooth choice of orientations for all the tangent spaces $T_x X$.

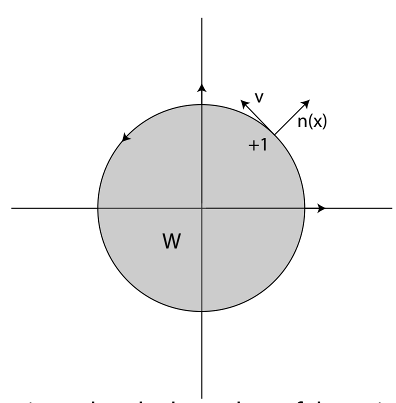
The choice of orientations on the tangent spaces is said to be *smooth* if for all $x_0 \in X$, there is a neighborhood U of x_0 and a coordinate map $\phi : U \longrightarrow \mathbb{R}^n$ such that for all $x \in U$ the differential $d\phi_x$ sends any positively oriented basis of $T_x X$ to a basis in the same equivalence class as the standard basis on $\mathbb{R}^n$.

Example 2.9. The circle S^1 is orientable. Its two orientations are clockwise and counterclockwise. The Möbius band M is not orientable, since one ordered basis can be smoothly slid around the band so that when it returns to its starting point, its orientation has reversed.

We can define an orientation on direct products. If V_1 and V_2 are two oriented vector spaces, the direct sum orientation on $V_1 \oplus V_2$ is defined as follows. If $v_1, \dots, v_n$ is a positively oriented basis for V_1 and $w_1, \dots, w_m$ is a positively oriented basis for V_2, then $v_1, \dots v_n, w_1, \dots, w_m$ is positively oriented for $V_1 \oplus V_2$. If X and Y are two oriented manifolds, for any point $(x, y) \in X \times Y$, $T_{(x,y)} X \times Y = T_x X \times T_y Y$, thus the product orientation on $X \times Y$ is defined by taking the direct sum orientation on tangent spaces.



Direct sum orientation on R x R



S^1 oriented as the boundary of the unit disk

If $X = \partial W$ and W is an oriented manifold, X inherits a natural boundary orientation from W. For a point $x \in X$, let $n(x)$ denote any outward pointing vector in T_xW. Then we say that a basis $v_1, \ldots, v_k$ of T_xX is positively oriented if and only if $n(x), v_1, \ldots, v_k$ is a positively oriented basis for T_xW.

3. Submanifolds and transversality

Let Z be a submanifold of Y. Then for any point $y \in Z$, and any vector v tangent to Z at y, the vector v is also tangent to Y at y. More precisely, the inclusion map $i : Z \longrightarrow Y$ is an immersion, meaning that for any point $y \in Y$, the map $di_y : T_yZ \longrightarrow T_yY$ is an injective linear map. Thus we can view T_yZ as a linear subspace of T_yY via the natural inclusion map di_y. In particular, if Z is a submanifold of $\mathbb{R}^n$, then for any $z \in Z$, T_zZ can be viewed as all the vectors in $\mathbb{R}^n$ starting at z and tangent to Z, i.e., as a linear subspace of $\mathbb{R}^n$ with origin translated to z.

Example 3.1. Consider the unit circle S^1 as a submanifold of $\mathbb{R}^2$. The tangent space to $\mathbb{R}^2$ at the point $(1, 0)$ is the set of vectors v starting at $(1, 0)$ and ending at any point in $\mathbb{R}^2$. Thus $T_{(1,0)}\mathbb{R}^2$ can be identified with $\mathbb{R}^2$ with the origin moved to $(1, 0)$. The tangent space to S^1 at $(1, 0)$ is the set of vectors starting at $(1, 0)$ and ending at any point $(1, x)$, and thus can be identified with the line $x = 1$.

Recall that if $f : X \longrightarrow Y$ is a smooth map, we say that $y \in Y$ is a regular value if for all $x \in f^{-1}(y)$, df_x is surjective. Since $T_y\{y\}$ is the zero vector space, we can reword the statement that y is a regular value as the following: for all $x \in f^{-1}(y)$,

$$\mathrm{im} df_x + T_y\{y\} = T_yY,$$

where the + sign denotes the linear span of two subspaces. This equation leads to the following generalization of regular values.

Definition 3.2. (Transversality) Let Z be a submanifold of Y. We say the map $f : X \longrightarrow Y$ is *transversal* to Z, denoted $f \pitchfork Z$, if for all $x \in f^{-1}(Z)$ with $y = f(x)$,

$$\mathrm{im} df_x + T_yZ = T_yY.$$

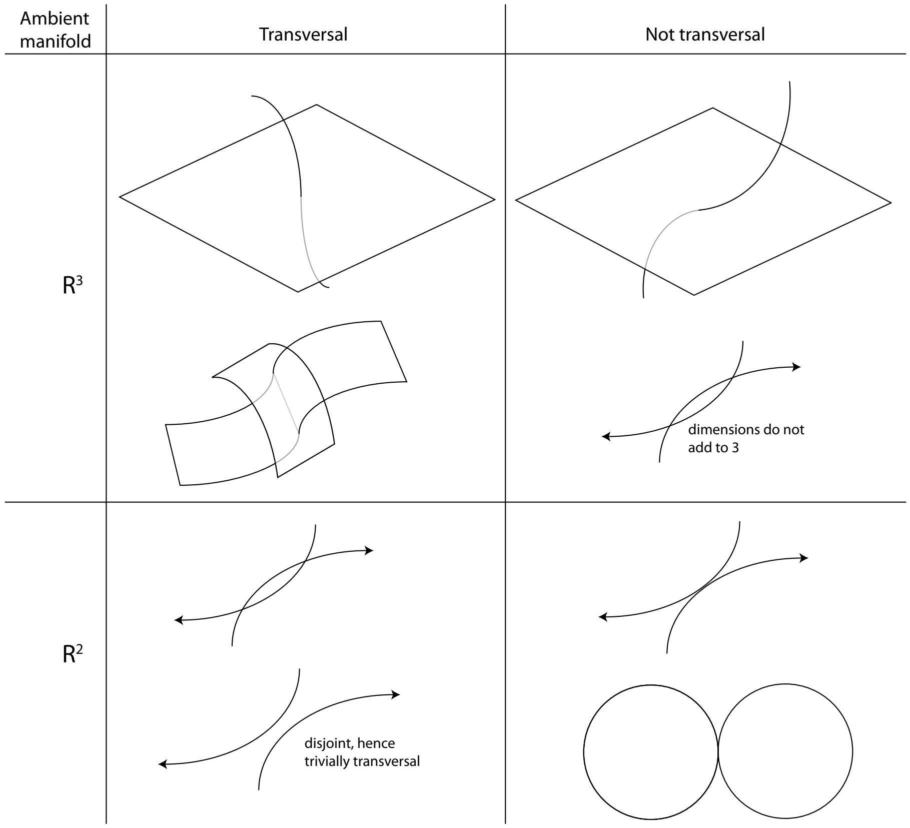
This means that at each point in $\mathrm{im} f \cap Z$, the vectors tangent to f and the vectors tangent to Z together span the ambient tangent space. See the figure on the next page for examples.

Example 3.3. Consider the submanifold S^1 as the unit circle in $\mathbb{R}^2$. The maps $\mathbb{R} \longrightarrow \mathbb{R}^2$ sending $x \mapsto (x, 0)$ or $x \mapsto (1/2, x)$ are transversal to S^1, but the maps sending $x \mapsto (x, x^2 + 1)$ and $x \mapsto (x, \sqrt{2} - x)$ are not transversal to S^1 because they are tangent to S^1. The map sending $x \mapsto (x, x^2 + 2)$ is trivially transversal to S^1 because it does not intersect S^1.

The following theorem illustrates how a submanifold transversal to f is analogous to a regular value.

Theorem 3.4. *(Transversal Submanifold Theorem) If $f : X \longrightarrow Y$ is transversal to the submanifold Z of Y, then $f^{-1}(Z)$ is a submanifold of X of codimension equal to the codimension of Z in Y.*

Proof. The idea of the proof is to rewrite $f^{-1}(Z)$ as the preimage of a regular value under another function, and then apply the Regular Submanifold Theorem. Suppose $\dim Y = n$ and $\dim Z = k$. Locally Z looks like $\mathbb{R}^k$ inside $\mathbb{R}^n$. Thus for



each $y \in Z$, there is a neighborhood U of y and a diffeomorphism $\phi : U \longrightarrow V \subset \mathbb{R}^n$ such that $\phi(y) = 0$ and

$$\phi(U \cap Z) = \{0_{n-k}\} \times \mathbb{R}^k.$$

If $\phi = (\phi_1, \dots, \phi_n)$, let

$$\psi = (\phi_1, \dots, \phi_{n-k})$$

be the first $n-k$ coordinates of ϕ. Then clearly $f^{-1}(Z) \cap f^{-1}(U) = (\psi \circ f)^{-1}(0)$. Thus it suffices to show that 0 is a regular value of $\psi \circ f$.

Since f is transversal to Z, for any $x \in f^{-1}(Z) \cap f^{-1}(U)$ with $f(x) = y$, we have

$$\operatorname{im} df_x + T_y Z = T_y Y.$$

Note that $\psi = \pi \circ \phi$ where π is the projection $\mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$ sending $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-k})$. Thus $d\psi_y = d\pi_0 \circ d\phi_y$ is the composition of the surjection $d\pi_0$ and the isomorphism $d\phi_y$, hence surjective. Furthermore, since ψ vanishes on Z, the kernel of $d\psi_y$ contains $T_y Z$. Thus applying $d\psi_y$ to both sides of the above equation gives

$$\operatorname{im} d(\psi \circ f)_x + 0 = T_0 \mathbb{R}^{n-k},$$

which shows that 0 is a regular value of $\psi \circ f$. □

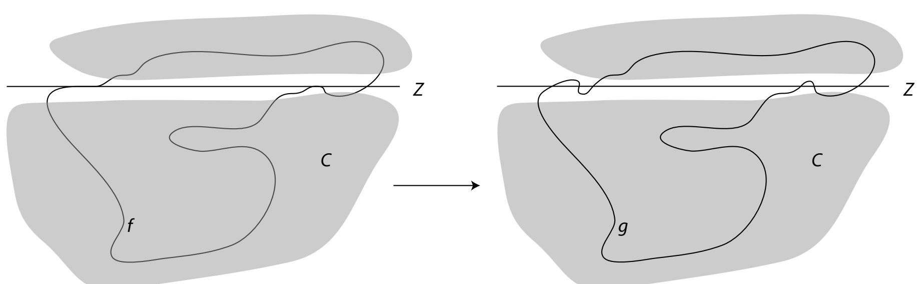
An important property of transversality is for any smooth map $f : X \longrightarrow Y$ and any submanifold Z of Y, it is possible to perturb f by an arbitrarily small amount such that it becomes transversal to Z.

Theorem 3.5. *(Transversality Homotopy Theorem) Let $f : X \longrightarrow Y$ be a smooth map with Z a submanifold of Y, where X may have boundary. Then there exists a smooth map $g : X \longrightarrow Y$ homotopic to f such that both g and ∂g are transversal to Z.*

The theorem is a corollary of Sard's Theorem. See [1] pages 68-70 for a proof. The claim is fairly intuitive, for the following reason. The map f is not transversal to Z only if for some point $x \in f^{-1}(Z)$ with $f(x) = y$ there is sufficient linear dependence between $\mathrm{im} df_x$ and T_yZ that the span of these two subspaces is not all of T_yY or $\dim X + \dim Z < \dim Y$. In the first case, the linear dependence can be removed by a slight perturbation. In the second case, it is possible to perturb f slightly so that it avoids Z altogether.

Example 3.6. Consider the map $x \mapsto (x, x^3)$ of $\mathbb{R} \longrightarrow \mathbb{R}^2$. This map is not transversal to the x-axis at $(0, 0)$, but it can be perturbed slightly so that it becomes transversal. The map $x \mapsto (x, x^3, 0)$ of $\mathbb{R} \longrightarrow \mathbb{R}^3$ is also not transversal to the x-axis, but it can be perturbed slightly so that it does not intersect the x-axis at all.

In fact, if we know that the map $f : X \longrightarrow Y$ is transversal to Z on some closed subset C of X, then it is possible to perturb f so that it is transversal to Z all over without changing its values on a neighborhood of C.



The map *f* can be made transversal to *Z* by perturbing it only outside of the closed region C.

Theorem 3.7. *(Transversal extension theorem) Let $f : X \longrightarrow Y$ and let Z be a closed submanifold of Y, where only X has boundary. Suppose that f is transversal to Z on a closed set $C \subset X$. Then there exists a smooth map $g : X \longrightarrow Y$ homotopic to f such that g and ∂g are transversal to Z and such that g agrees with f on a neighborhood of C.*

See [1] page 72 for a proof. The above theorem shows that it suffices to perturb f only on a neighborhood of the set on which transversally fails.

4. Intersection numbers

We now apply the previous ideas of orientation and transversality to topological properties of manifolds and smooth maps. Let $f : X \longrightarrow Y$ be a map of smooth manifolds, with Z a submanifold of Y and let only X have boundary. For this entire section, we shall assume that $f \pitchfork Z$ and that $\dim X + \dim Z = \dim Y$, that X is compact, and that Z, Y, Z are all oriented. We aim to study the points of intersection of f with Z, i.e., the solutions of the equation $f(x) \in Z$.

Since $\dim X + \dim Z = \dim Y$, the Regular Submanifold Theorem says that $f^{-1}(Z)$ is a 0-dimensional submanifold of X, hence a discrete set of points. Since X is compact, $f^{-1}(Z)$ is finite. Thus a naive approach would be just to count the number of points in $f^{-1}(Z)$. We would like however to come up with a number that depends only on the homotopy class of f. Clearly f can be perturbed to intersect Z arbitrarily many times. However, if we count these intersections with signs ± 1 depending on the direction f crosses Z, we can indeed obtain a number with the desired homotopy invariance. In order to assign signs to intersections, we use the orientations of X, Y, and Z.

For a point $x \in f^{-1}(Z)$ with $f(x) = y$, the dimensional constraint $\dim X + \dim Z = \dim Y$ and transversality implies that

$$\mathrm{im} df_x \oplus T_y Z = T_y Y$$

is a direct sum and that the map df_x is an injective linear map. Thus the isomorphism $df_x : T_x X \longrightarrow \mathrm{im} df_x$ induces an orientation on $\mathrm{im} df_x$ coming from the orientation on $T_x X$. We say that a basis of $v_1, \dots, v_n$ of $\mathrm{im} df_x$ is positively oriented if and only if it is the image under df_x of a positively oriented basis for $T_x X$.

Definition 4.1. (Orientation number) Let $f : X \longrightarrow Y$ be transversal to Z with X, Y, Z oriented and $\dim X + \dim Z = \dim Y$. The *orientation number* of f with respect to Z at a point $x \in f^{-1}(Z)$ with $f(x) = y$ (denoted $i_x(f)$) is defined to be $+1$ if the direct sum orientation on $\mathrm{im} df_x \oplus T_y Z$ agrees with the given orientation on $T_y Y$ and -1 otherwise.

Definition 4.2. (Intersection number) Let f be transversal to Z as above. The *intersection number* of f with respect to Z is defined to be

$$I(f, Z) := \sum_{x \in f^{-1}(Z)} i_x(f),$$

the sum of the orientation numbers of f with respect to Z at each point in $f^{-1}(Z)$.

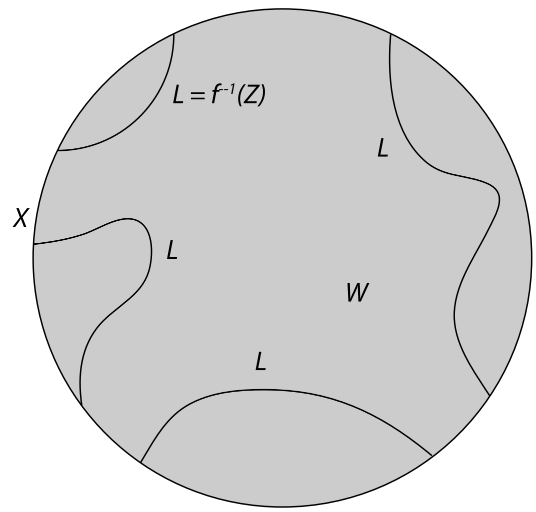
Our immediate goal is to show that intersection number is homotopy invariant in f. Suppose that $F : I \times X \longrightarrow Y$ is a homotopy of f_0 to f_1. The boundary of $I \times X$ is $X \times \{1\} - X \times \{0\}$, and thus

$$\begin{aligned} I(\partial F, Z) &= I(F\big|_{X \times \{1\} - X \times \{0\}}, Z) \\ &= I(f_1, Z) - I(f_0, Z) \end{aligned}$$

Thus to prove homotopy invariance it suffices to show that $I(\partial F, Z) = 0$, which follows from:

Theorem 4.3. *If $X = \partial W$ (with W compact and oriented) and $f : X \longrightarrow Y$ extends smoothly to W, then for any submanifold Z of Y satisfying $f \pitchfork Z$ and $\dim X + \dim Z = \dim Y$, we have $I(f, Z) = 0$.*

Proof. Let $F : W \longrightarrow Y$ be an extension of f. By the Transversal Extension Theorem, we may assume that F and ∂F are transversal to Z. Thus by the Transversal Submanifold Theorem, $F^{-1}(Z)$ is a submanifold of W of dimension 1. Furthermore, $F^{-1}(Z)$ is a closed submanifold of a compact manifold, hence compact. It is a well known theorem that any smooth compact 1-manifold is the disjoint union of circles and arcs; see the appendix of [1] for a proof. The circle components of $F^{-1}(Z)$ do not intersect X, and thus does not contribute to $I(f, Z)$.



Thus it suffices to show that each arc component of $L := F^{-1}(Z)$ contributes zero to $I(f, Z)$. To do this, we will show that the intersection number of f at a given endpoint x of one arc is ± 1 depending on whether the positively oriented tangent vector to L at x is inward or outward pointing on the boundary of W. Since on each arc, the positively oriented tangent vector to L points inward at one end and outward at the other, this will show the contribution from each arc is 0.

Before proceeding, we need orientations on X and L. First we define an orientation on L. Let $x \in L$ and $f(x) = y$. The idea is to show that dF_x is an isomorphism from any complementary subspace of $T_xL \subset T_yW$ onto a corresponding complementary subspace of $T_yZ \subset T_yY$. Let $v_0, \ldots v_k$ be a positively oriented basis for T_xW with v_0 tangent to L. Then $v_1, \ldots, v_k$ is a basis for a complementary subspace of $T_xL \subset T_xW$. Since $F \pitchfork Z$, we have

$$dF_x(v_0) + dF_x(v_1) + \ldots + dF_x(v_k) + T_yZ = T_yY.$$

But since F maps L into Z, dF_x maps v_0 into T_yZ. Hence the above equation becomes

$$dF_x(v_1) + \ldots + dF_x(v_k) + T_yZ = T_yY.$$

By the assumed dimensional constraints, $k + \dim Z = \dim Y$; hence the above sum is in fact a direct sum. We say that v_0 is positively oriented in T_xL if and only if the direct sum orientation on the left side of the above equation agrees with the given orientation on T_yY.

Next we orient X as the boundary of W. For any point $x \in X$, let $n(x)$ be any outward pointing vector. We say a basis $v_1, \ldots, v_k$ of T_xX is positively oriented if and only if $v_0, v_1, \ldots, v_k$ is positively oriented for T_xW.

Now let x be one endpoint of L. We will compute $i_x(f)$. Since L intersects X precisely at its endpoints, $x \in X$. Since L and X intersect at a discrete set of points and have complimentary dimension in W, T_xL and T_xX are complimentary in T_xW. At x, the positively oriented vector v_0 for T_xL is either an outward or inward pointing vector.

Suppose first that v_0 is an outward pointing vector. Let $v_1, \ldots, v_k$ be a positively oriented basis for T_xX. By definition of the boundary orientation on W, since v_0

is outward pointing,

$$v_0, v_1, \ldots, v_k$$

is a positively oriented basis for T_xW. Hence by the definition of the orientation on L defined above, the direct sum orientation on

$$dF_x(v_1) \oplus \ldots \oplus dF_x(v_k) \oplus T_yZ$$

agrees with the given orientation on T_yY. Since $dF_x = df_x$ when restricted to T_xX, the above statement is true if and only if the direct sum orientation on

$$\mathrm{im} df_x \oplus T_yZ = T_yY$$

agrees with the given orientation on T_yY, which by definition just means that the intersection number of f at x is $+1$.

If v_0 were instead inward pointing, the same argument shows that the intersection number of f at x would be -1. Since L is an arc, its positively oriented tangent vector must point outward at one endpoint and inward at the other endpoint. Hence the intersection numbers at the two endpoints are $+1$ and -1, which sum to 0. Adding the contributions from each arc, we have $I(f, Z) = 0$. $\square$

Thus $I(f, Z)$ is homotopy invariant in f. This allows us to extend the definition of intersection number to arbitrary maps f. By the Transversality Homotopy Theorem, there exists a map $\tilde{f} \pitchfork Z$ homotopic to f. We then define $I(f, Z)$ to be $I(\tilde{f}, Z)$.

5. Transversal Maps

We would like to show that the intersection number $I(f, Z)$ is homotopy invariant in not only f but also Z. If i is the inclusion map $Z \hookrightarrow Y$, we can view $I(f, Z)$ as the intersection number of the maps f and i. To this end, we define transversality and intersection numbers of maps.

Definition 5.1. (Transversality of maps) Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$. The maps f and g are transversal (denoted $f \pitchfork g$) if for all $x \in X$ and $z \in Z$ such that $f(x) = g(z) =: y$, we have

$$\mathrm{im} df_x + \mathrm{im} dg_z = T_yY.$$

The definitions of orientation number and intersection number for $f \pitchfork g$ are analogous to that for $f \pitchfork Z$.

Definition 5.2. (orientation number for maps) Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be transversal with $\dim X + \dim Y = \dim Z$. For points $x \in X$ and $z \in Z$ such that $f(x) = g(z) =: y$, the *orientation number* of f and g at (x, z) (denoted $i_{(x,z)}(f, g)$) is defined to be $+1$ if the direct sum orientation of $\mathrm{im} df_x \oplus \mathrm{im} dg_z$ agrees with the orientation of T_yY and -1 otherwise.

Definition 5.3. (intersection number of maps) The intersection number of the maps f and g is defined to be

$$I(f, g) := \sum_{f(x)=g(z)\in Z} i_{(x,z)}(f, g).$$

To prove that $I(f, g)$ is homotopy invariant in both f and g, we will relate $I(f, g)$ to $I(f \times g, \Delta(Y))$.

Theorem 5.4. *Let* $f : X \longrightarrow Y$ *and* $g : Z \longrightarrow Y$ *with* $\dim X + \dim Z = \dim Y$. *Then* $f \overline{\pitchfork} g \iff f \times g \overline{\pitchfork} \Delta$, *where* $\Delta(Y)$ *is the diagonal of* $Y \times Y$.

Proof. Suppose $f \overline{\pitchfork} g$. Then transversality and dimensional constraints imply that $\text{im} df_x \oplus \text{im} dg_z = T_y Y$ is a direct sum and both df_x and dg_z are injective. Hence

$$\begin{aligned} f \overline{\pitchfork} g &\iff \text{im} df_x \oplus dg_z = T_y Y \\ &\qquad \text{(by transversality and dimensional constraints)} \\ &\iff \text{im} df_x \cap \text{im} dg_z = 0 \\ &\iff \text{im} d(f \times g)(x, z) \cap \Delta(T_y Y) = \text{im} df_x \oplus \text{im} dg_z \cap \Delta(T_y Y) = 0 \\ &\iff \text{im} d(f \times g)(x, z) \oplus \Delta(T_y Y) = T_y Y \times T_y Y = T_{(y,y)}(Y \times Y). \end{aligned}$$

□

Hence if the maps $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ are transversal with $\dim X + \dim Y = \dim Z$, $f \times g : X \times Z \longrightarrow Y \times Y$ is transversal to $\Delta(Y)$. Since $\dim(X \times Z) + \dim(\Delta(Y)) = \dim(Y \times Y)$, we know by the Transversal Submanifold Theorem that $(f \times g)^{-1}(\Delta(Y))$ is a 0-dimensional submanifold of $X \times Z$. Thus the collection of points $(x, z) \in X \times Z$ such that $f(x) = g(z)$ is discrete. If X and Z are compact, this set is finite.

Theorem 5.5. *If* $f : X \longrightarrow Y$ *and* $g : Z \longrightarrow Y$ *are transversal,* $\dim X + \dim Y = \dim Z$, *and* X, Z *are compact, then*

$$I(f, g) = I(f \times g, \Delta(Y))(-1)^{\dim Z}.$$

This is just a computation. See [1] page 113 for a proof.

Thus $I(f, g)$ is homotopy invariant in both f and g. Furthermore, even if f is not transversal to g, we may define $I(f, g)$ by the above equation.

An immediate application of the homotopy invariance of $I(f, g)$ is showing that the degree of a smooth map $f : X \longrightarrow Y$ between n-manifolds is well-defined.

Definition 5.6. (degree) If $f : X \longrightarrow Y$ is a smooth map of n-dimensional manifolds and Y is connected, the degree of f is defined to be

$$\deg f = I(f, \{y\}),$$

where $y \in Y$ is any point.

Since Y is a connected manifold, it is path connected (this is true for manifolds, but not in general), so the inclusion maps of any two points $y \in Y$ are homotopic. Hence $\deg f$ does not depend on the point y chosen. The classic example is the map $S^1 \longrightarrow S^1$ sending $z \mapsto z^n$, which has degree n because it is an orientation preserving n-sheeted covering map (so the intersection number at each preimage point of y is $+1$). We shall use this definition of degree later in defining the index of a vector field at an isolated zero. For now, we turn to the Euler characteristic of a manifold.

6. Euler characteristic

If X is a compact CW-complex, its Euler characteristic is classically defined by the equation

$$\chi(X) = \#\text{ even cells } - \#\text{ odd cells}.$$

Instead of the above combinatorial definition, we shall define Euler characteristic using intersection theory.

Definition 6.1. If X is a compact manifold, the Euler characteristic of X is

$$\chi(X) = I(\Delta, \Delta).$$

For now, one should note how the above definition is immediately relevant to vector fields. If v is a vector field on X with finitely many zeros, the flow tangent to v will consist of a homotopic family of transformations $f_t : X \longrightarrow X$ with $f_0 = id$. Hence graph f_t is homotopic to graph id $= \Delta(X)$, so

$$I(f_t, \Delta) = I(\Delta, \Delta) = \chi(X).$$

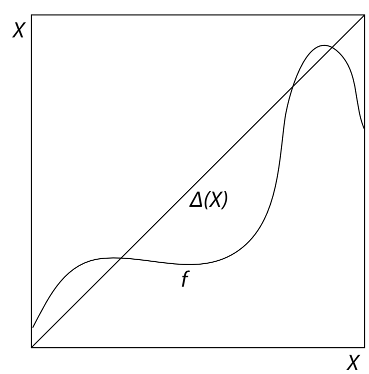
The intersection points of graph f_t and Δ are by definition the fixed points of f_t, and for small t, these fixed points are precisely the zeros of v. The Poincaré-Hopf Theorem will make precise this connection between $\chi(X)$ and the zeros of v. To this end, we will study the fixed points of maps.

7. Lefschetz fixed point theory

Let $f : X \longrightarrow X$ be a smooth self-mapping of a smooth $k-$manifold. We wish to study the fixed points of f. Note that a point $x \in X$ is a fixed point of f if and only if $(x, f(x)) \in \Delta$, where Δ is the diagonal of X. Thus fixed points of f correspond to intersection points of graph f and Δ, both of which are smooth k-dimensional submanifolds of the $2k$-dimensional manifold $X \times X$. Thus we can reformulate the study of fixed points in terms of an intersection number.

Definition 7.1. (Global Lefschetz number) Given a smooth self-mapping $f : X \longrightarrow X$, with X compact, the global Lefschetz number of f is defined to be

$$L(f) = I(\Delta, \text{ graph } f).$$



Note that if f is homotopic to g, the inclusion maps id $\times f$ and id $\times g$ of graph f and graph g are homotopic. Thus the global Lefschetz number of a map is homotopy invariant.

Definition 7.2. (Lefschetz map) The map $f : X \longrightarrow X$ is Lefschetz if graph $f \pitchfork \Delta$.

Lemma 7.3. *The map $f : X \longrightarrow X$ is Lefschetz at a fixed point $x \in X$ if and only if $df_x - I : T_xX \longrightarrow T_xX$ is an isomorphism.*

Proof. The statement that f is Lefschetz at x is

$$\operatorname{im}(d(id \times f)_{(x,x))} + T_{(x,x)}\Delta(X) = T_{(x,x)}(X \times X).$$

This can be rewritten as

$$\operatorname{im}(I \times df_x) + \Delta(T_x X) = T_x X \times T_x X.$$

Since $\operatorname{im}(I \times df_x)$ and $\Delta(T_x X)$ have complementary dimension inside $T_x X \times T_x X$, the above equation holds

$$\begin{aligned}
&\iff \operatorname{im}(I \times df_x) \cap \Delta(T_x X) = 0 \\
&\iff df_x \text{ has no nonzero fixed points} \\
&\iff \ker(df_x - I) = 0 \\
&\iff df_x - I \text{ is injective, and hence an isomorphism.}
\end{aligned}$$

□

Definition 7.4. (local Lefschetz number) If x is a Lefschetz fixed point of the map $f : X \longrightarrow X$, the local Lefschetz number of f at x (denoted $L_x(f)$) is defined to be the orientation number of (x, x) in the intersection of Δ with graph f, with Δ counted first.

Theorem 7.5. *If x is a Lefschetz fixed point of $f : X \longrightarrow X$, with X compact and oriented, then $L_x(f)$ is equal to $+1$ if the isomorphism $df_x - I$ preserves orientation and -1 if $df_x - I$ reverses orientation.*

Proof. Let $v_1, \ldots v_k$ be a positively oriented basis for $T_x X$. Then $v_1 \times v_1, \ldots, v_k \times v_k$ is a positively oriented basis for $T_{(x,x)}\Delta(X)$ and $v_1 \times df_x(v_1), \ldots v_k \times df_x(v_k)$ is a positively oriented basis for $T_{(x,f(x))}($ graph $f)$. Hence $L_x(f)$ is equal to the sign of the basis

$$v_1 \times v_1, \ldots, v_k \times v_k, v_1 \times df_x(v_1), \ldots, v_k \times df_x(v_k)$$

with respect to the product orientation on $T_{(x,x)}(X \times X)$. The above basis can be viewed as a matrix with $2k$ columns. Thus by Gaussian elimination we can subtract the first k columns from the last k columns to obtain the basis

$$v_1 \times v_1, \ldots v_k \times v_k, 0 \times (df_x - I)(v_1), \ldots 0 \times (df_x - I)(v_k)$$

with the same sign. Because $df_x - I$ is an isomorphism, the last k columns above span $0 \times T_x X$. Thus by Gaussian elimination again we obtain the basis

$$v_1 \times 0, \ldots v_k \times 0, 0 \times (df_x - I)(v_1), \ldots 0 \times (df_x - I)(v_k),$$

with the same sign. Hence the above basis is positively oriented in the product $X \times X$ if and only if the isomorphism $df_x - I$ preserves orientation. □

Note that if $f : X \longrightarrow X$ is a Lefschetz map, by definition

$$L(f) = \sum_{x=f(x)} L_x(f).$$

If $f : X \longrightarrow X$ is smooth, then f may have fixed points $x \in X$ that are not Lefschetz. In this case the total contribution of x to the global Lefschetz number $L(f)$ can be any integer (not just ± 1). Nonetheless, it is possible to perturb the map f on a neighborhood of x such that the fixed point x splits into finitely many Lefschetz fixed points. This property is described in the following theorem.

Theorem 7.6. *(Splitting of a fixed point into Lefschetz fixed points) Let $f : X \longrightarrow X$ be a smooth map and X be compact. Let U be a neighborhood of the fixed point x that contains no other fixed points. Then there exists a homotopy f_t of f such that each f_t equals f outside a compact subset of U and such that f_1 has only Lefschetz fixed points inside U.*

Proof. (From page 126 of [1].) Passing to charts, it suffices to prove the claim in $\mathbb{R}^k$. Let $f : U \longrightarrow \mathbb{R}^k$ be a smooth map fixing 0 but not fixing any other points, with U open in $\mathbb{R}^k$. Let $\phi : \mathbb{R}^k \longrightarrow [0,1]$ be a smooth map that is equal to 1 on a neighborhood V of 0 and with support contained in the compact $K \subset U$. We claim that for some point $a \in U$, the homotopy

$$f_t(x) = f(x) + t\phi(x)a$$

will allow us to split the fixed point x into finitely many Lefschetz fixed points. Since f has no fixed points on the compact $K \backslash V$, $|f(x) - x|$ has an absolute minimum $c > 0$ on this set. If $|a| < c/2$, then

$$|f_t(x) - x| = |f(x) - x + t\phi(x)| \geq |f(x) - x| - |\phi(x)| > c - c/2 = c/2,$$

so f_t has no fixed points for $x \in K \backslash V$ and $t \in [0,1]$.

Now by Sard's theorem, there exists a point a such that $|a| < c/2$ and such that a is a regular value of $id - f$. Now as above the only fixed points of f_t in U occur within V. Suppose in particular that x is a fixed point of f_1. Then $x \in V$, so $f_1(x) = f(x) + a$ near x. Hence $df_x = (df_1)_x$, so to show that x is a Lefschetz fixed point of f_1 it suffices to show that $df_x - I$ is nonsingular. Since $f_1(x) = f(x) + a = x$, then $a = x - f(x)$. Since a is a regular value of $id - f$, this implies that $I - df_x$ is surjective and hence an isomorphism. Thus $df_x - I$ is an isomorphism, which proves that x is a Lefschetz fixed point of f_1. □

The above theorem allows us to define local Lefschetz number for non-Lefschetz fixed points. If x is a fixed point of f and U a neighborhood of x containing no other fixed points, let f_1 be a map homotopic to f that agrees with f outside U and that has only Lefschetz fixed points in U. The local Lefschetz number of f at x is defined to be the sum of the local Lefschetz numbers of f_1 at all the Lefschetz fixed points in U. This definition is well-defined because global Lefschetz number is homotopy invariant

The following theorem is a key part of the proof of the Poincaré-Hopf Theorem, because it equates the local Lefschetz number at an isolated Lefschetz fixed point x of f to the degree of the map $\frac{f(x)-x}{|f(x)-x|}$ on a small sphere centered at x, and hence will allow us to equate the index at an isolated zero of a vector field v with the local Lefschetz number of its flow at that that point.

Theorem 7.7. *Let $f : U \longrightarrow \mathbb{R}^k$ be a smooth map, with U a neighborhood of 0 in $\mathbb{R}^k$ and such that 0 is a Lefschetz fixed point of f, where f has no other fixed points in U. Let B_ϵ be an epsilon neighborhood of 0 whose closure is contained in U. Then $L_0(f)$ is equal to the degree of the map*

$$\partial B_\epsilon \longrightarrow S^{k-1}$$
$$x \mapsto \frac{f(x) - x}{|f(x) - x|}.$$

Proof. (From page 128 of [1].) Taking a first order Taylor approximation of f, we have $f(x) = df_0(x) + R(x)$, where $R(x)/|x| \longrightarrow 0$ as $x \longrightarrow 0$. Since the map $df_x - I$

is an isomorphism, its kernel is 0 and hence it attains an absolute minimum $c > 0$ on the compact unit ball. Now choose a radius $\epsilon > 0$ such that $|R(x)|/|x| < c/2$ on B_ϵ. Define the homotopy

$$f_t(x) = df_0(x) + tR(x)$$

where $t \in [0,1]$. Note that for $|x| = \epsilon$,

$$|f_t(x) - x| = |df_0(x) - x + tR(x)| \geq |(df_0(x) - I)(x)| - |R(x)| \geq c\epsilon - c\epsilon/2 = c\epsilon/2,$$

so on ∂B_ϵ the map

$$x \mapsto \frac{f_t(x) - x}{|f_t(x) - x}$$

is a homotopy from the map

$$x \mapsto \frac{f(x) - x}{|f(x) - x|}$$

(whose degree is is $\deg_0(f)$) to the map

$$x \mapsto \frac{df_0(x) - I}{|df_0(x) - I|}$$

(whose degree is simply the sign ± 1 of the determinant of the linear map $df_0 - I$). By Theorem 7.5, the sign of $df_0 - I$ is simply $L_0(f)$. We are using here the easy linear algebra result that if A is an invertible linear map, the degree of $A(x)/|A(x)|$ on the unit ball is equal to ± 1 depending upon whether it preserves or reverses orientation. $\square$

Recall that we previously extended the definition of local Lefschetz number to non-Lefschetz fixed points by splitting the fixed point into finitely many Lefschetz fixed points by a local perturbation, and then letting $L_x(f)$ be the sum of the local Lefschetz numbers at these points. The above theorem provides alternative way to extend the definition of local Lefschetz number to non-Lefschetz fixed points. If x is an isolated, non-Lefschetz fixed point of f, we simply define $L_x(f)$ to be the degree of the map $x \mapsto \frac{f(x)-x}{|f(x)-x|}$ on a small sphere centered at x. The following theorem shows that that these two definitions agree

Theorem 7.8. *Let x_0 be an isolated fixed point of $f : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ and let B be a closed ball centered at x_0 that contains no fixed points other than x_0. Let f_1 be any map that agrees with f outside some compact subset of the interior of B that has only Lefschetz fixed points in B. Then*

$$L_{x_0}(f) = \sum_{f_1(x)=x\in U} L_x(f_1).$$

Proof. Let $x_1, \ldots, x_n$ be the fixed points of f_1 in B and let $B_1, \ldots, B_n$ be small disjoint spheres centered at $x_1, \ldots, x_n$ and contained in the interior of B. By Theorem 7.7, $L_x(f)$ is equal to the degree of

$$x \mapsto \frac{f(x) - x}{|f(x) - x|}$$

on ∂B. But on ∂B, this map is equal to

$$x \mapsto \frac{f_1(x) - x}{|f_1(x) - x|}.$$

Since f_1 has no fixed points on $B-(B_1\cup\ldots\cup B_n)$ and since $\partial(B-(B_1\cup\ldots\cup B_n)) = \partial B - (\partial B_1 \cup \ldots \cup \partial B_n)$, Theorem 4.3 tells us that

$$\begin{aligned}0 &= \deg(f_1|_{\partial(B-B_1\cup\ldots\cup B_n)})\\ &= \deg(f_1|_{\partial B}) - \sum_{i=1}^{n}\deg(f_1|_{\partial B_i})\\ &= L_{x_0}(f_1) - \sum_{i=1}^{n} L_{x_i}(f_1),\end{aligned}$$

where last line follows from Theorem 7.7. □

We have thus proven:

Corollary 7.9. *If $f : X \longrightarrow X$ is a smooth map with finitely many fixed points and X is compact, then*

$$L(f) = \sum_{f(x)=x} L_x(f).$$

The above equation holds even if f is not a Lefschetz map.

Proof. Let $B_1, \ldots, B_1$ be disjoint closed balls containing the fixed points $x_1, \ldots, x_n$ of f. Perturb f inside each ball to obtain a Lefschetz map f_1 homotopic to f. First since f and f_1 are homotopic, we have $L(f) = L(f_1)$.

Next since the claim is trivially true for the Lefschetz map f_1,

$$L(f_1) = \sum_{f_1(x)=x} L_x(f_1).$$

By Theorem 7.8, the sum of the local Lefschetz numbers of f_1 inside each B_i is simply $L_{x_i}(f)$, so the above sum is equal to

$$\sum_{f(x)=x} L_x(f).$$

Hence $L(f) = \sum_{f(x)=x} L_x(f)$, as claimed. □

8. Vectors fields

Definition 8.1. A *vector field* on a manifold X is a smooth assignment of a vector tangent to X at each point of x, i.e., a smooth map $v : X \longrightarrow TX$ such that $v(x) \in T_xX$ for all x.

Definition 8.2. (pullback of a vector field) Let $\phi : U \longrightarrow X$ be diffeomorphism and v a vector field on X. Then the pullback of v by ϕ is defined by

$$\begin{aligned}\phi^* v : U &\longrightarrow TU\\ u &\mapsto d\phi_u^{-1}\big(v(\phi(u))\big).\end{aligned}$$

Definition 8.3. (index at an isolated zero of a vector field) Let v be a vector field on $\mathbb{R}^k$. If z is an isolated zero of v, let B_ϵ be a ball around z containing no zeros of v other than z. The *index* of v at z (denoted $\mathrm{ind}_z v$) is defined to be the degree of the map

$$\begin{aligned}\partial B_\epsilon &\longrightarrow S^k\\ x &\mapsto \frac{v(x)-x}{|v(x)-x|}.\end{aligned}$$

The definition of index can immediately be extended to vector fields on an arbitrary smooth manifold X. If z is an isolated zero, choose a chart $\phi : U \longrightarrow V$ with V a neighborhood of z and U a neighborhood in $\mathbb{R}^k$ with $\phi(0) = z$. Now define $\mathrm{ind}_z v := \mathrm{ind}_0 \phi^* v$. It is somewhat nontrivial to show that that the index of a vector field is invariant under pullback by a diffeomorphism. For a proof, see [2] page 33.

Theorem 8.4. *(Poincaré-Hopf Index Theorem) If v is a smooth vector field on the compact oriented boundaryless manifold X with finitely many zeros, then the sum of the indices of the zeros of v is equal to the Euler characteristic of X.*

We shall prove the index theorem in the following way. The vector field v gives rise to a flow, a homotopic family of maps $f_t : X \longrightarrow X$ such that $f_0 = \mathrm{id}_X$ and such that for any fixed $x \in X$, the curve $f_t(x)$ is tangent to the vector field v. We shall show that the index of an isolated zero of v is equal to the local Lefschetz number of any of the flow maps f_t for small $t \neq 0$. Note that for small t the flow map f_t will have fixed points precisely at the zeros of v. Hence the sum of the indices at the zeros of v is equal to the sum of the local Lefschetz numbers of f_t. The sum of local Lefschetz numbers, however, is equal to the global Lefschetz number $L(f_t)$. Since f_t is homotopic to the identity transformation of X, then $L(f_t) = I(\Delta,\ \text{graph } f) = I(\Delta, \Delta) = \chi(X)$, proving the claim.

Lemma 8.5. *Let U be a neighborhood of 0 in $\mathbb{R}^k$ and let v be a vector field on U that vanishes only at 0. Let f_t be a homotopic family of maps with $f_0 = id_X$ and assume that for all $t \neq 0$, the map f_t has no fixed points in U except at 0. Furthermore, assume that the maps f_t are tangent to v at time zero, i.e., for all $x \in U$, the curve $f_t(x)$ is tangent to $v(x)$ at time $t = 0$. This means that at each point x, $\frac{\partial}{\partial t} f_t(x)|_{t=0}$ is a positive scalar multiple of v. Then for each f_t with $t \neq 0$, we have*

$$\mathrm{ind}_0 v = L_0(f_t).$$

Proof. (From page 135 of [1].) Taking a coordinate-wise second order Taylor expansion of $f_t(x)$, we may write

$$f_t(x) = f_0(x) + tv(x) + t^2 r_t(x),$$

where $r_t(x)$ is smooth in both t and x. Since for $t \neq 0$ we have $f_t(x) - x \neq 0$, we obtain

$$\frac{f_t(x) - x}{|f_t(x) - x|} = \frac{v(x) + tr_t(x)}{|v(x) - tr_t(x)|}.$$

Letting $t = 0$, the right hand side becomes $x \mapsto \frac{v(x)}{|v(x)|}$, the map whose degree is $\mathrm{ind}_0 v$. Letting $t \neq 0$, the left hand side gives us $L_0(f_t)$, by Theorem 7.7. Since the degree on the right and the Lefschetz number on the left are both homotopy invariant, we obtain the desired equality. □

We have thus proved that that the sum of the indices of a vector field v with finitely many zeros is the sum of the local Lefschetz numbers of its flow, which is just the Euler characteristic of X. Thus to prove Poincaré-Hopf, all that remains is to construct a deformation of the identity tangent to v.

Using the existence and uniqueness for solutions to ordinary differential equations, this construction would be trivial. However, we present a more elementary construction which makes use of the Tubular Neighborhood Theorem.

Construction 8.6. Let $v : X \longrightarrow TX$ be a smooth vector field, with X compact. Then there exists a homotopic family of maps $f_t : X \longrightarrow X$ tangent to v at time zero and such that $f_0 = id_X$ and for all $t \neq 0$, the fixed points of f_t are precisely the zeros of v.

Proof. (From page 137 of [1].) It is a theorem that any smooth connected manifold can be embedded as a submanifold of $\mathbb{R}^n$, for some n. Thus we may assume that X is a submanifold of $\mathbb{R}^n$. Since X is a compact submanifold of $\mathbb{R}^n$, then by the Tubular Neighborhood Theorem there exists $\epsilon > 0$ such that the normal bundle $N(X, \epsilon) := \{(x, v) | x \in X \text{ and } v \in (T_xX)^\perp \text{ and } |v| < \epsilon\}$ is diffeomorphic to the epsilon neighborhood $N_\epsilon := \{x \in \mathbb{R}^n | d(x, X) < \epsilon\}$ via the map $(x, v) \mapsto x + v$. Thus each point in the ϵ neighborhood of X can be written uniquely as $x + v$, where $x \in X$ and $v \in (T_xX)^\perp$. Hence there is a projection map $\pi : N_\epsilon \longrightarrow X$ sending $x + v \mapsto x$. It is not difficult to show that π is a submersion.

We can thus define the family of maps f_t by the formula

$$f_t : X \longrightarrow X$$

$$x \mapsto \pi(x + tv(x)).$$

At time t, the map f_t acts on a point $x \in X$ by sliding it along the vector $v(x)$ a distance of $t|v(x)|$ and then projecting the resulting point back onto X via the projection map π. Since the manifold X is compact, for sufficiently small t we can be sure that $x + tv(x)$ always lies in the tubular neighborhood, thus the map is well-defined for small t.

Note that by construction, the map f_0 is the identity of X. Furthermore, for $t \neq 0$, $x \in X$ is a fixed point of f_t if and only if π projects $x + tv(x)$ back down onto x. But this can only happen if $tv(x)$ is perpendicular to X, i.e., if $tv(x) \in (T_xX)^\perp$. But by definition, $tv(x)$ is a tangent vector to X at x, so it cannot be perpendicular to X unless it is zero. Thus $tv(x) = 0$ and since $t \neq 0$, we have $v(x) = 0$. This shows that for $t \neq 0$, fixed points of f_t are zeros of v. Conversely, if x is a zero of v, then trivially x is fixed by f_t.

It only remains to show that f_t is tangent to v at time zero. Fix a point $x \in X$ and consider the curve $f_t(x) = \pi(x + tv(x))$. By the differentiating with respect to t and using the chain rule we have

$$\frac{\partial}{\partial t}(f_t(x))\bigg|_0 = d\pi_x \circ v(x).$$

But π is the identity on X, so $d\pi_x$ is the identity map when restricted to T_xX. Since $v(x) \in T_xX$, we have $d\pi_x \circ v(x) = v(x)$, thus proving that $f_t(x)$ is tangent to v at time zero. □

This proves the Poincaré-Hopf theorem.

9. Conclusion: Some immediate applications

The Poincaré-Hopf Theorem provides a visually intuitive way to compute the Euler characteristic of a smooth manifold, simply by constructing a smooth vector field with finitely many zeros. Note that in two dimensions, the index of a vector field at a sink or source is $+1$, and the index at a saddle is -1.

It is easy to construct a vector field on S^2 with a source at the north pole and a sink at the south pole, thus $\chi(S^2) = 2$. Similarly, on the surface of genus g, a flow from one end to the other gives rise to a vector field with a source at one end,

$2g$ saddles, and a sink at the other end. Thus the surface of genus g has Euler characteristic $2 - 2g$.

Note that by the Poincaré-Hopf Theorem, any smooth orientable manifold that admits a non-vanishing vector field has Euler Characteristic zero. For even n, the sphere S^n has one 0-cell and one even n-cell, hence has Euler characteristic 2, which is non-zero. This proves the Hairy Ball Theorem in the smooth case: that any smooth vector field on S^n with n even has a zero.

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